M.A. Lavrent'ev

PROBLEMS OF CONTINUUM MECHANICS

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English Edition

CONTRIBUTIONS
IN HONOR OF
THE SEVENTIETH BIRTHDAY OF

ACADEMICIAN N. I. MUSKHELISHVILI

16th February 1961

PUBLISHED BY THE SOCIETY FOR
INDUSTRIAL AND APPLIED MATHEMATICS
Under a grant-in-aid from the National Science Foundation
PHILADELPHIA, PENNSYLVANIA
1961

RUSSIAN EDITION

Published by the Academy of Sciences of the U.S.S.R.

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Published by the Society for Industrial and Applied Mathematics

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Printed by N.V. Erven P. Noordhoff's Uitgeverszaak - Groningen

Printed in the Netherlands

PREFACE

TO THE ENGLISH EDITION

To honor Academician N. I. Muskhelishvili on his seventieth birthday, February 16, 1961, scientists and applied mathematicians from nine countries have contributed the articles collected here in this anniversary volume. The contents of this volume partly illustrate the scientific influence of Muskhelishvili as a teacher and a researcher through the numerous references to his outstanding books on the plane theory of elasticity and on integral equations; they also represent a breadth of technology in which Muskhelishvili has participated and in which he has exerted an influence which particularly manifests itself in his students, many of whom now play important roles in science and applied mathematics in the U.S.S.R. Indeed, the high standards of his achievements and his versatility have lead to the enthusiasm with which his colleagues everywhere have joined their most recent work in this single volume.

The diversity of topics surveyed and investigated here – problems in creep, crack formation, viscous flow, thermo- and visco-elasticity, the mechanics of soils, shells, and wave propagation; techniques in eigenvalue and boundary value problems; applications and properties of integral and differential equations – embody a cross-section of contemporary work in continuum mechanics. This branch of technology has attracted the attention of early mathematicians and scientists such as da Vinci, Galileo, Newton, Leibnitz, the Bernoullis, Euler, d'Alembert, Laplace, Lagrange, Gauss, Poisson and Cauchy whose collected works were often used as comprehensive treatises on the subjects which they embraced. But progress has been characterized by the gradual transition from this universal type of scientist to the specialist who focusses his attention on a narrow strip of the Universe, and these collected works are being replaced by a new kind of publication to fulfill the task of taking stock of the state of the art. Sometimes such status reports assume the form of carefully developed treatises, but at other times they bear closer resemblance to the random character of the collected works which they replace, as is the case here where the volume is to serve the additional purpose of honoring one of the acknowledged leaders in a field.

The simultaneous publication of the English and Russian editions of this volume, on February 16, 1961, is the result of a collaborative effort between the Academy of Sciences of the U.S.S.R. and the Society for Industrial and Applied Mathematics in the United States. The memorial volume was initiated by the Academy of Sciences which

invited contributions from applied mathematicians in China, Great Britain, Holland, India, Italy, Poland, Sweden, the United States and the U.S.S.R.

The Society for Industrial and Applied Mathematics is indebted to the Academy of Sciences of the U.S.S.R. for providing the Society with the complete set of manuscripts at the earliest possible date, to the contributors who have reviewed and proofread the English versions of their respective articles, to the National Science Foundation without whose support this English edition would not be possible, and especially to Dr. G. K. Mikhailov, Scientific Secretary of the U.S.S.R. National Committee on Theoretical and Applied Mechanics, whose constant attention to this project has made possible the close liaison and rapid communications necessary to achieve simultaneous bilingual publication.

I. E. BLOCK
J. R. M. RADOK
Philadelphia
February 16, 1961

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Н. Мускелишвили

NIKOLAI IVANOVICH MUSKHELISHVILI

Nikolai Ivanovich Muskhelishvili was born on February 16, 1891 in Tbilisi into the family of an army engineer. He received his secondary education at the Second Tbilisi High School (1901–1909) and his higher education at the University of Petrograd, in the Physico-Mathematical Faculty of which he graduated in 1914. In 1915, on presentation of a diploma thesis, N. I. Muskhelishvili was retained by the Department of Theoretical Mechanics for preparation for an academic career and he passed the master examination in 1917–1918.

In 1917, N. I. Muskhelishvili was elected assistant in the Department of Mechanics of the University of Petrograd and in 1919 an instructor in the same university. In addition, during the period from 1915 to 1920, he taught also at other higher educational institutions of Petrograd.

In 1920, N. I. Muskhelishvili moved to Tbilisi. His main work here took place in the Tbilisi University and in the Tbilisi Polytechnic Institute, where he was active at first as reader, and from 1922 onwards as professor. Since 1926, N. I. Muskhelishvili has held the post of dean almost continuously, at first in the Polytechnic Faculty of the University (until the transfer of this faculty to the Polytechnic Institute in 1928) and then in the Physico-Mathematical Faculty (until autumn 1936); during the period 1928–1930, he was occupied with the organization of the educational part of the Polytechnic Institute. Since 1938 N. I. Muskhelishvili has pursued educational work only at the University, where he is the head of the Department of Theoretical Mechanics.

Since 1933, N. I. Muskhelishvili has worked in the Georgian Division of the Transcaucasian Branch of the Academy of Sciences of the U.S.S.R. which was transformed in 1935 into the Georgian Branch of the Academy of Sciences of the U.S.S.R., and in 1941 into the Academy of Sciences of the Georgian S.S.R. From 1933–1935, he worked in the Geophysical Institute, and, after the organization of the Mathematical Institute of the Georgian Branch, was active there.

From 1935–1938, N. I. Muskhelishvili was deputy chairman of the Georgian Branch of the Academy of Sciences of the U.S.S.R., from 1938–1941 he was its chairman, and at the time of the foundation of the Academy of Sciences of the Georgian S.S.R. in 1941 he was an acting member and president of the Academy. For a number of years, N. I. Muskhelishvili has also been director of the Mathematical Institute of the Academy of Sciences of the AN Georgian S.S.R., named after A. M. Razmadze.

N. I. Muskhelishvili is largely responsible for the organization of the network of scientific institutions of Georgia. For example, due to his initiative and with his very intense participation, the Tbilisi Mathematical Institute as well as the Physical Institute of Tbilisi University were organized. N. I. Muskhelishvili took an active part in getting the work of the Georgian Polytechnic Institute under way, after it had been attached to the Polytechnic Faculty of the University. The Academy of Sciences of the Georgian S.S.R. which to-day represents a large complex of scientific research institutes was organized with his direct participation.

In 1934, N. I. Muskhelishvili was awarded the academic degree of doctor. In 1933, he became a corresponding member and in 1939 an active member of the Academy of Sciences of the U.S.S.R. He has been elected repeatedly to the staff of the Presidium of the Academy of Sciences of the U.S.S.R.

With the establishment in 1957 of the National Committee of the U.S.S.R. for Theoretical and Applied Mechanics, N. I. Muskhelishvili became its chairman. In 1956 he became a member of the International Committee of Congresses of Applied Mechanics and in 1960 a member of the Bureau of the International Union of Theoretical and Applied Mechanics.

N. I. Muskhelishvili is the author of over fifty scientific essays. They deal with different problems of the plane theory of elasticity, contact problems, torsion and bending of beams, the theory of singular integral equations, boundary value problems of analytic functions, et al. The methods developed in his work have been applied widely in contemporary mechanics and mathematics.

N. I. Muskhelishvili's fundamental monographs "Some basic problems of the mathematical theory of elasticity" and "Singular integral equations" were honored with Stalin prizes (1941 and 1946 respectively). These books have been translated into foreign languages and are widely known among specialists.

For his prominent scientific services in the region of mechanics, especially in the theory of elasticity, and also for his fruitful activity over many years in the training of scientific personnel, N. I. Muskhelishvili was awarded the title of Hero of Socialist Labor in 1945.

N. I. Muskhelishvili has also been very active in the political field. He was a member of the Central Executive Committee of the Transcaucasian Soviet Federative Republic of the VIIth Session (1935–1937), a delegate of the Georgian S.S.R. to the VIIIth Session of the Extraordinary All Union Congress of Soviets and a member of the Editorial Commission for the preparation of the final text of the Constitution of the U.S.S.R. N. I. Muskhelishvili is a deputy of the Supreme Soviet of the U.S.S.R. (Soviet of the Union) of all councils.

THE SCIENTIFIC WORKS

OF

N. I. MUSKHELISHVILI[1]

1915

On the equilibrium of elastic circular discs under the influence of stresses applied at points of their circumference and acting in their plane. Izv. Elektrotekhn. in-ta, Petrograd, 1915, *12*, 39–55 (co-authored with G. V. Kolosov). [Ann. Inst. électro techn., Pétrograd].

1916

On thermal stresses in the plane problem of the theory of elasticity. Izv. Electrotekhn. in-ta, Petrograd, 1916, *13*, 23–37, [Ann. Inst. électrotechn., Pétrograd].

1919

On the determination of a harmonic function from given values on a contour. Zh. Fiz.-mat. ob-va pri Permsk. un-te, 1918, (1919), No. 1., 89–93. [J. Soc. Phys. et mat. Univ. Perm].

Sur L'intégration de l'équation biharmonique. Izv. Rossiisk. AN, 1919, 13_6, No. 12–15, 663–686. [Bull. Russian Acad. Sci.]

1922

Lectures on analytic geometry, read at the Tiflis Polytechnic Institute, Pt. 1., Tiflis, Tiflissk. politekhn. in-t, 1922, 261 pp. (restricted).

Application des intégrales analogues à celles de Cauchy à quelques problèmes de la physique mathématique. Tiflis, Tiflissk. un-t, 1922, 157 pp.

Sulla deformazione piana di un cilindro elastico isotropo. Atti Accad. Lincei, 1922, 31_5, No. 12, 548–551.

1923

Sur l'équilibre des corps élastiques soumis à l'action de la chaleur. Bull. Univ. Tiflis, 1923, No. 3, 17–26.

1924

Solution of an integral equation encountered in the theory of black radiation. Zh. Russk. fiz. khim. ob-va fiz., 1924, *56*, No. 1., 30–39. [J. Russian Soc. Phys. Chem., Phys. Part].

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[1] This bibliography does not contain N. I. Muskhelishvili's works which bear a popular scientific character.

1925

Russian-Georgian and Georgian-Russian dictionary of mathematical terms. Pure and applied mathematics with theoretical mechanics. Tiflis, Gos. izd., 1925, 249 pp. (co-authored with G. N. Nikoladze and A. Kh. Kharadze,).

1926

Course of theoretical mechanics, Part I: Statics. Tiflis, Tiflissk. un-t, 1926, 292 pp. (in Georgian).

Lectures on analytic geometry, Part I, 2-nd revised edition, Tiflis, 1926, 208 pp.

1927

Sur la solution du problème biharmonique pour l'aire extérieur à une ellipse. Math. Z., 1927, *26*, No. 5, 700–705.

Sur l'intégration approchée de l'équation biharmonique. C. R. Acad. Sci., 1927, *185*, No. 22, 1184–1186.

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1928

Course of theoretical mechanics, Part II: Kinematics. Tiflis, Tiflissk. un-t., 1928, 253 pp. (in Georgian).

On periodic orbits and closed geodesic lines. (Abstract). Tr. Vcerossisk. s''ezda matematikov. (Moscow, April 27–May 4, 1927), Moscow-Leningrad, 1928, 189. [Proc. All-Russian Math. Congr.]

On certain boundary value problems of plane hydrodynamics. Ibid., 262.

Sur le problème fondamental d'hydrodynamique à deux dimensions. Atti Accad. Lincei, 1928, 7_6, No. 12, 995–1002.

1929

Zum Problem der Torsion der homogenen isotropen Prismen. Izv. Politekhn. in-ta Gruzii, 1929, *1*, No. 1, 1–20. [Bull. Inst. polytechn. Tiflis].

Sur le problème de torsion des cylindres élastiques isotropes. Atti Accad. Lincei, 1929, 9_6, No. 4, 295–300.

1930

Course of theoretical mechanics. Part I: Statics. Tiflis, Gos. izd., 1930, 216 pp. (in Georgian).

1931

Nouvelle méthode de réduction du problème biharmonique fondamental à une équation de Fredholm. C. R. Acad. Sci., 1931, *192*, No. 2, 77–79.

Théorèmes d'existence relatifs au problème biharmonique et aux problèmes d'élasticité à deux dimensions. Ibid., No. 4, 221–223.

1932

Course of theoretical mechanics, Part 2: Kinematics. 2-nd ed. Tbilisi, Gos. izd., Gruzii, 1932, 208 pp. (in Georgian).

On the problem of torsion and bending of elastic beams consisting of different materials. Izv. AN SSSR Otd. matem. i estestv. nauk, 1932, No. 7, 907–945. [Bull. Acad. Sci. URSS, Cl. sci. math. et natur.).

On the problem of torsion and bending of composite elastic beams. [Sur le problème de torsion et de flexion des poutres élastiques composées]. Izv. Inzh. str. in-ta Gruzii, 1932, No. 1, 123–127. [Ber. Bau-Inst. SSR Georgien, Tiflis].

[Published in French as: Sur le problème de torsion des poutres élastiques composées. C. R. Acad. Sci., 1932, *194*, No. 17, 1435–1437].

Recherches sur des problèmes aux limites relatifs à l'équation biharmonique et aux équations de l'élasticité à deux dimensions. Math. Ann., 1932, *107*, No. 2, 282–312.

1933

Course of analytic geometry, Part I. Course of analytic geometry using vectors. Moscow-Leningrad, 1933, 220 pp.

Course of theoretical mechanics. Part I: Statics. 2-nd ed., Tbilisi, Gos. izd. Gruzii, 1933, 190 pp. (in Georgian).

Some basic problems of the mathematical theory of elasticity. The basic equations, the plane problem, torsion and bending. (Forword by A. N. Krylov), Leningrad, Izd-vo AN SSSR, 1933, 397 pp.

Solution of the plane problem of the theory of elasticity for the solid ellipse. (Lösung des ebenen Problems der Elastizitätstheorie für die elliptische Scheibe). PMM (Old ser.), 1933, *1*, No. 1, 5–12.

Praktische Lösung der fundamentalen Randwertaufgaben der Elastizitätstheorie in der Ebene für einige Berandungsformen. Z. angew. Math. und Mech., 1933, *13*, No. 4, 264–282.

Sur l'équivalence de deux méthodes de réduction du problème plan biharmonique à une équation intégrale. C. R. Acad. Sci., 1933, *196*, No. 26, 1947–1948 (co-authored with V. A. Fok).

[Translation] A. G. Webster. The Dynamics of Particles and of Rigid, Elastic and Fluid Bodies. Lectures on mathematical physics. (Translation of 3rd English edition under the redaction of K. V. Melikov). Leningrad-Moscow, Gostekhizdat, 1933, 624 pp. (in cooperation with M. B. Savost'ianova and K. V. Melikov).

1934

Course of analytic geometry, Part 2. Leningrad-Moscow, Gostekhizdat, 1934, 276 pp.

A new general method of solution of the basic boundary problems of the plane theory of elasticity. (Nouvelle méthode générale pour résoudre les problèmes fondamentaux aux limites de la théorie de l'élasticité à deux dimensions). Dokl. AN SSSR, 1934, *3*, No. 1, 7–11 (C. R. Acad. Sci. URSS).

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1935

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On certain edge problems of the theory of elasticity. (Abstract), Ibid. 358.

1937

On the numerical solution of the plane problem of the theory of elasticity. Tr. Tbilissk. matem. in-ta, 1937, *1*, 83–87 (in Georgian with Russian summary). [Trav. Inst. math. Tbilisi].

[Editor] P. Frank and R. V. Mises. Die Differential und Integralgleichungen der Mechanik und Physik, Part 2, chaps. 7–9. Translation under the general editorship of L. E. Gurevich, Leningrad-Moscow, Gostekhizdat, 1937, 224–346.

1938

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1939

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1940

On the solution of the basic boundary problems of the theory of the Newtonian potential. [Über die Lösung der Fundamentalen Randwertaufgaben des Newtonischen Potentials] PMM, 1940, *4*, No. 4, 3–26.

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On the solution of the Dirichlet problem in the plane. Soobshch. Gruz. fil. AN SSSR, 1940, *1*, No. 2, 99–106. [Mitt. Georg. Abt. Akad. Wiss. USSR].

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1941

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Basic boundary value problems of the theory of elasticity for the half-plane. Soobshch. AN Gruz. SSR, 1941, *2*, No. 10, 873–880 [Mitt. Akad. Wiss. Georg. SSR.]

1942

Singular integral equations with Cauchy type kernels taken over open contours. Tr. Tbilissk. matem. in-ta AN Gruz. SSR, 1942, *11*, 141–172 (co-authored with D. A. Kveselava). [Trav. Inst. math. Tbilissi].

The basic boundary problems of the theory of elasticity for the half-plane with straight cuts. Soobshch. AN Gruz. SSR, 1942, *3*, No. 2, 103–110, [Bull. Acad. Sci. Georg. SSR].

On the problem of equilibrium of a rigid stamp on the boundary of an elastic half-plane in the presence of friction. Soobshch. AN Gruz. SSR, 1942, *3*, No. 5, 413–418 [Bull. Acad. Sci. Georg. SSR].

Systems of singular integral equations with Cauchy type kernels. Soobshch. AN Gruz. SSR, 1942, *3*, No. 10, 987–994 [Bull. Acad. Sci. Georg. SSR].

1943

The Riemann boundary value problem for several unknown functions and its applications to systems of singular integral equations. Tr. Tbilissk. matem. in-ta, 1943, *12*, 1–46 (co-authored with N. P. Vekua). [Trav. Inst. math. Tbilissi].

1946

Singular integral equations, boundary problems of function theory and their application to mathematical physics. Moscow-Leningrad, Gostekhizdat, 1946, 448 pp.

1947

Course of analytic geometry. 3-rd revised and augmented edition, Moscow-Leningrad, Gostekhizdat, 1947, 644 pp.

1949

Some basic problems of the mathematical theory of elasticity. The basic equations, the plane theory of elasticity, torsion and bending. 3-rd revised and substantially enlarged edition, Moscow-Leningrad, Izd-vo AN SSSR, 1949, 635 pp.

1951

Course of analytic geometry. Tbilisi, Izd-vo "Tekhnika da Shroma", 1951, 671 pp. (in Georgian).

1953

Some basic problems of the mathematical theory of elasticity. Groningen, P. Noordhoff, 1953, XXXI + 704 pp. (English translation of 3-rd edition, 1949).

Singular Integral Equations. Groningen, P. Noordhoff, 1953, 447 pp. (English translation of 1946 edition).

1954

Some basic problems of the mathematical theory of elasticity. 4th revised and augmented edition, Moscow, Izd-vo AN SSSR, 1954, 647 pp.

1955

Course of analytic geometry. Peking 1955, 633 pp. (Chinese translation of 3rd edition, 1947).

1956

Some basic problems of the mathematical theory of elasticity. Vols. 1–3. Bucharest Inst. studii rom. sov, 1956, 892 pp. (Rumanian translation of 3rd edition).

1958

Some basic problems of the mathematical theory of elasticity. Peking, 1958, 526 pp. (Chinese translation of 4th edition, 1954).

TORSION OF CIRCULAR RODS OF VARYING DIAMETER UNDER CONDITIONS OF UNSTEADY CREEP

N. KH. ARUTIUNIAN and M. M. MANUKIAN
Erevan, U.S.S.R.

This paper deals with the problem of torsion of circular rods of varying diameter under conditions of unsteady creep. In order to solve this problem we will start from the non-linear integral equations of the theory of creep, developed in [1]. The solution of the problem under consideration will be reduced to the solution of a nonlinear integro-differential equation, the solution and study of which for a power law of nonlinearity is presented in § 2. As an example, the solution of the problem of torsion of a circular and conical rod is obtained.

§ 1. Formulation of the problem and basic equations

We shall consider the problem of torsion of circular rods of varying diameter in a state of unsteady creep.

Let there be given a circular shaft of varying diameter, the lowest cross-section of which is clamped, while the upper is subject to the effect of the moment $M = M(t)$ which in the general case may depend on the time t. We will assume that the moment vector $M(t)$ is directed parallel to the axis of the rod. We direct the axis Oz of the cylindrical coordinate system (r, θ, z) along the axis of the rod, and place the plane $z = 0$ in the lower cross-section of the rod (the upper end being given by $z = l$).

We will propose, as in the theory of torsion of circular elastic rods of varying diameter, that

$$\sigma_r = \sigma_\theta = \sigma_z = \tau_{rz} = 0, \tag{1.1}$$

and that the state of stress is determined by the components of shear stress $\tau_{r\theta}$ and $\tau_{z\theta}$, which do not depend on θ.

Then the equilibrium equation in cylindrical coordinates assumes the form

$$\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{\partial}{\partial z}(r^2\tau_{z\theta}) = 0, \tag{1.2}$$

and it will be satisfied identically, if, as usually, one sets

$$\tau_{r\theta} = -\frac{1}{r^2}\frac{\partial U}{\partial z}, \quad \tau_{\theta z} = \frac{1}{r^2}\frac{\partial U}{\partial r}, \tag{1.3}$$

where the stress function U depends only on r, z and t.

Further, let Γ be the contour, bounding the simply connected region of the axial cross-section, then

$$U|_\Gamma = \text{const.}, \tag{1.4}$$

since the side surface of the rod is free from external loading.

On the end sections of the rod there must be given a distribution of the shear stress $\tau_{\theta z}$. Consequently, here likewise the stress function is known

$$U(0, r, t) = U(l, r, t) = \varphi(r, t). \tag{1.5}$$

For long rods this condition may be fulfilled by use of Saint-Venant's principle.

By the condition of static equivalence we will have

$$M(t) = 2\pi \int_0^{r_1} \tau_{\theta z} r^2 \, dr = 2\pi[U(r_1, z, t) - U(0, z, t)], \tag{1.6}$$

where r_1 is the radius of the cross-section of the shaft.

The condition of compatibility of displacements takes, by virtue of (1.1), the form

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\gamma_{\theta z}\right) - \frac{\partial}{\partial z}\left(\frac{1}{r}\gamma_{\theta r}\right) = 0. \tag{1.7}$$

As is known, in the general case of a three dimensional state of stress, the equations, interrelating the strain intensity $\varepsilon_i(t)$ and the stress intensity $\sigma_i(t)$ with consideration of nonlinear creep of the material and changes of its modulus of instantaneous deformation, have the form [1] [1]

$$\varepsilon_i(t) = \frac{\sigma_i(t)}{3G(t)} - \int_{\tau_1}^{t} \sigma_i(\tau) \frac{\partial}{\partial \tau}\left[\frac{1}{3G(\tau)}\right] d\tau - \int_{\tau_1}^{t} F[\sigma_i(\tau)]\sigma_i(\tau) \frac{\partial C(t, \tau)}{\partial \tau} \, d\tau. \tag{1.8}$$

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[1] We note that, as in many cases, the equation of the theory of flow, developed and widely used by L. M. Kachanov for the solution of problems of the theory of creep of metals, follows from the basic creep law (1.8).

If in (1.8) we put $G(t) = G = \text{const.}$, $C(t, \tau) = C(t) - C(\tau)$ and $F(\sigma_i) = \sigma_i^{m-1}$, we obtain after differentiation with respect to t the equation

$$\frac{d\varepsilon_i}{dt} = \frac{1}{E}\frac{d\sigma_i}{dt} + B(t)\sigma_i^m,$$

where

$$B(t) = \frac{dC(t)}{dt}, \quad E = 3G, \quad m > 1,$$

which are the initial equations, presented in the work of L. M. Kachanov [2].

Here $C(t, \tau)$ is the measure of the creep of the material in a uniaxial state of stress, $F(\sigma_i)$ is some function characterizing the nonlinear dependence between the creep stresses and strains for a given material, determined from experiments as the result of tests under simple creep and the normalized condition $F(1) = 1$, $G(t)$ is the modulus of instantaneous deformation by displacement and τ_1 is the age of the material at the instant of application of the load.

Under these conditions one has

$$\varepsilon_i(t) = \sqrt{\tfrac{2}{3}}\sqrt{(\varepsilon_x - \varepsilon_y)^2 + (\varepsilon_y - \varepsilon_z)^2 + (\varepsilon_z - \varepsilon_x)^2 + \tfrac{3}{2}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2)},$$

$$\sigma_i(t) = \frac{1}{\sqrt{6}}\sqrt{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)}. \tag{1.9}$$

In the case under consideration, the relations linking the strain components to the corresponding stress components assume by (1.1) the form

$$\gamma_{\theta z}(t) = \frac{\tau_{\theta z}(t)}{G(t)} - \int_{\tau_1}^{t} \tau_{\theta z}(\tau) \frac{\partial}{\partial \tau}\left[\frac{1}{G(\tau)}\right] d\tau - 3\int_{\tau_1}^{t} \tau_{\theta z}(\tau) F[\sigma_i(\tau)] \frac{\partial C(t, \tau)}{\partial \tau} d\tau,$$

$$\gamma_{r\theta}(t) = \frac{\tau_{r\theta}(t)}{G(t)} - \int_{\tau_1}^{t} \tau_{r\theta}(\tau) \frac{\partial}{\partial \tau}\left[\frac{1}{G(\tau)}\right] d\tau - 3\int_{\tau_1}^{t} \tau_{r\theta}(\tau) F[\sigma_i(\tau)] \frac{\partial C(t, \tau)}{\partial \tau} d\tau, \tag{1.10}$$

where

$$\sigma_i(t) = \sqrt{\tau_{\theta z}^2 + \tau_{r\theta}^2}. \tag{1.11}$$

We will let $F(\sigma_i)$ have the form

$$F(\sigma_i) = \alpha + \beta \sigma_i^{m-1}, \tag{1.12}$$

where α, β, m are constant parameters determined by experiment.

As has been shown by experimental studies, a power law of the form (1.12) describes experimental creep curves sufficiently well over a wide range of stress changes ($0 \leq \sigma \leq R$) for such materials as beton [3, 4], timber [5], coherent soil [6], plastics [7] and others. Simultaneously, these experiments show that for the stated materials the quantity β is small ($\beta < 1$).

Substituting the expression $F(\sigma_i)$ from (1.12) into (1.10) and taking (1.3) into consideration, we obtain

$$\gamma_{\theta z}(t) = \frac{1}{r^2}\frac{1}{G(t)}\frac{\partial U}{\partial r} - \int_{\tau_1}^{t} \frac{1}{r^2}\frac{\partial U(\tau)}{\partial r}\frac{\partial}{\partial \tau}\left[\frac{1}{G(\tau)}\right] d\tau -$$

$$- 3\int_{\tau_1}^{t}\left\{\alpha\frac{1}{r^2}\frac{\partial U(\tau)}{\partial r} + \beta\frac{1}{r^2}\frac{\partial U(\tau)}{\partial r}[\sigma_i(\tau)]^{m-1}\right\}\frac{\partial C(t,\tau)}{\partial \tau}\, d\tau,$$

(1.13)

$$\gamma_{r\theta}(t) = -\frac{1}{r^2}\frac{1}{G(t)}\frac{\partial U}{\partial z} + \int_{\tau_1}^{t} \frac{1}{r^2}\frac{\partial U(\tau)}{\partial z}\frac{\partial}{\partial \tau}\left[\frac{1}{G(\tau)}\right] d\tau +$$

$$+ 3\int_{\tau_1}^{t}\left\{\alpha\frac{1}{r^2}\frac{\partial U(\tau)}{\partial z} + \beta\frac{1}{r^2}\frac{\partial U(\tau)}{\partial z}[\sigma_i(\tau)]^{m-1}\right\}\frac{\partial C(t,\tau)}{\partial \tau}\, d\tau,$$

where

$$\sigma_i(t) = \frac{1}{r^2}\sqrt{\left(\frac{\partial U}{\partial r}\right)^2 + \left(\frac{\partial U}{\partial z}\right)^2}. \tag{1.14}$$

Introducing these expressions for the strain components into the compatibility equation (1.7), we obtain

$$\Delta U - G(t)\int_{\tau_1}^{t}\left\{\frac{\partial}{\partial \tau}\left[\frac{1}{G(\tau)}\right] + 3\alpha\frac{\partial C(t,\tau)}{\partial \tau}\right\}\Delta U\, d\tau =$$

$$= 3\beta G(t)\int_{\tau_1}^{t} L(U)\frac{\partial C(t,\tau)}{\partial \tau}\, d\tau, \tag{1.15}$$

where Δ and L are differential operators of the form

$$\Delta U = \frac{\partial}{\partial r}\left(\frac{1}{r^3}\frac{\partial U}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{1}{r^3}\frac{\partial U}{\partial z}\right), \tag{1.16}$$

$$L(U) = \frac{\partial}{\partial r}\left\{\frac{1}{r^{2m+1}}\left[\left(\frac{\partial U}{\partial r}\right)^2 + \left(\frac{\partial U}{\partial z}\right)^2\right]^{(m-1)/2}\frac{\partial U}{\partial r}\right\} + \tag{1.17}$$

$$+ \frac{\partial}{\partial z}\left\{\frac{1}{r^{2m+1}}\left[\left(\frac{\partial U}{\partial r}\right)^2 + \left(\frac{\partial U}{\partial z}\right)^2\right]^{(m-1)/2}\frac{\partial U}{\partial z}\right\}.$$

Thus, the solution of the problem of torsion of circular rods of varying diameter under conditions of unsteady creep has been reduced to the determination of the stress function $U(r, z, t)$ from the nonlinear integro-differential equation (1.15) for the boundary conditions (1.4) and (1.6).

Only for the simplification of the ensuing calculations we will assume that

$$G(t) = G = \text{const}. \tag{1.18}$$

Then equation (1.15) takes the form

$$\Delta U - \alpha\int_{\tau_1}^{t}\Delta U(\tau)K(t, \tau)d\tau = \beta\int_{\tau_1}^{t}L(U)K(t, \tau)d\tau, \tag{1.19}$$

where

$$K(t, \tau) = 3G\frac{\partial C(t, \tau)}{\partial \tau}. \tag{1.20}$$

§ 2. Solution of the basic integro-differential equation (1.19)

The solution of the equation (1.19) will be sought in the form

$$U(r, z, t) = U_0(r, z, t) + \beta U_1(r, z, t) + \beta^2 U_2(r, z, t) + \cdots. \tag{2.1}$$

Substituting (2.1) into (1.19) and comparing coefficients of equal powers of β, we obtain

$$\Delta U_0 - \alpha\int_{\tau_1}^{t}\Delta U_0(\tau)K(t, \tau)\,d\tau = 0, \tag{2.2}$$

$$\Delta U_1 - \alpha\int_{\tau_1}^{t}\Delta U_1(\tau)K(t, \tau)\,d\tau = \int_{\tau_1}^{t}L(U_0)K(t, \tau)\,d\tau, \tag{2.3}$$

$$\Delta U_2 - \alpha\int_{\tau_1}^{t}\Delta U_2(\tau)K(t, \tau)\,d\tau = m\int_{\tau_1}^{t}L(U_0, U_1)K(t, \tau)\,d\tau, \tag{2.4}$$

where $L(U_0, U_1)$ is a differential operator of the form

$$(2.5)\quad L(U_0, U_1) = \frac{\partial}{\partial r}\left\{\frac{1}{r^{2m+1}}\left[\left(\frac{\partial U_0}{\partial r}\right)^2 + \left(\frac{\partial U_0}{\partial z}\right)^2\right]^{\frac{m-1}{2}} \frac{\partial U_1}{\partial r} + \right.$$

$$+ r^{\frac{m-1}{2m+1}}\left[\left(\frac{\partial U_0}{\partial r}\right)^2 + \left(\frac{\partial U_0}{\partial z}\right)^2\right]^{\frac{m-1}{2}-1}\left(\frac{\partial U_0}{\partial r}\frac{\partial U_1}{\partial r} + \right.$$

$$\left.\left. + \frac{\partial U_0}{\partial z}\frac{\partial U_1}{\partial z}\right)\frac{\partial U_0}{\partial r}\right\} + \frac{\partial}{\partial z}\left\{\frac{1}{r^{2m+1}}\left[\left(\frac{\partial U_0}{\partial r}\right)^2 + \right.\right.$$

$$+ \left.\left(\frac{\partial U_0}{\partial z}\right)^2\right]^{\frac{m-1}{2}}\frac{\partial U_1}{\partial z} + r^{\frac{m-1}{2m+1}}\left[\left(\frac{\partial U_0}{\partial r}\right)^2 + \right.$$

$$\left. + \left(\frac{\partial U_0}{\partial z}\right)^2\right]^{\frac{m-1}{2}-1}\left(\frac{\partial U_0}{\partial r}\frac{\partial U_1}{\partial r} + \frac{\partial U_0}{\partial z}\frac{\partial U_1}{\partial z}\right)\frac{\partial U_0}{\partial z}\right\}.$$

Thus, the solution of the nonlinear integro-differential equation (1.19) reduces to the solution of a certain set of the recurrent linear integro-differential equations (2.2), (2.3), (2.4), $\cdots$ with the conditions

$$(2.6)\qquad U = \text{const.}$$

on the side surface ($r = r_1$) and

$$(2.7)\qquad U = 0$$

on the axis ($r = 0$) of the shaft.

Then the condition of static equivalence (1.6) takes the form

$$(2.8)\qquad U(r_1, z, t) = \frac{M(t)}{2\pi},$$

where r_1 is the radius of the cross-section of the shaft.

We will now turn to the solution of these equations. In the capacity of the solution of the integro-differential equation (2.2) we obtain

$$\Delta U_0(r, z, t) = 0,$$

or

$$(2.9)\qquad \frac{\partial^2 U_0(r, z, t)}{\partial r^2} - \frac{3}{r}\frac{\partial U_0(r, z, t)}{\partial r} + \frac{\partial^2 U_0(r, z, t)}{\partial z^2} = 0.$$

The solution of the equation (2.3) may be written in the form

$$(2.10)\quad \frac{\partial^2 U_1(r, z, t)}{\partial r^2} - \frac{3}{r}\frac{\partial U_1(r, z, t)}{\partial r} + \frac{\partial^2 U_1(r, z, t)}{\partial z^2} =$$

$$= r^3\left[\Psi_1(r, z, t) + \alpha\int_{\tau_1}^{t}\Psi_1(r, z, \tau)R(t, \tau, \alpha)\,d\tau\right],$$

where $R(t, \tau, \alpha)$ is the resolvent of the linear Volterra integral equation with the kernel $K(t, \tau)$ and

$$\Psi_1(r, z, t) = \int_{\tau_1}^{t} L(U_0)K(t, \tau)\, d\tau, \tag{2.11}$$

and the function $\Psi_1(r, z, t)$ is known, since the solution of the equation (2.9) is known.

If one accepts that the measure of the creep of the material is determined by the relationship

$$C(t, \tau) = \varphi(\tau)[1 - e^{-\gamma(t-\tau)}], \tag{2.12}$$

where $\varphi(\tau)$ is some monotonically decreasing function, characterizing the intensity of the change of the amount of creep of the material in dependence on its age τ and γ is a constant parameter, then the resolvent is determined by the formula [1]

$$\alpha R(t, \tau, \alpha) = \gamma - \eta'(\tau) + [\eta'^2(\tau) + \eta''(\tau) - \gamma\eta'(\tau)]\, e^{\eta(\tau)} \int_{\tau_1}^{t} e^{-\eta(x)}\, dx. \tag{2.13}$$

Here

$$\eta(\tau) = \gamma \int_{\tau_1}^{\tau} [1 + 3G\alpha\varphi(\tau)]\, d\tau. \tag{2.14}$$

We will assume that

$$\varphi(\tau) = \frac{A_1}{\tau} + C_0, \tag{2.15}$$

where A_1 and C_0 are certain parameters, characterizing the intensity of the change of the measure of the creep of the material $C(t, \tau)$ for the early and late stage of the material respectively.

In an analogous manner one may represent the solution of the equation (2.4) in the form

$$\frac{\partial^2 U_2(r, z, t)}{\partial r^2} - \frac{3}{r}\frac{\partial U_2(r, z, t)}{\partial r} + \frac{\partial^2 U_2(r, z, t)}{\partial z^2} = \tag{2.16}$$

$$= mr^3 \left[\Psi_2(r, z, t) + \alpha \int_{\tau_1}^{t} \Psi_2(r, z, \tau) R(t, \tau, \alpha)\, d\tau \right],$$

where

$$\Psi_2(r, z, t) = \int_{\tau_1}^{t} L(U_0, U_1) K(t, \tau)\, d\tau. \tag{2.17}$$

We note that the values U_0, U_1, U_2, $\cdots$ will be determined successively by integration of the differential equations (2.9), (2.10), (2.16), $\cdots$; for this purpose, the arbitrary integration functions, depending on the time t, are determined by the conditions (2.6), (2.7), (2.8).

§ 3. Torsion of a circular rod

In this case the stress function will only depend on the coordinates r, t and the equation (1.19) takes the form

$$\frac{\partial}{\partial r}\left(\frac{1}{r^3}\frac{\partial U}{\partial r}\right) - \alpha\int_{\tau_1}^{t} \frac{\partial}{\partial r}\left(\frac{1}{r^3}\frac{\partial U}{\partial r}\right) K(t, \tau)\, d\tau = \beta\int_{\tau_1}^{t} \frac{\partial}{\partial r}\left[\frac{1}{r^{2m+1}}\left(\frac{\partial U}{\partial r}\right)^m\right] K(t, \tau)\, d\tau. \tag{3.1}$$

Integrating this equation and using the relations (1.3), we obtain

$$\tau_{\theta z}(r, t) - \alpha\int_{\tau_1}^{t} \tau_{\theta z}(r, \tau) K(t, \tau)\, d\tau = \beta\int_{\tau_1}^{t} [\tau_{\theta z}(r, \tau)]^m K(t, \tau)\, d\tau + C_1(t) r, \tag{3.2}$$

where $C_1(t)$ is an arbitrary integration function. The equation (3.2) may likewise be written in the form

$$\tau_{\theta z}(r, t) - \alpha\int_{0}^{t} K(t, \tau)\tau_{\theta z}(r, \tau)\, d\tau - \beta\int_{\tau_1}^{t} [\tau_{\theta z}(r, \tau)]^m K(t, \tau)\, d\tau = g(r, t), \tag{3.3}$$

where

$$g(r, t) = C_1(t) r, \quad 0 < t,\ \tau < \infty. \tag{3.4}$$

In the integral equation (3.3) the functions $g(r, t)$ and $K(t, \tau)$ are continuous, α, β, m are numerical parameters and m is assumed to be an integer.

We will replace the equation (3.3) by its equivalent equation

$$\tau_{\theta z}(r, t) - \beta\int_{0}^{t} H(t, \tau)[\tau_{\theta z}(r, \tau)]^m\, d\tau = F(r, t), \tag{3.5}$$

where

$$F(r, t) = g(r, t) + \alpha \int_0^t R(t, \tau, \alpha) g(r, \tau)\, d\tau,$$

$$H(t, \tau) = K(t, \tau) + \alpha \int_\tau^t R(t, \xi, \alpha) K(t, \xi)\, d\xi. \tag{3.6}$$

We will assume that the solution of the equation (3.5) is an analytic function of the parameter β and we will determine the coefficients of its Taylor expansion in powers of β.

We will set

$$\tau_{\theta z}(r, t, \beta) = \sum_{k=0}^{\infty} W_k(r, t) \beta^k, \tag{3.7}$$

$$[\tau_{\theta z}(r, t, \beta)]^m = \sum_{k=0}^{\infty} H_k(r, t) \beta^k. \tag{3.8}$$

The coefficients $H_k(r, t)$ are expressed in terms of the coefficients $W_k(r, t)$ by the formulae

$$H_k(r, t) = \sum_k W_{i_1}(r, t) W_{i_2}(r, t) \cdots W_{i_m}(r, t), \tag{3.9}$$

where the sum is extended over all the non-negative solutions of the equation $i_1 + i_2 + \cdots + i_m = k$.

Substituting (3.7) into (3.5) and taking into consideration (3.8) we obtain

$$\sum_{k=0}^{\infty} W_k(r, t) \beta^k - \sum_{k=0}^{\infty} \beta \int_0^t H(t, \tau) H_k(r, \tau) \beta^k\, d\tau = F(r, t). \tag{3.10}$$

Comparing coefficients of equal powers of β, we find

$$W_{n+1}(r, t) = \int_0^t H(t, \tau) H_n(r, \tau)\, d\tau, \quad W_0(r, t) = F(r, t). \tag{3.11}$$

Substitution of (3.9) into (3.11) gives

$$W_{n+1}(r, t) = \sum_{i_1 + i_2 + \cdots + i_m = n} \int_0^t H(t, \tau) W_{i_1}(r, \tau) W_{i_2}(r, \tau) \cdots W_{i_m}(r, \tau)\, d\tau. \tag{3.12}$$

The formulae (3.12) represent recurrence relations which permit determination of all coefficients $W_n(r, t)$.

Let

$$F(r, t) = R_0(r)N_0(t), \tag{3.13}$$

then

$$W_0(r, t) = R_0(r)N_0(t). \tag{3.14}$$

It follows from (3.12) and (3.14) that

$$W_1(r, t) = \int_0^t H(t, \tau)R_0^m(r)N_0^m(\tau)\, d\tau = R_0^m(r) \int_0^t H(t, \tau)N_0^m(\tau)\, d\tau$$

or

$$W_1(r, t) = R_1(r)N_1(t), \tag{3.15}$$

where

$$R_1(r) = R_0^m(r), \quad N_1(t) = \int_0^t H(t, \tau)N_0^m(\tau)\, d\tau.$$

Analogously one may obtain

$$W_2(r, t) = R_2(r)N_2(t), \tag{3.16}$$

where

$$R_2(r) = R_0^{2m-1}(r), \quad N_2(t) = m \int_0^t H(t, \tau)N_0^{m-1}(\tau)N_1(\tau)\, d\tau.$$

In the general case we obtain

$$W_k(r, t) = R_k(r)N_k(t), \quad R_k(r) = R_0^{km-k+1}. \tag{3.17}$$

Let

$$|R_0(r)| \leq D, \tag{3.18}$$

then we will have

$$|R_k(r)| \leq D\left(\frac{E}{D}\right)^k, \quad E = D^m. \tag{3.19}$$

There remains to prove that the power series (3.7), where the coefficients $W_k(r, t)$ are determined from the recurrence relations (3.12), converges for sufficiently small $|\beta|$ and, the most important thing for the practical applicability of a method of solution, to state the bounds of β for which the series (3.7) continues to converge. For this purpose it is necessary to obtain an estimate of $|W_n(r, t)|$ as a function n, uniform with respect to t for some interval.

We will select the fixed time interval $0 \leq t \leq T$ and set

$$\max |K(t, \tau)| = K_T, \quad \max |R(t, \tau, \alpha)| = R_T, \quad 0 \leq t, \tau \leq T. \tag{3.20}$$

It follows then from (3.6) that

$$|H(t, \tau)| \leq K_T(1 + |\alpha| \, TR_T) = H_T. \tag{3.21}$$

In [8] it has been shown that

$$|N_k(t)| \leq \frac{1 \cdot m(2m - 1) \cdots [(k - 1)m - k + 2]}{k!} H_T^k t^k. \tag{3.22}$$

Then

$$|W_k(r, t)| \leq \frac{1 \cdot m(2m - 1) \cdots [(k - 1)m - k + 2]}{k!} H_T^k t^k D^{k(m-1)+1}, \tag{3.23}$$

$$k = 0, 1, 2 \cdots.$$

We consider now the series of numbers

$$\sum_{k=0}^{\infty} \frac{1 \cdot m(2m - 1) \cdots [(k - 1)m - k + 2]}{k!} H_T{}^k T^k \, |\beta| \, D^{k(m-1)+1} \tag{3.24}$$

with positive terms which by the strength of the estimates (3.23) is obviously a majorant for the power series (3.7) in the interval $0 \leq t \leq T$.

The radius of convergence r_T of the series (3.24) is determined by the known formula of Cauchy-Hadamard

$$\frac{1}{r_T} = \lim_{k \to \infty} \sqrt[k]{\frac{1 \cdot m(2m - 1) \cdots [(k - 1)m - k + 2]}{k!} H_T^k T^k D^{k(m-1)+1}}. \tag{3.25}$$

For the evaluation of the limit in the formula (3.25) we note that the following obvious estimate holds true:

$$\frac{1 \cdot m(2m - 1)(3m - 2) \cdots [(k - 1)m - k + 2]}{k!} \leq \leq \frac{1 \cdot m \cdot 2m \cdot 3m \cdots (k - 1)m}{k!} = \frac{m^{k-1}}{k}, \tag{3.26}$$

by the strength of which we obtain from (3.25)

$$\frac{1}{r_T} \leq \lim_{k \to \infty} H_T T_m^{\frac{k-1}{k}} k^{-\frac{1}{k}} D^{m + \frac{1}{k} - 1} = mH_T T D^{m-1}, \tag{3.27}$$

or

$$r_T \geq \frac{1}{mH_T T D^{m-1}}$$

Thus, in each case for those β which satisfy the condition

$$|\beta| < (mH_T TD^{m-1})^{-1}, \quad D \geq |R_0(r)|, \tag{3.28}$$

the numerical series (3.24) converges, and by the strength of the above statement the power series (3.7) converges absolutely and uniformly in the considered range $0 \leq t \leq T$; its sum is a regular analytic function of the parameter β inside the circle (3.28) and represents the solution of the basic nonlinear integral equation (3.5) or (3.3).

If in the general solution (3.7) of the integral equation (3.3) one restricts oneself to the first two approximations, we obtain, using the relations (3.12), (3.6) and (3.7),

$$\tau_{\theta z}(r, t) = r\left\{H_0(t, \tau_1) + \right. \tag{3.29}$$

$$\left. + \beta r^{m-1}\left[H_1(t, \tau_1) + \alpha \int_{\tau_1}^{t} H_1(\tau, \tau_1)R(t, \tau, \alpha)\, d\tau\right]\right\} + O(\beta^2),$$

where

$$H_0(t, \tau_1) = C_1(t) + \alpha \int_{\tau_1}^{t} C_1(\tau)R(t, \tau, \alpha)\, d\tau, \tag{3.30}$$

$$H_1(t, \tau_1) = \int_{\tau_1}^{t} [H_0(\tau, \tau_1)]^m K(t, \tau)\, d\tau. \tag{3.31}$$

We will have the solution of the integral equation (3.30) in the form

$$C_1(t) = H_0(t, \tau_1) + \alpha \int_{\tau_1}^{t} H_0(\tau, \tau_1)K(t, \tau)\, d\tau, \tag{3.32}$$

since $R(t, \tau, \alpha)$ is the resolvent of the kernel $K(t, \tau)$.

There remains to determine the unknown function $H_0(t, \tau_1)$. For this purpose one must substitute in the equation (3.29) for $\tau_{\theta z}$ the term $1/r^2\ \partial U/\partial r$ and integrate the equation obtained, using the condition

$$U = 0 \text{ for } r = 0. \tag{3.33}$$

Then we obtain

$$U(r, t) = H_0(t, \tau_1)\frac{r^4}{4} + \tag{3.34}$$

$$+ \beta \frac{1}{m+3} r^{m+3}\left[H_1(t, \tau_1) + \alpha \int_{\tau_1}^{t} H_1(\tau, \tau_1)R(t, \tau, \alpha)\, d\tau\right] + O(\beta^2).$$

Substituting the expression $U(r, t)$ from (3.34) into

$$M(t) = 2\pi[U(a, t) - U(0, t)], \tag{3.35}$$

where a is the radius of the cross-section of the shaft, we find

$$H_0(t, \tau_1) + \beta b_0 \int_{\tau_1}^{t} \Big\{ [H_0(\tau, \tau_1)]^m + \\ + \alpha \int_{\tau_1}^{\tau} [H_0(x, \tau_1)]^m R(t, x, \alpha)\, dx \Big\} K(t, \tau)\, d\tau = a_0(t). \tag{3.36}$$

Here

$$a_0(t) = \frac{2M(t)}{\pi a^4}, \quad b_0 = \frac{4}{m+3} a^{m-1}. \tag{3.37}$$

If one is to apply the above method to the equation (3.36) and to satisfy only the first two approximations, one obtains for the unknown function $H_0(t, \tau_1)$

$$H_0(t, \tau_1) = a_0(t) \Big\{ 1 + \beta b_0 a_0^{m-1}(t) \Big[3GC(t, \tau_1) - \\ - \alpha \int_{\tau_1}^{t} \int_{\tau_1}^{\tau} R(t, x, \alpha) K(t, \tau)\, d\tau\, dx \Big] \Big\} + O(\beta^2). \tag{3.38}$$

In an analogous manner, limiting oneself to the first three approximations in the general solution (3.7), one obtains

$$U(r, t) = H_0(t, \tau_1) \frac{r^4}{4} + \beta \frac{1}{m+3} r^{m+3} \Big[H_1(t, \tau_1) + \\ + \alpha \int_{\tau_1}^{t} H_1(\tau, \tau_1) R(\tau, \alpha)\, d\tau \Big] + \\ + \beta^2 \frac{m}{2(m+1)} r^{2(m+1)} \Big[H_2(t, \tau_1) + \\ + \alpha \int_{\tau_1}^{t} H_2(\tau, \tau_1) R(t, \tau, \alpha)\, d\tau \Big] + O(\beta^3), \tag{3.39}$$

where

$$H_2(t, \tau_1) = \int_{\tau_1}^{t} [H_0(\tau, \tau_1)]^{m-1} \Big\{ H_1(\tau, \tau_1) + \alpha \int_{\tau_1}^{\tau} R(t, x, \alpha) H_1(x, \tau_1)\, dx \Big\} K(t, \tau)\, d\tau. \tag{3.40}$$

§ 4. Torsion of a conical rod

We will consider the problem of torsion of a conical rod with an opening angle 2γ. We will direct the axis z of the cylindrical system of coordinates (r, θ, z) along the axis of the conical rod and place the origin of the coordinate system at the vertex of the cone.

In this case we represent the stress function $U(r, z, t)$ in the form of the series (2.1). The function $U_0(r, z, t)$ is determined from the equation (2.9) the solution of which is

$$U_0(r, z, t) = H_0(t) \left[\frac{z}{\sqrt{r^2 + z^2}} - \frac{1}{3} \left(\frac{z}{\sqrt{r^2 + z^2}} \right)^3 - \frac{2}{3} \right], \tag{4.1}$$

where $H_0(t)$ is an as yet unknown function.

Now we will proceed to finding the second approximation. For this purpose we substitute the expression $U_0(r, z, t)$ from (4.1) into (2.3). Then the differential equation (2.10) assumes the form

$$\frac{\partial^2 U_1(r, z, t)}{\partial r^2} - \frac{3}{r} \frac{\partial U_1(r, z, t)}{\partial r} + \frac{\partial^2 U_1(r, z, t)}{\partial z^2} = \frac{r^{m+1} z}{(r^2 + z^2)^{2m+\frac{1}{2}}} \left[H_1(t, \tau_1) + \alpha \int_{\tau_1}^{t} H_1(\tau, \tau_1) R(t, \tau, \alpha)\, d\tau \right], \tag{4.2}$$

where

$$H_1(t, \tau_1) = \int_{\tau_1}^{t} [H_0(\tau)]^m K(t, \tau)\, d\tau. \tag{4.3}$$

The equation (4.2) is readily solved by transition to spherical coordinates [9]. Setting

$$r = \rho \sin \theta, \quad z = \rho \cos \theta, \tag{4.4}$$

we reduce the equation (4.2) to the form

$$\frac{\partial^2 U_1}{\partial \rho^2} - \frac{2}{\rho}\frac{\partial U_1}{\partial \rho} - \frac{3}{\rho^2}\cot\theta\frac{\partial U_1}{\partial \theta} + \frac{1}{\rho^2}\frac{\partial^2 U_1}{\partial \theta^2} = \tag{4.5}$$

$$= B_m(t)\frac{\cos\theta \sin^{m+1}\theta}{\rho^{3m-1}},$$

where

$$B_m(t) = H_1(t, \tau_1) + \alpha\int_{\tau_1}^{t} H_1(\tau, \tau_1)R(t, \tau, \alpha)\,d\tau. \tag{4.6}$$

We will seek the solution of the equation (4.5) in the form

$$U_1(\rho, \theta, t) = \rho^{-3(m-1)}\Phi(\theta, t), \tag{4.7}$$

where $\Phi(\theta, t)$ is a new unknown function. Then (4.5) takes the form

$$\Phi''(\theta, t) - 3\cot\theta\Phi'(\theta, t) + 9m(m-1)\Phi(\theta, t) = \tag{4.8}$$

$$= B_m(t)\cos\theta \sin^{m+1}\theta.$$

Finally, setting

$$\Phi(\theta, t) = \varphi(\zeta, t), \tag{4.9}$$

where $\zeta = \cos\theta$, we reduce the equation (4.8) to the form

$$(\zeta^2 - 1)\varphi''(\zeta, t) - 2\zeta\varphi'(\zeta, t) - 9m(m-1)\varphi(\zeta, t) = \tag{4.10}$$

$$= -B_m(t)\zeta(1-\zeta^2)^{(m+1)/2}.$$

Without violating generality we will assume that the quantity m in (1.12) is an odd number, i.e., $m = 2k + 1$, where k is a positer integer. Then the equation (4.10) takes the form

$$(\zeta^2 - 1)\varphi''(\zeta, t) - 2\zeta\varphi'(\zeta, t) - 18k(2k+1)\varphi(\zeta, t) = \tag{4.11}$$

$$= -B_{2k+1}(t)\zeta(1-\zeta^2)^{k+1}.$$

As is known, the general solution of the equation

$$(\zeta^2 - 1)\frac{d^2y}{d\zeta^2} + 2(\nu+1)\zeta\frac{dy}{d\zeta} - (n-\nu)(n+\nu+1)y = 0, \tag{4.12}$$

when ν is a negative number, is

$$y(\zeta) = \left(\frac{d}{d\zeta}\right)^{\nu}[AP_n(\zeta) + DQ_n(\zeta)]. \tag{4.13}$$

Here $P_n(\zeta)$ is a Legendre function of the first kind, $Q_n(\zeta)$ is a Legendre function of the second kind, A and D are arbitrary constants.

Hence it follows that, if the solution of the homogeneous equation (4.11) is denoted by $\varphi_n^*(\zeta)$, then we will have for the solid conical rod

$$\varphi_n^*(\zeta) = A(t)\left(\frac{d}{d\zeta}\right)^{-2} P_n(\zeta), \tag{4.14}$$

where

$$\left(\frac{d}{d\zeta}\right)^{-2} f(\zeta) = \int_1^\zeta \int_1^\zeta f(\zeta)d\zeta d\zeta, \quad n = \frac{-1 \pm (12k+3)}{2}, \tag{4.15}$$

and $A(t)$ is some arbitrary function.

Further, we will choose that solution of the equation (4.11) for which n is a positive integer. Then the function $P_n(\zeta)$ will be a Legendre polynomial of degree n and it is determined by the Rodrigues formula

$$P_n(\zeta) = \frac{1}{2^n \cdot n!} \frac{d^n}{d\zeta^n}(\zeta^2 - 1)^n. \tag{4.16}$$

Using the formulae (4.7), (4.9), (4.14) and (4.16), the general solution of the homogeneous equation (4.2) may be represented in the form

$$U_1^*(r, z, t) = \frac{1}{2^n \cdot n!} \frac{A(t)}{(r^2+z^2)^{3k}} \frac{d^{n-2}}{d\zeta^{n-2}}(\zeta^2 - 1)^n, \tag{4.17}$$

where

$$\zeta = \frac{z}{\sqrt{r^2+z^2}}. \tag{4.18}$$

We will seek the particular solution of the differential equation (4.11), which we will denote by $\varphi^{**}(\zeta)$, in the form

$$\varphi^{**}(\zeta) = \sum_{i=0}^{k+1} a_{2i+1}(t)\zeta^{2i+1}, \tag{4.19}$$

where the coefficients $a_1(t), a_3(t), \cdots, a_{2k+3}(t)$ are determined by the formulae

$$\begin{aligned}
&[2(2k+3)k - 9k(2k+1)]a_{2k+3}(t) + (-1)^{k+1}B_{2k+1}(t) = 0,\\
&[2(2i+1)(i-1) - 9k(2k+1)]a_{2i+1}(t) +\\
&\qquad + (-1)^i C_{k+1}^i B_{2k+1}(t) = 2(i+1)(2i+3)a_{2i+3}(t),\\
&i = 0, 1, 2, \cdots k.
\end{aligned} \tag{4.20}$$

Here

$$C_{k+1}^{i} = \frac{(k+1)k(k-1)\cdots[(k+1)-(i-1)]}{1\cdot 2\cdot 3\cdots i}. \tag{4.21}$$

It follows from (4.7), (4.17) and (4.4) that in cylindrical coordinates the particular solution of the differential equation (4.2) will have the form

$$U_1^{**}(r,z,t) = \frac{1}{(r^2+z^2)^{3k}} \sum_{i=0}^{k+1} a_{2i+1}(t) \left(\frac{z}{\sqrt{r^2+z^2}}\right)^{2i+1}. \tag{4.22}$$

Then one may write the general solution (4.2) in the form

$$\begin{aligned} U_1(r,z,t) &= U_1^{*}(r,z,t) + U_1^{**}(r,z,t) = \\ &= \frac{1}{2^n\cdot n!} \frac{A(t)}{(r^2+z^2)^{3k}} \frac{d^{n-2}}{d\zeta^{n-2}}(\zeta^2-1)^n + \\ &+ \frac{1}{(r^2+z^2)^{3k}} \sum_{i=0}^{k+1} a_{2i+1}(t) \left(\frac{z}{\sqrt{r^2+z^2}}\right)^{2i+1}, \end{aligned} \tag{4.23}$$

where

$$\zeta = \frac{z}{\sqrt{r^2+z^2}}.$$

Thus, if one limits oneself to the first two approximations, the solution of the basic integro-differential equation (1.19) may be represented, by (4.1), (4.23) and (2.1), in the form

$$\begin{aligned} U(r,z,t) &= H_0(t)\left[\frac{z}{\sqrt{r^2+z^2}} - \frac{1}{3}\left(\frac{z}{\sqrt{r^2+z^2}}\right)^3 - \frac{2}{3}\right] + \\ &+ \beta\left[\frac{1}{2^n\cdot n!} \frac{A(t)}{(r^2+z^2)^{3k}} \frac{d^{n-2}}{d\zeta^{n-2}}(\zeta^2-1)^n + \right. \\ &\left. + \frac{1}{(r^2+z^2)^{3k}} \sum_{i=0}^{k+1} a_{2i+1}(t) \left(\frac{z}{\sqrt{r^2+z^2}}\right)^{2i+1}\right] + O(\beta^2). \end{aligned} \tag{4.24}$$

This solution satisfies the condition (2.7), since

$$\sum_{i=0}^{k+1} a_{2i+1}(t) = 0, \tag{4.25}$$

and for $\zeta = \cos\theta = 1$

$$\frac{d^{n-2}}{d\zeta^{n-2}}(\zeta^2-1)^n = 0, \tag{4.26}$$

where n is determined by (4.15).

This is especially easily seen in cases when k takes specific values;

for example, for $k = 1$ $(m = 3)$, the solution (4.24) becomes

$$(4.27)\quad U(r, z, t) = H_0(t)\left[\frac{z}{\sqrt{r^2+z^2}} - \frac{1}{3}\left(\frac{z}{\sqrt{r^2+z^2}}\right)^3 - \frac{2}{3}\right] +$$

$$+ \beta\frac{1}{2^n\cdot n!}\frac{A(t)}{(r^2+z^2)^3}\left[A_{14}^5\left(\frac{z}{\sqrt{r^2+z^2}}\right)^9 - 7A_{12}^5\left(\frac{z}{\sqrt{r^2+z^2}}\right)^7 +\right.$$

$$\left. + 21A_{10}^5\left(\frac{z}{\sqrt{r^2+r^2}}\right)^5 - 35A_8^5\left(\frac{z}{\sqrt{r^2+z^2}}\right)^3 + 35A_6^5\frac{z}{\sqrt{r^2+z^2}}\right] +$$

$$+ \beta\frac{1}{(r^2+z^2)^3}\sum_{i=0}^{2} a_{2i+1}(t)\left(\frac{z}{\sqrt{r^2+z^2}}\right)^{2i+1} + O(\beta^2),$$

where

$$A_m^n = m(m-1)\cdots[m-(n-1)].$$

In this case we obtain for $\zeta = 1$

$$(4.28)\quad \sum_{i=0}^{2} a_{2i+1}(t) = a_1(t) + a_3(t) + a_5(t) = (\tfrac{1}{17} - \tfrac{2}{17} + \tfrac{1}{17})B_3(t) = 0,$$

$$(4.29)\quad A_{14}^5 - 7A_{12}^5 + 21A_{10}^5 - 35A_8^5 + 35A_6^5 = 0.$$

In order that the solution (4.24) will become constant on the surface, i.e., that it will satisfy the condition (2.6), it is necessary that $A(t)$ satisfy for $\zeta = \cos\gamma$ the equation

$$(4.30)\quad \frac{A(t)}{2^n\cdot n!}\frac{d^{n-2}}{d\zeta^{n-2}}(\zeta^2-1)^n + \sum_{i=0}^{k+1} a_{2i+1}(t)\zeta^{2i+1} = 0.$$

Hence there follows

$$(4.31)\quad A(t) = -\frac{\sum_{i=0}^{k+1} a_{2i+1}(t)\cos^{2i+1}\gamma}{\frac{1}{2^n\cdot n!}\frac{d^{n-2}}{d\zeta^{n-2}}(\zeta^2-1)^n}.$$

We note that after the corresponding operations one has to replace ζ in (4.31) by its boundary value $\cos\gamma$. In the particular case $k = 1$ $(m = 3)$, we will have

$$(4.32)\quad A(t) = -\frac{\sum_{i=0}^{2} a_{2i+1}(t)\cos^{2i+1}\gamma}{\frac{1}{2^n\cdot n!}[A_{14}^5\cos^9\gamma - 7A_{12}^5\cos^7\gamma + 21A_{10}^5\cos^5\gamma - 35A_8^5\cos^3\gamma + 35A_6^5\cos\gamma]}.$$

The unknown function $H_0(t)$ is determined by the condition (2.8).

Substituting (4.24) into (2.8) and using the equation (4.30), we find

$$H_0(t) = -\frac{M(t)}{2\pi(\frac{2}{3} - \cos\gamma + \frac{1}{3}\cos^3\gamma)}, \tag{4.33}$$

where $M(t)$ is the magnitude of the twisting moment which in the general case may depend on the time t.

Substituting the value of $U(r, z, t)$ from (4.24) into (1.3) we obtain the following expressions for the shear stresses:

$$\tau_{r\theta} = -\frac{r^2}{(r^2+t^2)^{\frac{5}{2}}} H_0(t) + \beta\left\{\frac{\partial}{\partial z}\left[-\frac{1}{2^n\cdot n!}\,\frac{A(t)}{r^2(r^2+z^2)^{3k}}\times\right.\right. \tag{4.34}$$

$$\times \frac{d^{n-2}}{d\zeta^{n-2}}(\zeta^2-1)^n\Big] + \frac{1}{(r^2+z^2)^{3k+1}r^2z}\times$$

$$\left.\left.\times \sum_{i=0}^{k+1} a_{2i+1}(t)[6kz^2 - (2i+1)r^2]\left(\frac{z}{\sqrt{r^2+z^2}}\right)^{2i+1}\right\} + O(\beta^2),\right.$$

$$\tau_{\theta z} = -\frac{rz}{(r^2+z^2)^{\frac{5}{2}}} H_0(t) + \beta\left\{\frac{\partial}{\partial r}\left[\frac{1}{2^n\cdot n!}\,\frac{A(t)}{r^2(r^2+z^2)^{3k}}\times\right.\right. \tag{4.35}$$

$$\times \frac{d^{n-2}}{d\zeta^{n-2}}(\zeta^2-1)^n\Big] - \frac{1}{(r^2+z^2)^{3k+1}r}\times$$

$$\left.\left.\times \sum_{i=0}^{k+1} a_{2i+1}(t)(2i+1+6k)\left(\frac{z}{\sqrt{r^2+z^2}}\right)^{2i+1}\right\} + O(\beta^2).\right.$$

For $k = 1$ $(m = 3)$, the expressions for $\tau_{r\theta}$ and $\tau_{\theta z}$ assume the form

$$\tau_{r\theta} = -\frac{r^2}{(r^2+z^2)^{\frac{5}{2}}} H_0(t) + \beta\left\{\frac{6z}{r^2(r^2+z^2)^4}\left[A_{14}^5\left(\frac{z}{\sqrt{r^2+z^2}}\right)^9 - \right.\right. \tag{4.36}$$

$$- 7A_{12}^5\left(\frac{z}{\sqrt{r^2+z^2}}\right)^7 + 21A_{10}^5\left(\frac{z}{\sqrt{r^2+z^2}}\right)^5 - 35A_8^5\left(\frac{z}{\sqrt{r^2+z^2}}\right)^3 +$$

$$+ 35A_6^5\frac{z}{\sqrt{r^2+z^2}}\Big] + \frac{1}{z(r^2+z^2)^4}\left[9A_{14}^5\left(\frac{z}{\sqrt{r^2+z^2}}\right)^9 - \right.$$

$$- 7\cdot 7A_{12}^5\left(\frac{z}{\sqrt{r^2+z^2}}\right)^7 + 21\cdot 5A_{10}^5\left(\frac{z}{\sqrt{r^2+z^2}}\right)^5 -$$

$$- 35\cdot 3A_8^5\left(\frac{z}{r^2+z^2}\right)^3 + 35A_6^5\frac{z}{\sqrt{r^2+z^2}}\Big] +$$

$$\left.\left. + \frac{1}{(r^2+z^2)^4r^2z}\sum_{i=0}^{2} a_{2i+1}(t)[6z^2 - (2i+1)r^2]\left(\frac{z}{\sqrt{r^2+z^2}}\right)^{2i+1}\right\} + O(\beta^2),\right.$$

$$(4.37)\quad \tau_{\theta z} = -\frac{rz}{(r^2+z^2)^{\frac{1}{2}}} H_0(t) + \beta\left\{\frac{1}{r(r^2+z^2)^4}\left[15A_{14}^5\left(\frac{z}{\sqrt{r^2+z^2}}\right)^9 - \right.\right.$$

$$-7\cdot 13A_{12}^5\left(\frac{z}{\sqrt{r^2+z^2}}\right)^7 + 11\cdot 21A_{10}^5\left(\frac{z}{\sqrt{r^2+z^2}}\right)^5 -$$

$$\left.\left. - 35\cdot 9A_8^5\left(\frac{z}{\sqrt{r^2+z^2}}\right)^3 + 35\cdot 7A_6^5\left(\frac{z}{\sqrt{r^2+z^2}}\right)\right] - \right.$$

$$\left. - \frac{1}{(r^2+z^2)^4}\sum_{i=0}^{2} a_{2i+1}(t)(2i+7)\left(\frac{z}{\sqrt{r^2+z^2}}\right)^{2i+1}\right\} + O(\beta^2).$$

Using this method, one may obtain also the solutions of a number of other problems of the torsion of shafts of variable cross-section under conditions of unsteady creep.

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ON SOME BASIC IDEAS OF THE THEORY OF EQUILIBRIUM CRACKS, FORMING DURING BRITTLE FRACTURE

G. I. BARENBLATT

Moscow, U.S.S.R.

The study of the equilibrium of brittle bodies [1] containing cracks – surfaces of discontinuity of the material, i.e., surfaces of discontinuity of the displacements, has attracted increased attention in recent times. There is a solid foundation for the supposition that in many more cases than has been suspected hitherto the development of cracks, especially at low temperatures, occurs by a brittle or quasi-brittle mechanism.

The problem of such a study, however, does not belong to the traditional problems of the classical theory of elasticity. The fact is that, in contrast to the usual formulations of problems of the theory of elasticity, in the present case the region in which the solution is to be constructed *is not given beforehand.* The dimensions and the shape of the cracks the surface of which forms part of the boundary of the body depends on the loading applied to the body, and it must be determined in the course of the solution of the problem. This feature makes the problem under consideration essentially nonlinear, and conceptually links it to problems which arise in the theory of jets and the theory of waves of finite amplitude in hydrodynamics, to the Stefan problem in the theory of heat conduction, and others.

The present paper considers the formulation of problems of the equilibrium of brittle bodies containing cracks of normal rupture. After a short survey of the most important work in this field, an analysis of the physical pattern near the ends of cracks will be presented, which forms the basis of the following study. The results of this analysis permit the formulation of two hypotheses referring to end regions of cracks in which cohesion forces act, – a hypothesis regarding the dimension of this region and a hypothesis regarding the independence with respect to the loading of the shape of the surface of the cracks applied in the end regions. Further, starting from the principle of virtual displacements, there is established the finiteness of the rupture stresses at the edges of the cracks, and likewise the smooth closing up of opposite sides of the

[1] As usually, brittle bodies are here understood to be bodies which before failure possess the property of linear elasticity.

cracks at the edges which proves hypotheses first proposed by S. A. Khristianovich. Relying on the above two hypotheses and the established finiteness of the stresses at the edges of the cracks, the boundary condition at the edges of the cracks is obtained; it contains the forces of molecular cohesion only in an integrated form by means of a new macroscopic characteristic of the given material under given conditions, which has been called the *modulus of cohesion*. A relation is established between the modulus of cohesion and the density of surface energy, exerted in the formation of the cracks (which for purely brittle fracture is equal to the surface tension) and the elastic constants of the material. In conclusion, the paper discusses briefly certain new problems.

In this paper we devote little time to the solution of concrete problems; but, on the contrary, we consider only principal questions, relating to the formulation of problems. For the reader's convenience, a detailed bibliography of the work of the author in which, in particular, one may find a study of the solution of a number of concrete problems of the theory of equilibrium cracks is attached at the end of the paper.

In the proposed formulation, the problems of the equilibrium of elastic bodies with cracks may be solved by the usual methods of the classical theory of elasticity. This makes it possible to obtain effective solutions of several general and sufficiently important classes of such a family of problems and to make special deductions of general character, in particular referring to the stability of cracks and systems of cracks, on the influence of boundaries of bodies on crack formation, on the role of forces of molecular cohesion, etc.

It should be noted that recent progress in the U.S.S.R. and abroad in the theory of equilibrium cracks is essentially linked to the classical results of N. I. Muskhelishvili and his school, presented in the monographs by N. I. Muskhelishvili [1, 2]. It may be hoped that new studies will lead to more accurate quantitative ideas relating to the strength of classes of real materials, important in practice.

§ 1. The problem of cracks of brittle fracture was given early consideration by investigators in mechanics and physics, thanks to the nontrivial physical picture of the situation and its practical importance. Attempting in no measure to give an exhaustive survey, we will state the most important papers which left their mark on the study of the problems under consideration.

At the beginning of the studies of cracks, one has to place the work of C. E. Inglis [3], in which there is considered within the framework of the classical linear theory of elasticity the problem of the equilibrium of an infinite body with an isolated rectilinear cut in a homogeneous

stress field. In the work of N. I. Muskhelishvili [4], likewise within the framework of the classical theory of elasticity, there is investigated the equilibrium of an infinite body with an elliptic cut. In particular, in [4], there was also obtained in a simple and effective form the solution of the problem of the equilibrium of an infinite body with a rectilinear cut in an arbitrary stress field. However, the solutions of [3, 4] have two properties: first, the length of the cut remains undetermined, since it is found that the solution may be constructed for an arbitrary value of this parameter. Further, in the problem of C. E. Inglis, the stresses become infinitely large at the ends of the cut; *generally speaking*, such a condition will apply also in the problem considered by N. I. Muskhelishvili. Clearly, solutions involving infinite stresses at the edges of the cuts are physically unacceptable for the model of a brittle body.

In the work of A. A. Griffith [5] which is justly considered fundamental for the theory of cracks, a method is stated for the determination of the dimensions of cracks. It became clear to A. A. Griffith that an important role in this problem is played by molecular cohesion forces which act near the ends of the cracks and which were neglected in [3] (and likewise in [4]). However, as regards the distribution of these forces, in essence, nothing is known and, at the same time, the work spent on overcoming these forces in the formation of a unit of surface of a crack is (under certain reasonable assumptions) a completely determined constant of the material – the surface tension; A. A. Griffith introduced energy considerations for the determination of the dimensions of cracks. He determined the dimension of the crack from the condition of an extremum of the free energy of the body [1] which is, apart from a constant term, exactly equal to the difference between the decrease of the elastic energy of the body at the expense of the creation of the crack and the surface energy of the crack. However, the surface energy of the crack is very simply expressed – it is equal to the product of the surface tension of the material on the area of the surface of the crack. Thus, the energy method established by A. A. Griffith reduced the determination of the dimensions of cracks to the finding of the derivative of the above decrease of the elastic energy of the body with respect to the dimension of the crack, – i.e., the rate of release of elastic energy.

A. A. Griffith himself considered only the case of an isolated recti-

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[1] In the case considered by A. A. Griffith a maximum of the free energy occurs Ia. I. Frenkel' [6] stated incorrectly that a maximum as well as a minimum of the free energy occurs.

linear crack in an infinite body under the influence of a homogeneous stress field; using the results of C. E. Inglis [3] he obtained an expression for the equilibrium dimension of the crack.

The distribution of the stresses and the configuration of the crack near its end were not elucidated in the work of A. A. Griffith; his study of this problem is quite unsatisfactory and, to a considerable extent, may be recognized to be untrue. This circumstance has brought about serious critical comments with regard to A. A. Griffith's theory.

The complexity of the energy approach proposed by A. A. Griffith led to the fact that attempts were only made to solve the problem in essentially the simplest cases of isolated rectilinear, and likewise of isolated penny-shaped axisymmetric cracks in an infinite body subject to a homogeneous field, and in the current literature there is even much vagueness about the formulation of problems. However, the problem of the structure of cracks and the fields of stress and deformation near the ends of cracks has not hitherto been cleared up.

In the following important stage of the theory of cracks there appeared the work of E. Orowan [7, 8] and G. R. Irwin [9, 10] in which the concept of quasi-brittle fracture was developed. The idea of E. Orowan and G. R. Irwin consisted of the fact that in the majority of cases of formation of cracks, the crack is not established purely as the result of brittle rupture of the material, but due to plastic deformations concentrated in very narrow regions near the surface of the cracks. For those cases one may use all the relations of the theory of brittle fracture.

Another idea, expressed first by S. A. Khristianovich [11], had a very important influence. Considering an isolated crack in a rock massif under the influence of rock pressure he likewise came in contact with the impossibility of determining the length of the crack from the differential equations and boundary conditions of the problem. However, S. A. Khristianovich turned his attention to the fact that there exists one single value of the length of the crack for which the rupture stresses at the ends of the crack are finite, and for this value of the length (and only for it) the opposite sides of the crack touch smoothly at its edges. He advanced the hypothesis of the finiteness of the stress at the edges of the crack, or of the smoothness of contact of the opposite sides at the edges of the crack as the basic condition determining the length of the crack. The utilization of this hypothesis permitted the solution of important problems of the formation and development of cracks in a rock massif [11–15]; it served as starting point for the studies discussed in the present paper.

There must also still be noted the work of P. A. Rebinder [16] where

in qualitative form, and not in relation to the formulation of problems of mechanics, the wedge shape of the ends of cracks, has been discussed for the first time.

In the papers [11–15], no consideration was given to the forces of molecular cohesion of the material. For the study of cracks in rock massifs, such an omission is apparently completely justified, since the action of the rock pressure manifests here a considerably stronger effect than the forces of molecular cohesion. However, in the majority of cases, the forces of cohesion play an important role and their consideration has a basic meaning for the elucidation of the mechanism of the development and equilibrium of cracks.

In work completed in recent times [17–23] there has been proposed and utilized a new force approach to problems of the theory of equilibrium cracks which is based on a detailed analysis of the situation near the edges of the crack. This analysis offered the possibility of formulating local conditions determining the position of the edges of the crack. The difficulty associated with the lack of knowledge of the distribution of the forces of molecular cohesion could be passed by as the result of certain assumptions; – it was found possible to characterize with fully sufficient completeness this distribution by a certain material constant, called the modulus of cohesion, and to relate this constant simply to the density of surface energy and the elastic constants of the material. The force approach turned out to be sufficiently effective, since it made it possible to use the methods and results of the classical theory of elasticity directly.

At the present time, relying on the latest studies, one may give a sufficiently general and, in addition, sufficiently simple study of the foundations of the theory of equilibrium cracks, formed during brittle fracture. An attempt at such a study will be presented in this paper.

§ 2. Basically, we will concern ourselves with cases in which on the surfaces of cracks only discontinuities in the normal displacements occur and points of the surface of the cracks before application of the loading remain in the plane of the crack; this plane will be called the plane of the crack. Such a situation applies when the applied load is symmetric about the plane of the crack; consideration of such cracks is of fundamental importance, since in the general case one obtains similar conditions near the edges of cracks.

The surface of the crack is naturally divided into two parts (Fig. 1). In the first part, which we will call *the internal region of the crack*, opposite sides of the crack are sufficiently far apart from each other so that their interaction is negligibly small.

In the second part, adjoining the edge of the crack, which we will call *the end region of the crack*, opposite sides of the crack come very close to each other so that their interaction is very significant as regards the magnitude of the forces of cohesion. As is known, the intensity f of the force of cohesion, acting between two pieces of material previously

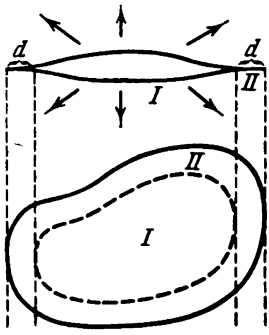

Fig. 1

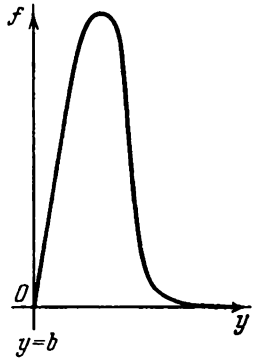

Fig. 2

constituting a single unit and afterwards separated, depends strongly on the distance y between these segments (Fig. 2). For distances y, equal to the normal interatomic distance b, the intensity f is equal to zero; as the distance increases to approximately one and a half to two normal interatomic distances, it rises to a certain maximum value, of the order of magnitude of Young's modulus; subsequently, it decreases rapidly with increasing distance. The determination of the forces of cohesion, acting in the end region of the cracks, demanded knowledge of the relation represented in Fig. 2, which for real materials, as a rule, is unknown. Even if this dependence were given, the determination of the system of cohesion forces acting in the end regions would demand solution of a complicated and essentially nonlinear problem of the theory of elasticity. However, the above property of the molecular forces of cohesion permits circumvention of this difficulty. In fact, the analysis of the dependence of the forces of cohesion on the distance leads to conclusions which may be formulated in the form of the following two fundamental hypotheses:

First hypothesis. *The dimension d of the end region of a crack is small compared with the dimension of the entire crack.*

Second hypothesis. *The shape of the normal section of the surface of a crack in the end region (and, consequently, the distribution of the cohesion forces) does not depend on the applied loading and, for a given material under given conditions (temperature, composition and pressure of the*

atmospheric environment, etc.) is always the same (we understand here by normal section the cross-section of the plane, normal to the contour of the crack).

The first hypothesis is essentially a definition of the class of cracks to be considered. Of course micro-cracks exist for which this hypothesis is not fulfilled. However, inasmuch as the dimension d of the end region is very small, the first hypothesis is already true for very small cracks and, definitely, for all real macro-cracks. However, we emphasize that we will always assume the dimension d to be sufficiently large in comparison with micro-dimensions, so that at distances of the order d one may use the methods of the mechanics of continuous media.

According to the second hypothesis, the ends of all cracks of the type under consideration are always the same in a given material under given conditions. As the cracks widen the end region simply moves into a different position, but the shape of its normal section remains the same; this shape corresponds to the maximum possible strength. The possibility of the adoption of the second hypothesis is connected with the fact that the stresses in the end region arising from molecular cohesion forces significantly exceed the stresses acting in this region as the result of the applied loading. Therefore one may disregard the changes of the stresses acting in the end regions, and consequently also the changes of shape of the normal cross-section of the end regions which arise from changes of the loading. It must be noted that in view of the fact that for the majority of materials the cracks are not reversible, the second hypothesis is only applicable to those equilibrium cracks which are formed as a result of the first rupture of the brittle body, but not to those cracks which are formed as a result of an artificial cut without subsequent widening or by the decrease of loading arising from the crack existing for some large load. For the latter types of cracks the stresses in the end region will be different.

§ 3. We will now turn to the analysis of the conditions determining the configuration of systems of cracks. We consider a brittle body in which a system of equilibrium cracks are maintained as the result of the action of a certain load. Generally speaking, the cracks may have a non-symmetric shape so that in the general case the configuration of the cracks is determined by some (finite or infinite) number of parameters $c_1, c_2, \cdots, c_n, \cdots$ (determining the positions of the surfaces of the cracks, or some points of the surfaces of the cracks, etc.) and functions of the coordinates of points of the body $f_1, f_2, \cdots, f_m, \cdots$ (determining the surfaces of the cracks, their boundaries, etc.). A peculiar feature of the problem under consideration consists of the fact that the differential

equations of problems in the theory of elasticity may be solved for the corresponding boundary conditions for a choice of the parameters c_i and the functions f_j which is arbitrary within known limits. The above parameters and functions thus form a set of undetermined elements of the problem which we will denote by M; above all we have now to obtain conditions which will fix these undetermined elements, i.e., which will determine the configuration of the system of cracks.

We will denote the set of variations of the undetermined elements $(\delta c_1, \delta c_2, \cdots, \delta c_n, \cdots, \delta f_1, \delta f_2, \cdots, \delta f_m, \cdots)$ by δM. Let $\mathbf{u}$ represent the field of displacements, corresponding to a fixed configuration of the system of cracks M, $\delta_1\mathbf{u}$ the variation of this field which is in agreement with geometric connections imposed on the system and corresponds to the same configuration M, and $\delta_2\mathbf{u}$ the variation of the displacement field corresponding to the variation δM of the set of undetermined elements.

We will start from the principle of virtual displacements, as the most general formulation of the basic law of statics of elastic systems

$$\delta W - \delta A = 0, \tag{3.1}$$

where δW is the variation of the elastic potential of the system W and δA is the variation of the work done by the external forces (including the reactions as well as the molecular forces of cohesion) for a given possible state of the elastic system. The state, characterized by the displacement field $\mathbf{u} + \delta_1\mathbf{u} + \delta_2\mathbf{u}$ is a possible state of the elastic system. Therefore the relation (3.1) assumes the form

$$\delta_1 W - \delta_1 A + \delta_2 W - \delta_2 A = 0, \tag{3.2}$$

where $\delta_1 W$ and $\delta_1 A$ are the variations corresponding to the variation $\delta_1\mathbf{u}$ of the displacement field for a fixed configuration of the system of cracks, i.e., corresponding to fixed M, and $\delta_2 W$ and $\delta_2 A$ are the variations corresponding to some variation δM. In view of the independence of the variations $\delta_1\mathbf{u}$ and δM, it follows from (3.2) that

$$\delta_1 W - \delta_1 A = 0, \tag{3.3}$$

$$\delta_2 W - \delta_2 A = 0. \tag{3.4}$$

From the relation (3.3) one obtains in the usual manner (cf. e.g., [24]) the differential equations and boundary conditions of the theory of elasticity, corresponding to an arbitrary fixed configuration of a system of cracks, i.e., to an arbitrary fixed set of undetermined elements M.

It is known that, if the field satisfies the differential equations of equilibrium and the boundary conditions, Clapeyron's theorem [24]

applies according to which

$$2W = A \tag{3.5}$$

for any M, whence we obtain

$$2\delta_2 W = \delta_2 A \tag{3.6}$$

and, substituting (3.6) in (3.4),

$$\delta_2 W = 0. \tag{3.7}$$

The relation (3.7) is the general condition, determining the configuration of the system of cracks, i.e., the set of undetermined elements M. Thus, *there is indeed accomplished such a configuration of a system of cracks in a brittle body which corresponds to an extreme value of the elastic potential.*[1] Inasmuch as

$$\delta_2 W = \frac{\partial W}{\partial c_1}\delta c_1 + \frac{\partial W}{\partial c_2}\delta c_2 + \cdots + \frac{\partial W}{\partial c_n}\delta c_n + \cdots + \delta_{f_1} W + \delta_{f_2} W + \cdots,$$

where $\delta_{f_i} W$ is the variation of the elastic potential, corresponding to some variation of the function f_i and fixed remaining undetermined elements, we obtain from (3.7)

$$\frac{\partial W}{\partial c_i} = 0 \ (i = 1, 2, \cdots n \cdots), \qquad \delta_{f_j} W = 0, \ (j = 1, 2, \cdots m \cdots). \tag{3.8}$$

§ 4. We will now consider an arbitrary point O of the boundary of one of the cracks of the system under consideration. We choose some neighbourhood of this point, large compared with the dimension d of the end region, but small compared with the dimensions of the crack as a whole; such a neighbourhood may be defined by the first hypothesis of § 2.

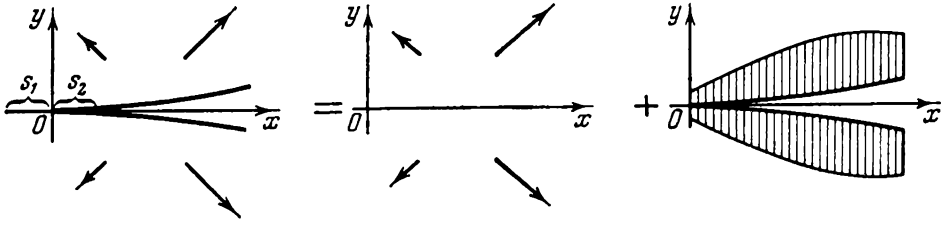

Fig. 3

[1] The result established has obviously a completely general meaning, since the presence of elements of a solution, which are not determined by the differential equations and boundary conditions, are not only characteristic for problems of the theory of cracks, but also for contact problems and a number of other problems of the theory of elasticity and, generally, of mechanics of continuous media.

The deformation in this region may be assumed to be plane and to correspond to a semi-infinite rectilinear cut in an infinite body (Fig. 3, the plane of deformation is the normal plane to the boundary of the crack at the point O). We will consider this configuration in greater detail. First of all, we note that the stress and displacement field may be represented in the form of the sum of two fields (Fig. 3) the first of which corresponds to the continuous body subject to the same loading and the second to the body with a cut subject to normal symmetric loading over the surface of the cut. For this the shape of the deformed surface of the cut is completely determined by the second state of stress, since for the first state of stress the normal displacements in the region of the cut obviously are equal to zero. The analysis of the first state of stress presents no difficulties; it is readily executed by the ordinary methods of the theory of elasticity, so that we will assume this state to be known and turn to the study of the second state of stress. Let the line of the cut correspond to the positive semi-axis Ox, then the distribution of symmetric normal stresses, applied to the surface of the cut in the second state of stress, $g(x)$, itself represents the difference of the stresses, applied to the surface of the cut in the combined field, and the stresses in the region of the cut, corresponding to the first state of stress.

For the analysis of the second state of stress we will use the method of N. I. Muskhelishvili [1]. The boundary condition has the form

$$X_y^{(2)} = 0, \quad Y_y^{(2)} = g(x), \quad (0 \leq x < \infty), \tag{4.1}$$

where $X_x^{(2)}$, $Y_y^{(2)}$, $X_y^{(2)}$ are the components of the stress tensor of the second state of stress. One has the relations (cf. [1], § 120)

$$X_x^{(2)} + Y_y^{(2)} = 4 \operatorname{Re} \{\Phi(z)\}, \tag{4.2}$$

$$Y_y^{(2)} - iX_y^{(2)} = \Phi(z) + \Omega(\bar{z}) + (z - \bar{z})\overline{\Phi'(z)}, \tag{4.3}$$

$$2\mu(u^{(2)} + iv^{(2)}) = \kappa\varphi(z) - \omega(\bar{z}) - (z - \bar{z})\overline{\Phi(z)}, \quad \kappa = 3 - 4\nu, \tag{4.4}$$

where $u^{(2)}$, $v^{(2)}$ are the displacement components along the Ox and Oy axes for the second state of stress, μ is the shear modulus, ν is Poisson's ratio and the functions φ, ω are expressed in the following manner:

$$\varphi(z) = \int_0^z \Phi(z)\, dz, \quad \omega(z) = \int_0^z \Omega(z)\, dz. \tag{4.5}$$

In the case of symmetric loading under consideration, tangential

stresses are absent and it may be shown that one has the formulae

$$\Phi(z) = \Omega(z) = \frac{1}{2\pi i\sqrt{z}} \int_0^\infty \frac{\sqrt{t}\, g(t)\, dt}{t - z}, \tag{4.6}$$

$$\varphi(z) = \omega(z) = -\frac{1}{2\pi i} \int_0^\infty g(t) \log \frac{\sqrt{t} + \sqrt{z}}{\sqrt{t} - \sqrt{z}}\, dt. \tag{4.7}$$

We obtain from the formulae (4.2)–(4.4) that on the cut itself and on its continuation there are fulfilled the relations

$$\begin{aligned} &X_x^{(2)} = Y_y^{(2)} = 2\,\mathrm{Re}\,\{\Phi(z)\}, \\ &X_y^{(2)} = 0, \\ &v^{(2)} = \frac{\kappa + 1}{2\mu}\,\mathrm{Im}\,\{\varphi(z)\}. \end{aligned} \tag{4.8}$$

Near the end of the cut, on its continuation $z = -s_1$, where s_1 is a small, positive quantity, we have

$$\mathrm{Re}\,\{\Phi(z)\} = -\frac{1}{2\pi\sqrt{s_1}} \int_0^\infty \frac{g(t)\, dt}{\sqrt{t}} + O(1). \tag{4.9}$$

Near the end of the cut on the cut itself $z = s_2$, where s_2 is a small, positive quantity, we have

$$\mathrm{Im}\,\{\varphi(z)\} = \pm\frac{\sqrt{s_2}}{\pi} \int_0^\infty \frac{g(t)\, dt}{\sqrt{t}} + O(s_2^{\frac{1}{2}}) \tag{4.10}$$

(the minus and plus signs corresponding to the upper and lower sides of the cut). Thus, we obtain an expression for the rupture stress near the end of the cut and for the distribution of normal displacements near the end of the cut in the form

$$\begin{aligned} &Y_y = -\frac{1}{\pi\sqrt{s_1}} \int_0^\infty \frac{g(t)\, dt}{\sqrt{t}} + O(1), \\ &v = \pm\frac{4(1 - \nu^2)\sqrt{s_2}}{\pi E} \int_0^\infty \frac{g(t)\, dt}{\sqrt{t}} + O(s_2^{\frac{1}{2}}). \end{aligned} \tag{4.11}$$

§ 5. The study of the preceding section elucidates the distribution of stresses and displacements near the boundary of an arbitrary surface of discontinuity of the type under consideration. In fact, the following

expressions for the rupture stress and the normal displacements apply at points near this boundary:

$$(5.1)\qquad Y_y = \frac{N}{\sqrt{s_1}} + O(1),\ v = \pm\frac{4(1-\nu^2)N\sqrt{s_2}}{E} + O(s_2^{\frac{3}{2}}),$$

where N is a quantity depending only on the acting stresses and the shape of the contour as well as on the coordinate of the point O, s_1 is the small distance of the point considered in the plane of the crack from its boundary, s_2 is the small distance of the considered point of the surface of the crack from its contour (the minus and plus signs corresponding, as before, to the upper and lower sides of the crack).

We will now consider a possible state of the elastic system under consideration which differs from the initial state only by some change of the shape of the boundary of the crack near the point O of the contour (Fig. 4). The new contour represents a circle with a small radius R with center at the point O, lying in the plane of the original crack up to the intersection with the former contour; in all the remaining parts the contours of all cracks remain unchanged. The distribution of the normal displacements of points of the new surface of the crack obviously have the form

$$(5.2)\qquad v = \pm\frac{4(1-\nu^2)N\sqrt{s_1}}{E} + O(s_1^{\frac{3}{2}}),$$

where s_1 is the distance of the considered point of the surface from the contour. The corresponding variation of the elastic potential may be expressed in the following manner:

$$(5.3)\quad \delta_2 W = \tfrac{1}{2}\cdot 2\int_{\delta S} Y_y\,|v|\,dS = \frac{4(1-\nu^2)\pi N^2}{E}\int_{-R}^{+R} dx\int_0^h \sqrt{\frac{h-y}{y}}\,dy =$$

$$= \frac{(1-\nu^2)\pi^2 N^2 R^2}{E} = \frac{2(1-\nu^2)N^2\pi}{E}\,\delta S,\ h = \sqrt{R^2-x^2},$$

where δS is the size of the area of the projection of the new surface of the crack onto its plane. By the strength of the fundamental relation (3.7), the variation $\delta_2 W$ must vanish, whence it follows from (5.3) that $N = 0$. Setting $N = 0$ in (5.1) we arrive at the following very important result:

1) *The rupture stress at the contour of the crack is finite;*

2) *Opposite sides of the crack close smoothly on its contour.*

Thus, near the contour of the crack its cross-section, normal to the contour, has the shape shown in Fig. 5. We note still that the condition

of the finiteness of the stresses and the smoothness of the closing of the crack was first stated in the form of a hypothesis by S. A. Khristianovich [11]. We likewise note that reasoning, analogous to the deduction of the formulae (5.3) for the case of plane deformation, without link to the finiteness of the stresses and the smoothness of the closing up on the contour of the crack was given in the work of G. R. Irwin [10].

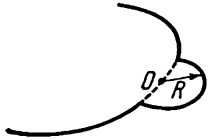

Fig. 4

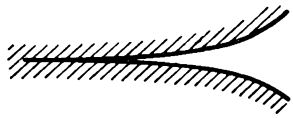

Fig. 5

The fundamental hypotheses formulated in § 2 and the condition of the finiteness of the stresses, just proved, form the basis of the proposed force approach to the theory of steady state cracks. Using this approach, it has been possible to solve a number of new problems among which there is that of the isolated rectilinear crack in an arbitrary stress field [18], the problem of the isolated axi-symmetric penny-shaped crack in an arbitrary stress field [17], the problem of rupture of a brittle body by a resting wedge and by a wedge moving with constant velocity [18, 23] and different problems of cracks in bounded bodies [18, 22]. There have also been given in general form the stability conditions of equilibrium cracks [19], an elucidation of the influence of the boundaries of a body on the development of cracks [18, 22] and the deduction of an estimate of the influence of the forces of molecular cohesion on the stress and deformation fields [17, 18].

§ 6. By the strength of the fact that the brittle body under consideration is by definition linearly elastic, the state of stress in such a body with cracks can be represented in the form of the sum of two states of stress, the state of stress evaluated without consideration of cohesion forces and the state of stress corresponding to cohesion forces only. In correspondence with this the quantity N, entering into (5.1), may be represented in the form of the sum

$$N = N_0 + N_m, \tag{6.1}$$

where N_0 corresponds to the same applied loading and the same configuration of cracks, but without consideration of cohesion forces, and N_m corresponds to the same configuration of cracks taking only account of the forces of molecular cohesion. By strength of the first hypothesis of § 2, the dimension d of the end region in which there act forces of molecular cohesion is small compared with the dimensions of the crack as a whole. Therefore one may represent for the determination of the

quantity N_m the state of stress by the corresponding scheme of an infinite body with a semi-infinite rectilinear cut under the conditions of plane strain, considered in § 4. Thus it follows from the results of § 4 that

$$N_m = -\frac{1}{\pi}\int_0^{\infty} \frac{G(t)\,dt}{\sqrt{t}} = -\frac{1}{\pi}\int_0^{d} \frac{G(t)\,dt}{\sqrt{t}}, \tag{6.2}$$

where $G(t)$ is the distribution of cohesion forces which differ from zero only in the end region $0 \leq t \leq d$. By the strength of the second hypothesis of § 2 the distribution of the cohesion forces does not depend on the applied loading, so that the integral on the right hand side of (6.2) *is a constant characteristic for the given material under given conditions.* This constant, introduced in the paper [17], will be denoted by K and called *modulus of cohesion.* The modulus of cohesion of a material characterizes the resistance of the material to the development of cracks. The relation (6.2) will be rewritten in the form

$$N_m = -\frac{K}{\pi}, \tag{6.3}$$

whence follows from (6.1) and the fact that $N = 0$ that

$$N_0 = \frac{K}{\pi}. \tag{6.4}$$

Thus, one may formulate the following general condition: on the contour of the crack the shearing stress evaluated without consideration of the forces of molecular cohesion becomes infinite according to the law

$$Y_y = \frac{K}{\pi\sqrt{s}}, \tag{6.5}$$

where s is the small distance of the point considered in the plane of the crack from the contour. This condition permits one to exclude, in general, from the consideration the forces of molecular cohesion, when determining the configuration of cracks. Special estimates have shown that the influence of the forces of molecular cohesion on the stress and displacement field is only essential in the end regions of the cracks and that its sphere of influence is of the order of the dimension of the end regions. Not being specially interested in this region, for the analysis of which one has to deal in greater detail with the distribution of the forces of molecular cohesion, one may thus solve a problem of the theory of elasticity, neglecting the forces of molecular cohesion. The configuration of cracks is determined for this by the conditions (6.5), so that the forces of molecular

cohesion enter into the considerations only through the macroscopic integral characteristic – the modulus of cohesion.

For experimental determination of the modulus of cohesion one may propose a series of experimental schemes, based on solutions of concrete problems. One of the simplest schemes is the following (Fig. 6). A wide wedge of constant thickness $2h$ is hammered into a plate of the material under consideration; by means of the wedge a crack of length L is formed. As is shown by calculations [18], the modulus of cohesion is determined in this experiment by the relation

$$K = \frac{Eh}{2(1-\nu^2)\sqrt{L}}. \tag{6.6}$$

The plate must be sufficiently thick, so that the state of stress may be assumed to correspond to plane strain, and sufficiently large compared with the length of the formed crack. The values of the modulus of cohesion obtained make it possible to estimate the magnitude of the dimension of the end region d. For several high-polymers which were investigated this quantity was found to be of the order of one to several tens of microns, i.e., sufficiently large compared with the mean-atomic distances, but sufficiently small compared even with small cracks.

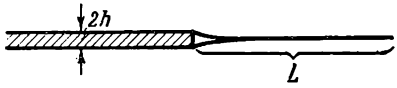

Fig. 6

The fundamental hypotheses formulated in § 2 are the only assumptions on which the proposed force approach to the theory of steady state cracks of brittle fracture has been based. These hypotheses, as is readily seen, do not contradict the prerequisites on which the energy approach is based. Therefore there must exist a relation between the modulus of cohesion K, on the one hand, and the surface tension T and the elastic characteristics of the material E and ν on the other hand. For the purpose of elucidating this connection it is sufficient to compare the relations determining the dimensions of the steady state crack, obtained by energy and force methods.

In the simplest case of the isolated straight crack in an infinite body in a homogeneous field of tension p_0, normal to the plane of the crack, the half length l of the equilibrium crack is determined by the relation

$$l = \frac{2K^2}{\pi^2 p_0^2}, \tag{6.7}$$

obtained by the force method [18], and by the relation

$$l = \frac{2ET}{\pi(1-\nu^2)p_0^2}, \tag{6.8}$$

obtained by the energy method [5][1]. Comparing (6.7) and (6.8), we obtain a universal relation linking the modulus of cohesion to the surface tension and the elastic characteristics of the material in the form

$$K^2 = \frac{\pi E T}{1 - \nu^2}. \tag{6.9}$$

Thus, through the definition of the modulus of cohesion, each experiment permits determination of the density of surface energy. Comparison of the obtained density of surface energy with the order of the magnitude of the surface tension, natural for the material under consideration, makes it possible to determine whether the mechanism of failure is brittle or not.

The examination presented completely elucidates the situation for plane cracks of normal failure, i.e., cracks on which the normal displacement suffers a discontinuity and, in addition, for which points of the surface before application lie in one plane. This stage of the investigation is important because it embraces a range of problems which are essential in practice, and it likewise offers the possibility of considering certain general qualitative relationships, characteristic for cracks in general. However, this stage of the study does not at all exhaust the problems as a whole; it is only the beginning. Therefore it will serve some purpose to note new formulations of problems.

We will first dwell on the problem of quasi-static formation of cracks. If the loading is symmetric about the plane of the crack and grows proportionally to some parameter, the configuration of the system of cracks does not depend on the history of the process of change of the load and it is determined entirely by the maximum load attained. In the general case of curvilinear cracks this is not true – the configuration of the system of cracks depends on the entire history of the process of change of the loading. The formulation of the general problem of development of systems of cracks within the framework of the plane theory of elasticity may be conceived in the following manner. Let a certain system of equilibrium cracks occur in a brittle body for a given system of loads; an analysis of the conditions which must be fulfilled for this may be executed without special difficulties by relying on the results already deduced. We will assume that a loading acts on the body which is proportional to some parameter which may be taken to have a real or complex value and which for the initial state is equal to unity. We will give this parameter an infinitesimal increase, real or

[1] We note a certain notational disagreement: sometimes the surface tension is denoted by $2T$, equal to twice the density of surface energy.

complex; the problem is now to determine the corresponding change of the surface of the cracks. Knowing the solution of this problem, one may by integration obtain the complete conditions of the development of a system of cracks dependent on the course of the process of continuous changes of the loading. One may likewise determine the condition of stability of the given system of equilibrium cracks. Within the frame work of the plane theory of elasticity the solution of the problem thus formulated is completely perspective, having in mind application of the variational methods of conformal mapping of M. W. Lavrent'ev [25]. The difference of the formulation of the problem in the three dimensional case consists of the fact that for the parameter, characterizing the change of the loading, one has to introduce a second pseudo unit.

The development of the system of cracks has hitherto been supposed to pass through a series of consecutive equilibrium states. Indeed, such a character of the development is possible only for infinitely small changes of the loading and only if these intermediate equilibrium states are stable. In the case of sudden application of a finite load, or for the transition of the system of cracks from an unstable state to another, stable one, the process of formation of the system of cracks bears an essentially dynamic character and it passes through a series of states which are not of the equilibrium type. Certain problems of dynamic development of cracks have been studied [23]; however, these problems concern the stationary development of cracks for the splitting of a brittle body by a wedge which is moving with constant velocity. The study of unsteady dynamic problems presents great difficulties and, of course, any prospect of success for rectilinear cracks only. For dynamic problems, the same indefiniteness of the lengths of cracks is characteristic in accordance with the differential equations, the boundary and the initial conditions as is for the static problem. In order to get rid of this indeterminateness one has to obtain supplementary conditions, analogous to the conditions of § 3 and § 5, starting already from Hamilton's variational principle. The solution of dynamic problems would permit the elucidation of many phenomena which are of practical interest such as, for example, that of the formation of cracks during explosions.

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THE APPLICATION OF THE METHODS OF N. I. MUSKHELISHVILI TO THE SOLUTION OF SINGULAR INTEGRAL EQUATIONS IN QUANTUM FIELD THEORY

N. N. BOGOLIUBOV, B. V. MEDVEDEV, and A. N. TAVKHELIDZE
Moscow, U.S.S.R.

The methods of solution of singular integral equations developed by N. I. Muskhelishvili have found wide use in many regions of pure mathematics as well as in its applications to mechanics and physics. We wish to report here on applications which these methods have found in quantum field theory. With the mathematical readers in mind, we will start with a short physical introduction explaining the character and formulation of the problems which arise here.

§ 1. Quantum field theory deals with the study of the properties of so-called "elementary" or "fundamental" particles, and, mainly, with their interactions and mutual transformations. The state of a free elementary particle is completely specified by saying what kind of a particle is under consideration and by giving the value of its momentum and, perhaps, several discrete quantum numbers such as the spin of an electron, the state of polarization of a photon, etc. The formulation of the problems of the interaction of elementary particles is characterized by several distinctive features. The fact is that, because of the extremely short distances with which one has to deal and the limitations imposed by the indeterminacy principle, we cannot experimentally observe the detailed behavior of elementary particles during the process of their interaction. Only the asymptotic properties of the process are accessible to observation. In fact, the problem of the interaction of elementary particles is formulated in the following manner. At the beginning, in the "initial state", we have a certain number of free elementary particles which lie far away from each other and which therefore are not interacting. Subsequently, these particles approach one another, interact with each other in some manner (exchange energy, momentum, and possibly transform into each other); finally, they separate again to great distances when the "final state" is formed which once again consists of a number of free, noninteracting particles. Only the initial and the final states are accessible to experimental observation.

Thus, the problem which experiment poses to the theory is to determine,

for a given initial state, the probability of the occurrence of any possible final state.

We will write the initial state $|\text{in}\rangle$, represented by a vector in a multi-dimensional Hilbert space, in the form

$$|\text{in}\rangle = |\boldsymbol{k}_1, \alpha_1; \cdots; \boldsymbol{k}_n, \alpha_n\rangle, \tag{1.1}$$

where $\boldsymbol{k}_1, \cdots, \boldsymbol{k}_n$ are the momenta of the participating particles and $\alpha_1, \cdots, \alpha_n$ are indices, specifying their type and other discrete quantum numbers. As a result of the process of interaction this state becomes a certain final state $|\text{fin}\rangle$. In accordance with the general rules of quantum mechanics the probability amplitude [1] for this final state to be a certain definite state which is formed by particles of type $\alpha_1', \cdots, \alpha_m'$ with momenta $\boldsymbol{k}_1', \cdots, \boldsymbol{k}_m'$ will be given by the scalar product

$$iF_{\omega'\omega} = \langle \boldsymbol{k}_1', \alpha_1'; \cdots; \boldsymbol{k}_m', \alpha_m' | \text{fin}\rangle, \tag{1.2}$$

where ω' denotes the whole set of the variables $\boldsymbol{k}_1', \alpha_1'; \cdots; \boldsymbol{k}_m', \alpha_m'$, and ω the whole set of the variables $\boldsymbol{k}_1, \alpha_1; \cdots; \boldsymbol{k}_n, \alpha_n$.

The basic entity with which one operates in quantum field theory is the operator S which transforms the initial state $|\text{in}\rangle$ into the final state $|\text{fin}\rangle$

$$|\text{fin}\rangle = S\,|\text{in}\rangle. \tag{1.3}$$

For historical reasons this operator is called the scattering matrix. With the help of this operator the expression (1.2) for the probability amplitude assumes the form

$$iF_{\omega'\omega} = \langle \boldsymbol{k}_1', \alpha_1'; \cdots; \boldsymbol{k}_m', \alpha_m' |\, S\, | \boldsymbol{k}_1, \alpha_1; \cdots; \boldsymbol{k}_n, \alpha_n\rangle$$

or, more precisely,

$$iF_{\omega'\omega} = \langle \boldsymbol{k}_1', \alpha_1'; \cdots; \boldsymbol{k}_m', \alpha_m' |\, S - 1\, | \boldsymbol{k}_1, \alpha_1; \cdots; \boldsymbol{k}_n, \alpha_n\rangle, \tag{1.4}$$

since we must exclude from the transition probability the probability of the event that no interaction at all occurs, and that the state $|\boldsymbol{k}', \alpha'\rangle$ simply happens to coincide with the initial state $|\boldsymbol{k}, \alpha\rangle$. Thus, the probability of transition between any pair of states is expressed in terms of the square of the absolute value of the matrix element of the scattering matrix, taken between these states, and, if the scattering matrix is known to us, we may obtain complete information on the probability of any scattering process.

How can one attack the problem of finding the scattering matrix? In principle, this problem has been solved; in quantum field theory one has a

[1] It is usually referred to as the transition probability amplitude or the scattering amplitude.

system of equations in terms of the solution of which the operator S could be constructed. However, these equations are non-linear operator equations and exact methods for their solution are unknown at the present time; the development of the theory is still held up by unsurmountable mathematical difficulties. The sole regular method for dealing with these equations is the method of perturbation, in which the non-linear terms are assumed to be small and the solution is sought in the form of a series, in which the expansion parameter is found to be a dimensionless quantity which is related to some type of charge, for example, in the case of electrodynamics it turns out to be the quantity $e^2/(\hbar c) = 1/137$. If, as is the case in this example, this expansion parameter is small, then we attain, in a number of cases, strikingly accurate predictions of experimental results by evaluating only the first few terms of the series (one, two, very rarely three).

However, two essential defects are inherent in the perturbation method. First, one cannot prove with its help the convergence of the series; in fact, there are many arguments in support of the view that at best the series converges only asymptotically. Therefore a number of important properties of the solution, especially in the region of high energies and momenta, are not accessible to investigation. Second, the perturbation method fails to be of use if the expansion parameter is large. Even if the series were to converge, one would have to include in the summation, in the case of a large expansion parameter, a number of terms which by far exceeds the limits of practical possibilities; however, in the case of a large expansion parameter, the evaluation of the first few terms of the asymptotic series makes, generally speaking, no sense. And yet many processes of the greatest practical importance, and, in particular, processes in which nuclear particles, i.e. — nucleons and the so-called π mesons which are responsible for the interaction between them, take part, are just the ones which lead to a large expansion parameter (≈ 15).

These difficulties have brought about the development of another approach to the problem. If, because of mathematical difficulties, we cannot find the complete solution of the problem, i.e., construct the scattering matrix, perhaps it is possible that we might establish at least certain of its properties by use of some type of reasoning. For this purpose it is natural to start from the general requirements which must be imposed on the scattering matrix from a physical point of view. Such requirements are: 1) relativistic covariance, i.e., the requirement following from the theory of relativity that the results must be independent of the choice of the coordinate system (in particular, it

follows from this that the operator S must be invariant), 2) unitarity, i.e.,

$$SS^+ = S^+S = 1, \tag{1.5}$$

which is necessary in order to preserve the norm of the vector of state. In physical terms, this means that the probability of finding the system after scattering in some (no matter which) state must be equal to unity. The condition of unitarity usually also includes the requirement of the existence of a complete system of states (in the sense of the possibility of expanding any state in terms of it); otherwise it would be impossible to interpret the meaning of the product of operators in (1.5). Finally, a third fundamental requirement which in a certain sense has the character of a hypothesis on the nature of micro-interaction, is the condition of the strict validity of causality in all, however small, space time regions. This is usually referred to as the requirement of 3) micro-causality. It is clear that the requirement of micro-causality, which refers to space-time relationships, can be conveniently stated in terms of the coordinate rather than the momentum representation. In order to accomplish the transition to this representation we must subject our scattering amplitudes to $n + m$ four-dimensional Fourier transformations after which the scattering amplitude will be found to be a function of $n + m$ points $x_a{}^i$ of four-dimensional space-time ($a = 1, \cdots, n + m$; $i = 0, 1, 2, 3$). The requirement of micro-causality will reduce to the condition that certain quantities simply related to the scattering amplitude, and called retarded and advanced amplitudes, must vanish for negative or for positive values of the time x^0.

It has turned out that one may actually obtain from these requirements certain information regarding the properties of the scattering matrix without solving the problem of finding the matrix. In this context, a basic role is played by the last condition — the condition of causality. In order to demonstrate the manner in which, by starting from the condition of causality imposed on the scattering amplitude, one may arrive at certain relations which it must satisfy, we will consider the classical one-dimensional example in which the scattering amplitude $F(t)$ depends only on the time $t = x^0$. Then it may be said that the system receives "signals" $f(t)$ and gives "responses" $g(t)$ to them, and the condition of causality reduces to the obvious requirement that the "response" may not be given before the "signal" is received. If, in addition, it is assumed that the system is linear, the "responses" must be represented by a linear functional of the "signals",

$$g(t) = \int_{-\infty}^{\infty} F(t - t')f(t')dt', \tag{1.6}$$

whose kernel $F(t-t')$ depends only on the difference $t-t'$, on the basis of considerations of invariance with respect to displacements in time. In this example the "signals" correspond to the initial states, the "responses" to the final states, and the kernel $F(t)$ to the scattering amplitude.

The condition of causality may be written in the form

$$F(t) = 0 \text{ for } t < 0. \tag{1.7}$$

If we now construct the Fourier transform of the scattering amplitude

$$\tilde{F}(E) = \int_{-\infty}^{\infty} F(t) e^{iEt}\, dt, \tag{1.8}$$

and let the quantity E assume the complex values $E = x + iy$ in the upper half-plane ($y > 0$), then the cut-off factor $\exp(-yt)$, will appear in the integrand and the integral will converge and determine an analytic function without essential singularities at infinity. [1]

Then one may apply Cauchy's integral formula to this function using a contour, formed by a straight line immediately above the real axis tending from $-\infty$ to $+\infty$ and a large semi-circle in the upper half-plane. By letting the point at which Cauchy's formula determines the value of the function approach the real axis, by neglecting the integral along the large semi-circle [2] and by separating real and imaginary parts, we arrive at the following integral relation for the scattering amplitude $\tilde{F}(E)$,

$$\operatorname{Re} \tilde{F}(E) = \frac{1}{\pi} \mathrm{P} \int_{-\infty}^{+\infty} \frac{\operatorname{Im} \tilde{F}(E')}{E' - E}\, dE', \tag{1.9}$$

which expresses its real part in terms of the integral of its imaginary

[1] Naturally, one has to impose for this purpose certain conditions on the function $F(t)$. It may be shown that if $F(t)$ is a generalized function, integrable over a certain class of smooth functions, then for real E the integral (1.8) will exist in the form of a generalized function, and for complex E it will be in the upper half-plane an analytic function increasing at infinity not faster than a certain polynomial the degree of which is determined by the class of functions over which $F(t)$ is integrable.

[2] Of course, this may be done only if $\tilde{F}(E)$ decreases at infinity faster than a constant. If, as is usually the case, $\tilde{F}(E)$ decreases slower than some positive power, Cauchy's integral formula must be applied not directly to it, but to the function

$$G(E) = (E - E_0 + it)^{-(n+1)} \tilde{F}(E).$$

The relation (1.9) then becomes somewhat complicated; there appear under the integral sign additional powers of E' and, besides, a polynomial in E is added to the right hand side.

part. In quantum field theory relations of this type are called dispersion relations (the name is due to the fact that such relations were first used in physics in optical dispersion theory).

In going over from a model example to the realistic case of a physical process a number of complicating circumstances arise. First of all, the number of variables increases sharply. Thus, if we undertake the study of a process in which, as a result of an interaction n particles transform into m particles, we will have altogether $(n + m - 1)$ independent four-vectors (the law of conservation of total energy and of total momentum imposes on the $(n + m)$ vectors one relation between the coordinates or the momenta of all the particles). By the strength of considerations of relativistic covariance the amplitude of the process may not depend on the components of these vectors, but only on their $\frac{1}{2}(n + m - 1)(n + m)$ squares and scalar products. But in the initial and final states our particles are free, and therefore the squares of their four-momenta must be equal to the squares of the corresponding masses. Consequently, $(n + m)$ variables will be fixed and, as a result, the amplitude corresponding to the reaction resulting in the transformation of n particles into m particles will depend on $\frac{1}{2}(n + m - 3)(n + m)$ variables. In the simplest case of the transformation of two particles into two particles $(n = m = 2)$, we already obtain two variables.

The following complication, which leads in the general case to great algebraic difficulties, is connected with the fact that the causality condition does not impose a limitation of the type (1.7) directly on these variables, on which the function under consideration depends. As has just been said, the scattering amplitude depends explicitly on the relativistically invariant squares and scalar products, while the causality condition imposes limitations on its dependence on the zero components ("time") of the corresponding vectors. It will be convenient to explain another, and perhaps the most essential complication below.

Due to these complications, a general derivation of dispersion relations of the type (1.9) for an arbitrary process and with respect to all the variables occurring in it has hitherto not be achieved. At the cost of invoking a number of subtle theorems from the borderline field lying between the theory of functions of several complex variables and the theory of generalized functions, it has been possible to attain only much more modest results — to prove the dispersion relations with respect to one variable in the case of scattering processes or in the case of transformation of two particles into two others for fixed and not too large values of a second variable. Even this result could not be established for arbitrary processes of this type, but only for certain ones (the applicability of the

proof is determined by the masses of the particles participating in the reaction). It must be said that physicists adopt a very optimistic point of view with regard to this problem: often dispersion relations are written down and used even for such processes for which they have not as yet been proved (usually no great difficulty is experienced in writing down the dispersion relations with the help of non-rigorous arguments). This optimism in many cases finds support in comparison with experiment. Such a state of affairs is now stimulating intensified attempts to extend by various methods the region of proved dispersion relations.

In the majority of cases the variable in terms of which the dispersion relations are formulated is the energy E of the relative motion of the system. Therefore we will separate in our formula (1.4) this variable from the set of the remaining variables ω, ω' and we will show it explicity in the argument of the scattering amplitude by writing $F_{\omega'\omega}(E)$. The role of complex conjugation will now be played by Hermitian conjugation, and instead of the real and imaginary parts, appearing in the relation (1.9), we will now have the Hermitian and the anti-Hermitian parts $D_{\omega'\omega}(E)$ and $A_{\omega'\omega}(E)$

$$D_{\omega'\omega}(E) = \frac{F_{\omega'\omega}(E) + F^{+}_{\omega'\omega}(E)}{2},$$

$$A_{\omega'\omega}(E) = \frac{F_{\omega'\omega}(E) - F^{+}_{\omega'\omega}(E)}{2i}, \tag{1.10}$$

$$F^{+}_{\omega'\omega}(E) = (F_{\omega\omega'}(E))^{*}.$$

On assuming again, in order not to complicate the discussion, that the rate of increase of n at infinity is sufficiently small (cf. note 2 on p. 43), we may now write down the dispersion relations for a real physical case in the form

$$D_{\omega'\omega}(E) = \frac{1}{\pi} \mathrm{P} \int_{-\infty}^{\infty} \frac{A_{\omega'\omega}(E')}{E' - E}\, dE'. \tag{1.11}$$

The values occurring in the above integral of the anti-Hermitian part of the scattering amplitude corresponding to the physically unrealizable negative energies may, as is always the case in quantum field theory, be expressed in terms of the corresponding quantity for the anti-particles; we will not dwell on the details of this transformation. As a result, the integration in (1.11) will be carried out with respect to only the positive values of E, but a second term $A'_{\omega'\omega}$, related to the scattergin

of anti-particles, will be added to the integrand

$$D_{\omega'\omega}(E) = \frac{1}{\pi} \mathrm{P} \int_0^\infty \frac{A_{\omega'\omega}(E')}{E' - E} dE' + \frac{1}{\pi} \int_0^\infty \frac{A'_{\omega'\omega}(E')}{E' + E} dE'. \tag{1.12}$$

The further reasoning may be pursued in two different directions. First of all, although the formula (1.12) does not represent an equation from which one could determine both D and A, nevertheless it establishes between them a definite relation. The Hermitian and the anti-Hermitian parts of the scattering amplitudes $D_{\omega'\omega}$ and $A_{\omega'\omega}$ are closely connected with the observed probabilities of processes (as the physicists say — with the effective cross-sections); in certain important cases both the right and the left hand sides of (1.12) may be determined from experimental data. Then, by verifying whether the relation (1.12) is fulfilled, we obtain a means for verifying the validity of the basic physical assumptions made in the derivation of this relation, and in the first place, of the fundamental requirement of micro-causality which plays a basic role in the derivation of the dispersion relations. The experimental data now available (we have in mind first of all the process of the scattering of π-the mesons by nucleons) are in good agreement with the dispersion relations, and for the present do not give grounds for doubting that the conditions of micro-causality are satisfied.

On the other hand, one might wonder whether it might not be possible to extract some additional information from the fundamental requirements formulated above, which, when added to the dispersion relations (1.12), would convert them into equations determining the scattering amplitude. One must now, of course turn to the condition of unitarity which has hardly yet been used and which by the strength of its non-linear character establishes a relation between scattering amplitudes for different processes.

It may be seen from the formula (1.4) which determines the scattering amplitude that the scattering amplitudes are matrix elements not of the matrix S itself, but rather of the operator

$$T = \frac{S - 1}{i}. \tag{1.13}$$

The condition of unitarity (1.5), rewritten for this operator, assumes the form

$$\frac{T - T^+}{2i} = - TT^+, \tag{1.14}$$

which leads to the following relation for its matrix elements — the scattering amplitudes

$$\frac{F_{\omega'\omega} - F^{+}_{\omega'\omega}}{2i} = A_{\omega'\omega} = -\sum F_{\omega' n} F^{+}_{n\omega}. \tag{1.15}$$

The summation is here performed over a complete system of states each of which is briefly denoted by the single symbol n.

Thus, the condition of unitarity expresses the anti-Hermitian part of the amplitude of any concrete process in terms of the sum of products of the amplitudes of all the processes, possible (by the strength of the conservation laws) for the given initial and final states.

Now we enter into the region of the hypothetical. If we knew how to prove the dispersion relations for any process, then, by writing them down and by expressing in every one of them the anti-Hermitian parts in the integrands in terms of sums of the form (1.15), we would arrive at an infinite system of coupled equations by solution of which we could find all the scattering amplitudes. Thereby the basic problem of quantum field theory would be solved.

In such an approach, we encounter two difficulties. First, as has already been stated, the dispersion relations have been proved for far from all processes. Second, even if they had been proved, we would not be able to solve the infinite system. Therefore, in order to proceed further, one has to resort to approximating assumptions.

The following assumption is usually employed. If we consider a process in which only a few particles of not too high energies take part, we might expect that in the sum (1.15) the higher terms corresponding to states with large numbers of particles will not play an essential role. If this is so, then they may be neglected and the sum may be terminated after a few terms. Thus, for example, when dealing with the process of the scattering of two particles or of the transformation of two particles into two others, the simplest and, of course, the crudest approximation is obtained by restricting the sum with respect to n likewise to two particles states. Proceeding in this manner we arrive either at a single integral equation determining the amplitude of the process of interest to us, or at a system of several (two, three) integral equations, jointly determining the amplitudes of several closely interrelated processes.

It is clear from the form of the dispersion relations that without fail these will be equations with singular kernels of the type

$$\mathrm{P}\frac{1}{x - x'} \quad \text{or} \quad \frac{1}{x - x' - i\varepsilon}. \tag{1.16}$$

In a number of important cases these equations are found to belong to a class studied by N. I. Muskhelishvili, and they can be solved by means of the methods developed by him. At the present time, N. I. Muskhelishvili's method is being used successfully by physicists in our country as well as abroad both for practical calculations, and for the study of the very important and interesting problems of the uniqueness of solutions. With its help we have investigated processes of photo-production of pions on nucleons, the production of pions in pion-nucleon collisions, processes of the decay of hyperons, etc. In the course of these investigations, new, previously inaccessible results have been obtained which are of great interest. In order to demonstrate the effectiveness of these methods, we will present several concrete examples selected from the large number of cases which have already been studied.

§ 2. As the first example, we will study the process of photo-production of π-mesons on nucleons which consists of the absorption of a photon (γ — quantum) by a nuclear particle — a nucleon (proton or neutron), and the subsequent radiation of a π-meson.

In our notation the initial and the final states of the system may be written in the form

$$\begin{aligned} |\text{in}\rangle &= |\boldsymbol{k}_1, \alpha_1; \boldsymbol{k}_2, \alpha_2\rangle, \\ |\text{fin}\rangle &= |\boldsymbol{k}_1', \alpha_1'; \boldsymbol{k}_2', \alpha_2'\rangle, \end{aligned} \tag{2.1}$$

where $\boldsymbol{k}_1$, $\boldsymbol{k}_1'$ are the momenta of the nucleon, α_1 and α_1' specify the spin and the charge states (proton or neutron) of the nucleon before and after the reaction, $\boldsymbol{k}_2$, α_2 are the momentum and the polarization of the absorbed photon and $\boldsymbol{k}_2'$, α_2' are the momentum and the charge state (π^+, π^0, π^-) of the emitted π-meson.

Substituting (2.1) in (1.4) we obtain for the photo-production amplitude

$$iF_{\omega'\omega}(E, \mathbf{p}^2) = \langle \boldsymbol{k}_1', \alpha_1'; \boldsymbol{k}_2', \alpha_2' | S-1 | \boldsymbol{k}_1, \alpha_1; \boldsymbol{k}_2, \alpha_2 \rangle, \tag{2.2}$$

where ω and ω' characterize only the spin and the charge states of the system, while the dependence on the two independent scalar products of the vectors $\boldsymbol{k}_1, \cdots, \boldsymbol{k}_2'$ (as such products we select the energies of relative motion E and the momentum transfer $\mathbf{p}^2$ in a certain particular frame of reference) is explicitly shown in the argument. Due to the presence of the discrete variables ω', ω it is clear that the amplitude for the reaction of photo-production is specified by several scalar functions the number of which, generally speaking, is rather large. Thus, for example, for given values of energy and momentum the system of nucleon + photon

may exist in 8 different spin states, while the system of nucleon + π-meson may exist in 12 different spin and charge states. If the system could make a transition from the given initial state to any final state, then the number of independent components of the amplitude for photo-production would be equal to $8 \times 12 = 96$. However, taking into consideration the laws of conservation of total charge and of the total angular momentum of the system, the number of amplitudes may be reduced from 96 to 12.

The above method of dispersion relations makes it possible to obtain approximate equations for the determination of the 12 components of the photo-production amplitude. At the present time the region of small energies, when the nucleons may be treated non-relativistically, has been studied in greatest detail. In this case the amplitude of the process is specied by only three functions (the contribution from the remaining ones will be small) for which dispersion relations of the type (1.12) will hold.

In order to convert these relations into equations one must, as has been stated above, resort to the condition of unitarity (1.15) which permits evaluation of the anti-Hermitian parts of the reaction amplitude. Since we do not wish to obtain an infinite system of coupled equations, we retain in the expansion (1.15) only the first two terms [1] $n = \omega_1$ and $n = \omega_2$ (ω_1 denotes the single nucleon state, ω_2 the state sonsisting of a nucleon + 1 meson). Then the condition of unitarity may be written in the form

$$A_{\omega'\omega} = - F_{\omega'\omega_1} F^{+}_{\omega_1\omega} - F_{\omega'\omega_2} F^{+}_{\omega_2\omega}.$$

The contribution from the first term in the dispersion relations can be evaluated directly.

The second term has the form of a product of the meson-nucleon scattering amplitude $F_{\omega'\omega_2}$ and a quantity which is the Hermitian conjugate of the photo-production amplitude $F_{\omega_2\omega}$. In fact, if ω denotes the nucleon+photon state and ω_2 the nucleon + 1 meson state, then by definition $F_{\omega_2\omega}$ is the photo-production amplitude. Taking into consideration that ω' as well as ω_2 denote the nucleon + 1 meson state, we see that $F_{\omega'\omega_2}$ is the meson-nucleon scattering amplitude.

One may assume the meson-nucleon scattering amplitude to be a known function of the energy either as the result of corresponding experiments or of the solution of the equations to which the dispersion relations

[1] It may turn out that in the region of low energy the states with large numbers of particles will not play an essential role. Such an assumption leads, for example, in the case of scattering of pions by nucleons to satisfactory agreement with experiments.

lead in the case of the process of the scattering of mesons by nucleons (equations of F. E. Low and G. F. Chew). Therefore the sole unknown quantities will be the photo-production amplitudes $y_i(E)$ for which we obtain the system of linear singular integral equations of the form

$$(2.3)\qquad y_i(x)\cos\delta_i(x) = f_i^0(x) + \frac{1}{\pi}\mathrm{P}\int_1^a \left[\frac{y_i(x')\sin\delta_i(x')}{x'-x} + \right.$$

$$\left. + \sum_{j=1}^{3} K_{ij}(x, x')\sin\delta_j(x')y_j(x')\right] dx',$$

where K_{ij} is a regular kernel and $f_i^0(x)$ and $\delta_i(x)$ are known functions; $\delta_i(x)$ characterizes the process of meson-nucleon scattering.

The equation (2.3) belongs to the class of linear singular integral equations studied by N. I. Muskhelishvili. Using the methods developed by him we will show that the system of singular equations (2.3) may be reduced to regular Fredholm equations and that, if the functions $\delta_i(x)$ satisfy the condition

$$(2.4)\qquad \delta_i(1) = 0,\ \delta_i(a) < \pi,$$

then the transition to the regular equations is unique. It is natural to require for this purpose that the solution y_i is to be bounded in the entire interval $1 \leq x \leq a$ which we will denote by L, and to satisfy on L a Hölder condition.

The functions f_i^0 and K_{ij} satisfy the Hölder condition

$$(2.5)\qquad |\varphi(x_1) - \varphi(x_2)| \leq A\,|x_2 - x_1|^{\mu},\ \mu > 0.$$

Omitting the index i, we rewrite the system (2.3) in the form

$$(2.6)\qquad y(x)\cos\delta(x) = f_0(x) + \frac{1}{\pi}\mathrm{P}\int_1^a \frac{y(x')\sin\delta(x')}{x'-x}\,dx',$$

where the function

$$(2.7)\qquad f_0(x) \equiv f_{i0}(x) = f_i^0(x) + \frac{1}{\pi}\int_1^a \sum_{j=1}^{3} K_{ij}(x, x')\sin\delta_j(x')y_j(x')dx'$$

satisfies a Hölder condition on L. We will introduce the function of the complex variable

$$(2.8)\qquad \Psi(z) = \frac{1}{2\pi i}\int_1^a \frac{y(x)\sin\delta(x)}{x-z}\,dx.$$

By the strength of the requirements, imposed on y and δ, this function will be

1) holomorphic everywhere except on L,
2) analytically continued on L from above and below,
3) bounded on L,
4) $\Psi(\infty) = 0$.

We introduce the notation

$$\Psi^{\pm}(x) = \lim_{\varepsilon \to 0} \Psi(x \pm i\varepsilon), \quad \varepsilon > 0. \tag{2.9}$$

It follows from (2.8) that

$$y(x) \sin \delta(x) = \Psi^{+}(x) - \Psi^{-}(x). \tag{2.10}$$

Using the relations (2.8)—(2.10), we obtain from the expression (2.6)

$$ye^{i\delta} = 2i\Psi^{+} + f_0, \quad ye^{-i\delta} = 2i\Psi^{-} + f_0, \tag{2.11}$$

$$\Psi^{+} = e^{2i\delta}\Psi^{-} + f_0 ie^{i\delta} \sin \delta. \tag{2.11'}$$

We will construct the function

$$X(z) = (z - a)^n \exp\left(\frac{1}{\pi} \int_1^a \frac{\delta(x)}{x - z}\, dx\right), \tag{2.12}$$

where $2i\delta$ denotes some branch of $\log e^{2i\delta}$ which we choose, for example, so that

$$\delta(1) = 0; \tag{2.13}$$

and n is an integer satisfying the condition

$$0 \leq \frac{\delta(a)}{\pi} + n < 1. \tag{2.14}$$

The function $X(z)$ has the properties 1)—3) and satisfies the boundary condition

$$X^{+} = e^{2i\delta}X^{-}. \tag{2.15}$$

In any finite part of the plane, $X(z)$ may vanish only at $z = 0$, while for $z \to a$

$$X(z) = (z - a)^{[\delta(a)/\pi + n]}\Omega(z), \quad \Omega(z) \neq 0. \tag{2.16}$$

For $z \to \infty$,

$$X(z) \to z^n. \tag{2.17}$$

With the help of the function $X(z)$ one may rewrite the condition (2.11) in the form

$$\frac{\Psi^+(x)}{X^+(x)} - \frac{\Psi^-(x)}{X^-(x)} = \frac{if_0(x)\sin\delta(x)}{X(x)}. \tag{2.18}$$

The function $\Psi(z)/X(z)$ has the properties 1), 2) and for $z \to a$ it may have, by the strength of (2.14), the removable singularity $(z-a)^\alpha$, $\alpha < 1$. If one requires, in addition, that it is to have a finite order pole at infinity, then it is not difficult to write down its general expression, whence we obtain the general expression for the function $\Psi(z)$, which satisfies the conditions 1)—3) and has a pole of finite order at infinity,

$$\Psi(z) = \frac{X(z)}{2\pi}\int_1^a \frac{f_0(x)\sin\delta(x)}{X(x)(x-z)}\,dx - X(z)\mathrm{P}(z). \tag{2.19}$$

In order to satisfy the condition 4) for $n > 0$, one must set $\mathrm{P}(z) \equiv 0$. Moreover, one must yet satisfy the conditions

$$\int_1^a \frac{x^k f_0(x)\sin\delta(x)}{X(x)}\,dx = 0, \quad (k = 0, 1, \cdots, n-1). \tag{2.20}$$

Taking into consideration (2.19), we obtain on the basis of (2.11) for the determination of the function y the regular Fredholm equation

$$y(x) = f_0(x)\cos\delta(x) + X(x)\frac{1}{\pi}\mathrm{P}\int_1^a \frac{f_0(x')\sin\delta(x')}{(x'-x)X(x')}\,dx' \tag{2.21}$$

and the supplementary conditions (2.20).

The negative values $n < 0$ for which the transition to regular equations is not unique are excluded due to the conditions (2.4) and (2.14).

Thus, in the example under consideration of the photo-production of π-mesons on nucleons, the method of N. I. Muskhelishvili permits one to establish under definite conditions the uniqueness of the solutions of the singular integral equations to which in this case the dispersion relations reduce and to reduce these singular equations to well known regular Fredholm equations which may be solved, for example, by numerical methods [1].

[1] Such computations have been perform d and they have led to good agreement with experiment in the low energy regione.

§ 3. We will now proceed to another example in which the method of N. I. Muskhelishvili permits one not only to reduce the singular equations to regular ones, but also to find the solution of the problem in analytic form.

The recently discovered heavy Bose particle — "the K-meson" — is not stable and may decay by various modes. We will consider the decay of a neutral K-meson into two π-mesons.

The reaction amplitude for this process is specified by two independent functions $F_I(E)$ for which one may write the rather simple dispersion relations

$$F_I(E) = \frac{E}{\pi} \int_{\mu}^{\infty} \frac{\operatorname{Im} F_I(E')}{E'(E' - E - it)}\, dE' + F_I(0), \tag{3.1}$$

where the index I assumes the values 0, 2 and characterizes the charge state of the system, and μ is the mass of the π-meson.

For the evaluation of the anti-Hermitian part $\operatorname{Im} F_I$, just as in the preceding example, one must utilize the unitarity condition (1.15). Taking only the two meson states in the expansion (1.15) into account, [1] we obtain for the anti-Hermitian part $\operatorname{Im} F_I$ the approximate expression

$$\operatorname{Im} F_I(E) = h_I(E) F_I(E), \tag{3.2}$$

where $h_I(E)$ is conveniently written in the form

$$h_I(E) = e^{-i\delta_I(E)} \sin \delta_I(E),$$

and $\delta_I(E)$ may be determined from scattering experiments of π-mesons by π-mesons. Substituting (3.2) into (3.1) we obtain for the determination of the amplitude for the decay of a neutral K-meson into two π-mesons the singular integral equation

$$F_I(E) = \frac{E}{\pi} \int_{\mu}^{\infty} \frac{h_I(E') F_I(E')}{E'(E' - E - it)}\, dE' + F_I(0), \tag{3.3}$$

the solutions of which one may readily obtain with the help of the method of N. I. Muskhelishvili.

We consider for this purpose the function of the complex variable

$$T_I(z) = \frac{z}{\pi} \int_{\mu}^{\infty} \frac{h_I(E') F_I(E')}{E'(E' - z)}\, dE'. \tag{3.4}$$

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[1] Cf. Footnote 1 on p. 49.

It follows from the definition of the function $T_I(z)$ that

$$T_I(E_+) - T_I(E_-) = [1 - e^{-2i\delta_I(E)}]F_I(E)\theta(E - \mu), \tag{3.5}$$

where

$$E_\pm = E \pm i\varepsilon, \ \varepsilon > 0,$$

$$\theta(x) = \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{for } x < 0. \end{cases}$$

Consequently, the function $T_I(z)$ is the solution of the following Hilbert boundary value problem:

$$T_I(E_+)e^{-2i\delta_I(E)\theta(E-\mu)} - T_I(E_-) = 0. \tag{3.6}$$

If it is still assumed that $T_I(z)$ increases at infinity not faster than a constant, the solution of our problem becomes

$$F_I(z) = F_I(0) \exp\left\{\frac{E}{\pi} \mathrm{P} \int_\mu^\infty \frac{\delta_I(x)}{x(x - E)} dx + i\delta_I(E)\theta(E - \mu)\right\}, \tag{3.7}$$

where $F_I(0)$ is a real constant which may be determined from experiment.

The above examples are typical. Whatever process we may consider with the help of the method of dispersion relations, we find that after utilization of the condition of unitarity and restriction of the set of intermediate states involved, the solution of the problem considered reduces for sufficiently low energies and independently of the magnitude of the coupling constant to the study of a system of integral equations with singular kernels of the type $1/(z - z')$. In actual fact, if we had not restricted ourselves to the non-relativistic case, then as a natural generalization we would have arrived at the utilization of the double spectral representations introduced by Mandel'shtam which, taken in conjunction with the two-particle approximations, would have again led us to a system of singular integral equations of the same type. Thus, we arrive in this way at a new method of solution for solving at least some problems of quantum field theory, which is independent of perturbation theory. For the development of this method, the procedures developed by N. I. Muskhelishvili for the solution of singular integral equations have been seen to have played a very important role.

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AN ASYMPTOTIC METHOD FOR THE STUDY OF THE PROBLEM OF EIGENVALUES FOR RECTANGULAR REGIONS

V. V. Bolotin
Moscow, U.S.S.R.

This paper presents an asymptotic method for finding the eigenfunctions and eigenvalues for one class of linear, self-adjoint operators, defined in a rectangular region. The method is based on the representation of the solution in the form of the sum of a certain generating solution and of corrective solutions by the help of which the boundary conditions are fulfilled. As regards its properties, the corrective solution is analogous to an integral of the edge effect type [1, 2] or of the boundary layer type [3], used in the construction of asymptotic solutions of differential equations with small parameters as factors of the highest derivatives. The asymptotic expression for the eigenfunctions is found by joining the solutions for each of the boundary regions. The error involved in such a procedure decreases rapidly with an increase of the number of functions (of the wave number). In contrast to the known asymptotic estimates for eigenvalues and eigenfunctions of linear operators [4], suitable only for sufficiently high wave numbers, the present method gives an effective solution also for low wave numbers. Application of the method is illustrated by problems of natural vibrations of elastic plates and shells.

§ 1. We consider the homogeneous linear self-adjoint boundary value problem

$$L(\varphi) - \lambda M(\varphi) = 0 \tag{1.1}$$

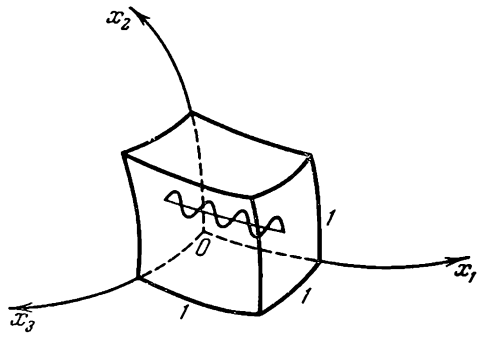

Fig. 1

for a rectangular region in the space m of the variables x_1, $x_2, \cdots, x_m$ (Fig. 1). Without restriction of generality one may assume the range of variation of the variables to be $0 \leq x_\alpha \leq 1$ $(\alpha = 1, 2, \ldots, m)$. Here $L(\varphi)$ and $M(\varphi)$ are the linear differential operators with constant coefficients of order $2n$ and $(2n - 1)$ respec-

tively, not containing odd order derivatives:

$$L(\varphi) = a_0\varphi + \sum_{\beta=1}^{n} \sum_{\beta_1,\beta_2,\cdots,\beta_m} a^{(\beta)}{}_{\beta_1,\beta_2,\cdots,\beta_m} \frac{\partial^{2\beta}\varphi}{\partial x_1{}^{2\beta_1}\partial x_2{}^{2\beta_2}\cdots\partial x_m{}^{2\beta_m}},$$

(1.2)

$$M(\varphi) = b_0\varphi + \sum_{\beta=1}^{n-1} \sum_{\beta_1,\beta_2,\cdots,\beta_m} b^{(\beta)}{}_{\beta_1,\beta_2,\cdots,\beta_m} \frac{\partial^{2\beta}\varphi}{\partial x_1{}^{2\beta_1}\partial x_2{}^{2\beta_2}\cdots\partial x_m{}^{2\beta_m}}.$$

In these expressions a_0, $a^{(\beta)}{}_{\beta_1,\beta_2,\cdots,\beta_m}$, b_0, $b^{(\beta)}{}_{\beta_1,\beta_2,\cdots,\beta_m}$ are constant real coefficients, $\beta_1, \beta_2, \cdots, \beta_m$ are integers or zero, where $\beta_1 + \beta_2 + \cdots \beta_m = \beta$. The functions φ must satisfy conditions which, generally speaking, are different for each border of the region. We assume that these conditions have the form $N_\alpha{}^{(\beta)}(0) = 0$, $N_\alpha{}^{(\beta)}(1) = 0$ $(\alpha = 1, 2, \cdots, m, \beta = 1, 2, \cdots, n)$, where $N_\alpha{}^{(\beta)}(\varphi)$ is some linear differential expression with constant coefficients. The objective is to determine the eigenvalues λ and eigenfunctions φ of this boundary value problem.

The equation (1.1) will be satisfied, if one writes

$$\varphi = \sin k_1(x_1 - \xi_1) \sin k_2(x_2 - \xi_2) \cdots \sin k_m(x_m - \xi_m), \tag{1.3}$$

where $k_1, k_2, \cdots, k_m$, $\xi_1, \xi_2, \cdots, \xi_m$ are certain constants. In what follows the constants $k_1, k_2, \cdots, k_m$ will be called *wave numbers*. The corresponding values of the parameter λ will be

$$\lambda = \frac{a_0 + \sum_{\beta=1}^{n} \sum_{\beta_1,\beta_2,\cdots,\beta_m} a^{(\beta)}{}_{\beta_1,\beta_2,\cdots,\beta_m}(-1)^{\beta_1,\beta_2,\cdots,\beta_m} k_1{}^{2\beta_1}k_2{}^{2\beta_2}\cdots k_m{}^{2\beta_m}}{b_0 + \sum_{\beta=1}^{n-1} \sum_{\beta_1,\beta_2,\cdots,\beta_m} b^{(\beta)}{}_{\beta_1,\beta_2,\cdots,\beta_m}(-1)^{\beta_1,\beta_2,\cdots,\beta_m} k_1{}^{2\beta_1}k_2{}^{2\beta_2}\cdots k_m{}^{2\beta_m}}. \tag{1.4}$$

This procedure will, in general, not satisfy the boundary conditions.

One may try to find an approximate solution of the problem by considering the expression (1.3) as some *generating solution* which is only true for an internal region which is sufficiently distant from the boundary. This will be possible in the case when for values of the parameter λ, determined by (1.4), one can construct for every boundary $n - 1$ independent solutions which decay rapidly with the distance from the boundary. Then we have at our disposal sufficient arbitrary constants to satisfy all the boundary conditions. Altogether there will be constructed $2m$ different solutions satisfying the differential equations (1.1) and the conditions on the corresponding boundaries. All these solutions tend to the generating solution (1.3) as the inward distance from the boundary of the region increases. Joining the solutions found we obtain m conditions for finding the wave numbers of the generating solution $k_1, k_2, \cdots, k_m$. The error involved in such a procedure in

each case may be estimated and usually it decreases rapidly with increasing wave numbers $k_1, k_2, \cdots, k_m$.

For example, we will consider the boundary $x_\alpha = 0$. For this boundary the solution will be sought in the form

$$\varphi_\alpha = \Phi_\alpha(x_\alpha) \sin k_1(x_1 - \xi_1) \cdots \tag{1.5}$$
$$\cdots \sin k_{\alpha-1}(x_{\alpha-1} - \xi_{\alpha-1}) \sin k_{\alpha+1}(x_{\alpha+1} - \xi_{\alpha+1}) \cdots \sin k_m(x_m - \xi_m).$$

The function Φ_α must satisfy the equation

$$L_\alpha(\Phi_\alpha) - \lambda M_\alpha(\Phi_\alpha) = 0, \tag{1.6}$$

where

$$L_\alpha(\Phi_\alpha) = a_0\Phi_\alpha + \sum_{\beta=1}^{n} \sum_{\beta_1,\beta_2,\cdots,\beta_m} a^{(\beta)}{}_{\beta_1,\beta_2,\cdots,\beta_m} \frac{d^{2\beta_\alpha}\Phi_\alpha}{dx_\alpha{}^{2\beta_\alpha}} \cdot$$
$$\cdot (-1)^{\beta_1,\cdots,\beta_{\alpha-1},\beta_{\alpha+1},\cdots,\beta_m} k_1^{2\beta_1} \cdots k_{\alpha-1}^{2\beta_{\alpha-1}} k_{\alpha+1}^{2\beta_{\alpha+1}} \cdots k_m^{2\alpha_m},$$

$$M_\alpha(\Phi_\alpha) = b_0\Phi_\alpha + \sum_{\beta=1}^{n-1} \sum_{\beta_1,\beta_2,\cdots,\beta_m} b^{(\beta)}{}_{\beta_1,\beta_2,\cdots,\beta_m} \frac{d^{2\beta_\alpha}\Phi_\alpha}{dx_\alpha{}^{2\beta_\alpha}} \cdot$$
$$\cdot (-1)^{\beta_1,\cdots,\beta_{\alpha-1},\beta_{\alpha+1},\cdots,\beta_m} k_1^{2\beta_1} \cdots k_{\alpha-1}^{2\beta_{\alpha-1}} k_{\alpha+1}^{2\beta_{\alpha+1}} \cdots k_m^{2\alpha_m},$$

and λ is given by (1.4). The corresponding characteristic equation, obtained from (1.5) by the substitution

$$\Phi_\alpha = C\, e^{rx_\alpha},$$

has the order $2n$; we will represent this equation by the symbol $\Delta_\alpha(r^2) = 0$.

Obviously, the equation $\Delta_\alpha(r^2) = 0$ has the two purely imaginary roots $r_{1,2} = \pm ik_\alpha$. Detaching these roots, corresponding to the generating solution (1.3), we obtain an equation of order $2(n-1)$ which we will represent by the symbol $\Delta_\alpha{}^*(r^2) = 0$. Thus

$$\Delta_\alpha(r^2) = \Delta_\alpha{}^*(r^2)(r^2 + k_\alpha{}^2). \tag{1.7}$$

The possibility of constructing an asymptotic solution depends exclusively on the properties of the equation $\Delta_\alpha{}^*(r^2) = 0$. We assume that all the roots r are either real or complex conjugate. Among them exactly $(n-1)$ roots will have negative real parts. We denote them by $r = -p_\beta + iq_\beta$, where $p_\beta > 0$, $\beta = 3, 4, \cdots, n+1$. The expression

$$\Phi_\alpha = C_{\alpha1} \sin k_\alpha x_\alpha + C_{\alpha2} \cos k_\alpha x_\alpha + \sum_{\beta=3}^{n+1} C_{\alpha\beta} \exp(-p_\beta + iq_\beta)x_\alpha \tag{1.8}$$

contains $(n+1)$ constants with the help of which one may satisfy n con-

ditions on the boundary $x_\alpha = 0$. The remaining constant may be used for the normalization

$$C_{\alpha 1}^2 + C_{\alpha 2}^2 = 1.$$

Then, for $x_\alpha = 0$, the solution (1.5) with the function Φ_α, determined by (1.8), will satisfy all the conditions $N_\alpha^{(\beta)}(0) = 0$ ($\beta = 1, 2, \cdots, n$), and for $x_\alpha \to \infty$ it will tend to the generating solution (1.3).

The expression (1.8) only applies in the case of simple characteristic roots. In the case of multiple roots the expression will contain terms of the type

$$x_\alpha^\gamma \exp(-p_\beta + iq_\beta)x_\alpha,$$

where the highest power γ is equal to the multiplicity of the root minus unity; all the remaining properties of the solution (1.8) are maintained.

The corrective solution

$$\Psi_\alpha(x_\alpha/0) = \sum_{\beta=3}^{n+1} C_{\alpha\beta} \exp(-p_\beta + iq_\beta)x_\alpha \tag{1.9}$$

describes the deviation of the solution from the generating solution near the edge $x_\alpha = 0$. This deviation will be called *dynamic edge effect.* If all $q_\beta \equiv 0$, we will talk of *a non-oscillatory* edge effect. Otherwise we will call the edge effect *oscillatory*. If it is impossible to find among the r roots $(n - 1)$ roots with negative real parts, we will speak of a *degenerate* dynamic edge effect.

It is not difficult to establish necessary and sufficient conditions under which the edge effect does not degenerate. We consider the equation

$$\Delta_\alpha^*(r^2) = c_0 r^{2(n-1)} + c_1 r^{2(n-2)} + \cdots + c_{n-2} r^2 + c_{n-1} = 0. \tag{1.10}$$

In order that among the roots of equation (1.10) there will be $(n - 1)$ roots with negative real parts, it is necessary and sufficient that the equation

$$\Delta_\alpha^*(\rho) = c_0 \rho^{n-1}(-1)^{n-1} + c_1 \rho^{n-2}(-1)^{n-2} + \cdots - c_{n-2}\rho + c_{n-1} = 0, \tag{1.11}$$

where $\rho = -r^2$, has no positive or zero real roots. By a known theorem of Descartes the equation (1.11) has no positive real or zero roots, if and only if all its coefficients have the same sign. Thus the edge effect will not degenerate, if and only if the coefficients of the equation (1.10) form a series with alternating signs.

Another criterion for the absence of degeneracy may also turn out to be useful. Let $g_0, g_1, \cdots, g_n$ be the coefficients of the equation

$$\Delta_\alpha(r^2) = g_0 r^{2n} + g_1 r^{2(n-1)} + \cdots + g_{n-1} r^2 + g_n = 0.$$

It is readily shown that the edge effect corresponding to the wave number k_α will not degenerate, if and only if the expressions

$$g_0,\ g_1 - g_0k_\alpha^2,\ g_2 - (g_1 - g_0k_\alpha^2)k_\alpha^2,\ g_3 - [g_2 - (g_1 - g_0k_\alpha^2)k_\alpha^2]k_\alpha^2 \cdots$$

form an alternating sign sequence.

§ 2. We assume that for each boundary of the region $x_\alpha = 0$ and $x_\alpha = 1$ $(\alpha = 1, 2, \cdots, m)$ a solution of the type (1.5) has been constructed. Each of these solutions satisfies exactly the differential equation (1.1) and the conditions on one border; as x_α tends into the region, all of them approximate the generating solution (1.3). We will find the wave numbers $k_1, k_2, \cdots, k_m$ by joining the solutions, arising from two opposite borders of the region. We consider the two solutions

$$\begin{aligned} \Phi_\alpha(x_\alpha/0) &= C_{\alpha1}\sin k_\alpha x_\alpha + C_{\alpha2}\cos k_\alpha x_\alpha + \Psi_\alpha(x_\alpha/0), \\ \Phi_\alpha(x_\alpha/1) &= D_{\alpha1}\sin k_\alpha(1 - x_\alpha) + D_{\alpha2}\cos k_\alpha(1 - x_\alpha) + \Psi_\alpha(x_\alpha/1), \end{aligned} \tag{2.1}$$

the second of which satisfies all conditions on the boundary $x_\alpha = 1$. By analogy with (1.9)

$$\Psi_\alpha(x_\alpha/1) = \sum_{\beta=3}^{n+1} D_{\alpha\beta}\exp(-p_\beta + iq_\beta)(1 - x_\alpha).$$

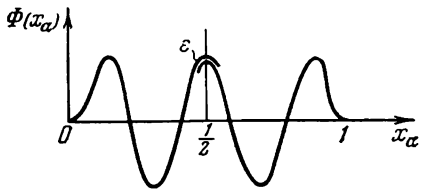

Fig. 2

The constants $C_{\alpha\beta}$ and $D_{\alpha\beta}$ are connected by the normalization condition

$$C_{\alpha1}^2 + C_{\alpha2}^2 = D_{\alpha1}^2 + D_{\alpha2}^2.$$

Since inside the internal region the functions $\Psi_\alpha(x_\alpha/0)$ and $\Psi_\alpha(x_\alpha/1)$ are small compared with the basic terms in the generating solution (Fig. 2), the condition for the joined expression may be written in the form

$$\begin{aligned} C_{\alpha1}\sin k_\alpha x_\alpha + C_{\alpha2}\cos k_\alpha x_\alpha = \\ = D_{\alpha1}\sin k_\alpha(1 - x_\alpha) + D_{\alpha2}\cos k_\alpha(1 - x_\alpha). \end{aligned} \tag{2.2}$$

Here, obviously, an error ε having the order

$$\varepsilon \sim \exp(-\tfrac{1}{2}\min p_\beta), \tag{2.3}$$

is admitted, where $(-\min p_\beta)$ is that negative real part among the roots of the equation $\Delta_\alpha^*(r^2) = 0$ which has the smallest absolute value. The estimate (2.3) is obtained under the assumption that the joining together

is executed at the middle of the region (for $x_\alpha = \frac{1}{2}$). If the generating solution satisfies the boundary conditions on one of the boundaries exactly, we obtain instead of (2.3) the error estimate

$$\varepsilon \sim \exp(-\min p_\beta). \tag{2.4}$$

We will consider in greater detail the condition for the combination (2.2). Taking into consideration (2.1), it may be written in the form

$$C_{\alpha 1} = -D_{\alpha 1} \cos k_\alpha + D_{\alpha 2} \sin k_\alpha. \tag{2.5}$$

For example, let the boundary conditions for $x_\alpha = 0$ and $x_\alpha = 1$ be the same. Then $C_{\alpha 1} = \pm D_{\alpha 1}$, where the positive sign corresponds to solutions symmetric about the point $x_\alpha = \frac{1}{2}$, the negative sign to antisymmetric solutions. Instead of (2.5) we obtain the two ratios

$$\cot \frac{k_\alpha}{2} = \frac{D_{\alpha 2}}{D_{\alpha 1}}, \quad \tan \frac{k_\alpha}{2} = -\frac{D_{\alpha 2}}{D_{\alpha 1}}. \tag{2.6}$$

They may be united in the following manner:

$$k_\alpha = 2 \arctan\left(-\frac{D_{\alpha 2}}{D_{\alpha 1}}\right) + s_\alpha \pi, \qquad (s_\alpha = 1, 2, \cdots). \tag{2.7}$$

Odd values of s_α correspond to symmetric, even values of s_α to antisymmetric solutions, where one must take for arctan its principal value.

The coefficients $C_{\alpha 1}$, $D_{\alpha 1}$, and $D_{\alpha 2}$ in the equations (2.5), (2.6) and (2.7) depend, generally speaking, on all the wave numbers $k_1, k_2, \cdots, k_m$. Combining the solutions for all x_α ($\alpha = 1, 2, \cdots, m$), we obtain m equations which are sufficient for finding the wave numbers. Subsequently the asymptotic expressions for the eigenvalues λ and the eigenfunctions φ are found from the formulae (1.3) and (1.4); the more accurate expressions for the eigenfunctions near the boundary (except in the neighborhood of corners) are found from formulae of the type (1.5). It may be anticipated, and computations confirm this (cf. § 3) that the accuracy in the determination of the eigenvalues has the same order as the error ε. This assumption still requires a proof.

§ 3. As an example we consider the natural vibrations of an elastic plate of constant thickness and square plan-form (Fig. 3). The equation of motion has the form

$$\Delta\Delta w + \frac{\gamma h a^4}{gD} \frac{\partial^2 w}{\partial t^2} = 0, \tag{3.1}$$

where $w(x_1, x_2, t)$ is the normal deflection, x_1 and x_2 are non-dimensional coordinates, t is the time, a the length of the sides of the plate, h its

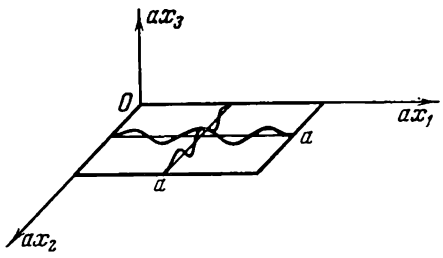

Fig. 3

thickness, D the flexural rigidity, γ the mass density of the material, g the gravitational acceleration. The substitution

$$w = \varphi(x_1, x_2) \exp(-i\omega t)$$

reduces the equation (3.1) to the form

$$\Delta\Delta\varphi - \frac{\gamma h a^4 \omega^2}{gD}\varphi = 0. \tag{3.2}$$

Obviously the equation (3.2) satisfies all the conditions, formulated in § 1. Representing the solution in the form (1.3), we obtain for the frequency ω a formula of the type (1.4)

$$\omega = \frac{k_1^2 + k_2^2}{a^2}\sqrt{\frac{gD}{\gamma h}}. \tag{3.3}$$

For the boundaries $x_1 = 0$ and $x_2 = 1$ we obtain the characteristic equation

$$\Delta_1(r^2) = r^4 - 2k_1^2 r^2 - 2k_1^2 k_2^2 - k_1^4 = 0,$$

whence

$$\Delta_1^*(r^2) = r^2 - k_1^2 - 2k_2^2.$$

A solution near the boundary $x_1 = 0$, which contains $(n + 1) = 3$ constants, will be

$$\Phi_1(x_1) = C_{11} \sin k_1 x_1 + C_{12} \cos k_1 x_1 + C_{13} \exp[-x_1(k_1^2 + 2k_2^2)^{\frac{1}{2}}]. \tag{3.4}$$

Thus, for plates, there always exists an edge effect which is a non-oscillatory effect.

We will assume that the plate is supported elastically along its entire contour with a coefficient of the stiffness of the support c. On the side $x_1 = 0$ one has to fulfill the conditions

$$\varphi(0, x_2) = 0, \quad \frac{\partial^2\varphi(0, x_2)}{\partial x_1^2} - \mu\frac{\partial^2\varphi(0, x_2)}{\partial x_2^2} = \frac{ca}{D}\frac{\partial\varphi(0, x_2)}{\partial x_1},$$

i.e.,

$$\Phi_1(0) = 0, \quad \Phi_1''(0) - 2c_0\Phi_1'(0) = 0,$$

where $c_0 = ca/2D$. Satisfying these conditions, we find that

$$\frac{C_{12}}{C_{11}} = -\frac{C_{13}}{C_{11}} = -\frac{c_0 k_1}{k_1^2 + k_2^2 + c_0(k_1^2 + 2k_2^2)^{\frac{1}{2}}}. \tag{3.5}$$

We obtain the analogous expression for the side $x_2 = 0$ by changing the subscripts.

We find the wave numbers k_1 and k_2 from the equations

$$k_1 = 2\arctan\frac{c_0 k_1}{k_1^2 + k_2^2 + c_0(k_1^2 + 2k_2^2)^{\frac{1}{2}}} + s_1\pi,$$

$$k_2 = 2\arctan\frac{c_0 k_2}{k_1^2 + k_2^2 + c_0(2k_1^2 + k_2^2)^{\frac{1}{2}}} + s_2\pi, \tag{3.6}$$

$$(s_1, s_2 = 1, 2, \cdots).$$

The equations (3.6) reduce to a single equation involving the ratio k_1/k_2. This equation is readily solved by a method of successive approximations. Figure 4 shows the natural frequencies, found by means of the solutions of the equations (3.6).

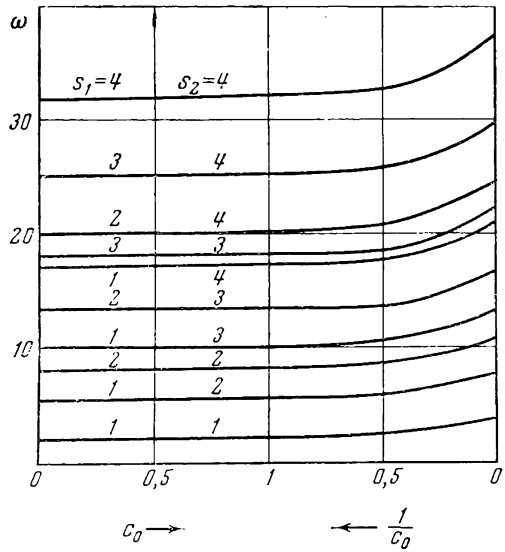

Fig. 4

Setting $c_0 \to 0$ in the equations (3.6), we arrive at the case of the hinged plate. In this case the values $k_1 = s_1\pi$, $k_2 = s_2\pi$ correspond to the exact solution. If $c_0 \to \infty$, we obtain the case of the rigidly clamped plate. It is interesting that for $k_1 = k_2$ the equations (3.6) admit the exact solution:

$$k_1 = k_2 = (s + \tfrac{1}{3})\pi.$$

Hence we obtain from the formula (3.3) for the natural frequency

$$\omega = \frac{2\pi^2(s + \frac{1}{3})^2}{a^2}\sqrt{\frac{gD}{\gamma h}}, \quad (s = 1, 2, \cdots). \tag{3.7}$$

We will give an estimate of the accuracy of (3.7) for $s = 1$. In this case we have the sufficiently reliable results of S. Iguchi [5], which, applying a variational method and taking into account six terms of a series, he obtained as the result of very lengthy calculations in the formula

$$\omega = \frac{\pi^2\bar{\omega}}{a^2}\sqrt{\frac{gD}{\gamma h}}$$

with the coefficient $\bar{\omega} = 3.646$. The formula (3.7) gives the value $\bar{\omega} = 32/9 = 3.556$. The discrepancy amounts to 2.53%; in this context one must take into consideration that the analysis of Iguchi, by its character, must give an over-estimate of the frequency coefficient.

By (2.3) the error in the combination of the solutions has the order

$$\varepsilon \sim \exp\left[-\tfrac{1}{2}(k_1^2 + 2k_2^2)^{\frac{1}{2}}\right] = \exp\left[-\frac{\pi\sqrt{3}}{2}(s + \tfrac{1}{3})\right]. \tag{3.8}$$

For $s = 1$ we find that $\varepsilon \sim 2.66\%$ in complete accordance with the results, obtained above. A more complete representation of the accuracy of the solutions obtained is given in Table I. In this table the last column gives the magnitude of the error ε, evaluated from the formula (3.8). One must have in mind for the comparison of the results that the values

TABLE 1

s_1	s_2	k_1/π	k_2/π	$\bar{\omega}$		ε,%	
				From (3.7)	Iguchi [5]	Comparison with Iguchi	From (3.8)
1	1	4/3	4/3	3.556	3.646	2.53	2.66
1	2	1.2027	2.4372	7.386	7.437	0.69	0.93
2	2	7/3	7/3	10.889	10.965	0.70	0.18
1	3	1.1420	3.4688	13.337	13.398	0.42	0.24
2	3	2.2556	3.4012	16.656	16.717	0.37	0.07
3	3	10/3	10/3	22.222	—	—	0.01
1	4	1.1084	4.4816	21.313	—	—	0.05
2	4	2.2038	4.4366	24.540	24.631	0.36	0.02
3	4	3.2784	4.3832	29.960	—	—	0.00
4	4	13/3	13/3	37.556	—	—	0.00

obtained by Iguchi are over-estimates and that the error involved in the method employed by him grows with increasing wave numbers.

We will assume that along the edges $x_1 = 0$ and $x_2 = 0$ the plate is clamped and that along $x_1 = 1$ and $x_2 = 1$ it is hinged (Fig. 5, case III). Then we obtain as before

$$k_1 = \arctan \frac{k_1}{(k_1^2 + 2k_2^2)^{\frac{1}{2}}} + s_1\pi,$$

$$k_2 = \arctan \frac{k_2}{(2k_1^2 + k_2^2)^{\frac{1}{2}}} + s_2\pi, \tag{3.9}$$

$$(s_1, s_2 = 1, 2, \cdots).$$

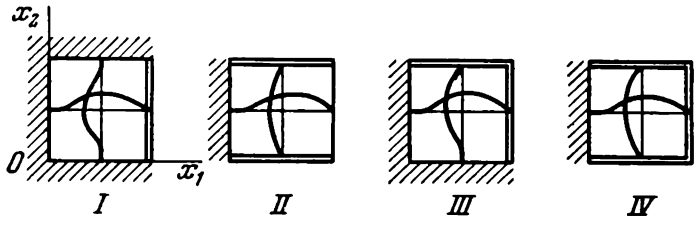

Fig. 5

For $k_1 = k_2$ we find the solution

$$k_1 = k_2 = (s + \tfrac{1}{6})\pi.$$

Consequently

$$\omega = \frac{2\pi^2(s + \frac{1}{6})^2}{a^2}\sqrt{\frac{gD}{\gamma h}}, \quad (s = 1, 2, \cdots). \tag{3.10}$$

The difference between the frequencies found from (3.10) for $s = 1$ and the frequency determined by a variational method amounts to 0.86%. The error estimate (2.4) gives in this case

$$\varepsilon \sim \exp\left[-(k_1^2 + 2k_2^2)^{\frac{1}{2}}\right] = \exp\left[-\pi\sqrt{3}(s + \tfrac{1}{6})\right],$$

i.e., $\varepsilon \sim 0.18\%$.

The values of the fundamental frequencies of vibration for four cases of support (Fig. 5), evaluated by the asymptotic method and the Ritz method, are given in Table 2.

We will now compare the results obtained with the asymptotic estimates belonging to R. Courant and H. Weyl [4]. These estimates are easily found, for example from (3.6) and (3.9), by taking into consideration that for large s_α

$$k_\alpha = s_\alpha\pi + O(1), \quad (\alpha = 1, 2).$$

TABLE 2

Form of support (cf. Fig. 5)	$\pi^2\bar{\omega}$		ε,%
	From (3.3)	Ritz method	
I	31.37	31.96	1.88
II	28.92	28.99	0.24
III	26.87	27.10	0.86
IV	23.64	23.70	0.25

Substituting these values in (3.3), we obtain for the non-dimensional frequency the estimate

$$\bar{\omega} = \pi^2(s_1^2 + s_2^2) + O(1),$$

from which the estimate of Courant and Weyl is readily obtained. The method studied here gives the estimate

$$\bar{\omega} = \pi^2(k_1^2 + k_2^2) + O(\varepsilon),$$

where k_1 and k_2 are to be determined from the equations (2.6) or (2.7), and the order of ε is shown by formulae of the types (2.3) and (2.4).

§ 4. As a second example we consider the problem of vibrations of circular cylindrical shells (Fig. 6). We will assume that the wave numbers are sufficiently large, so that one may use the well known variant of the equations for states with large indices of variation [1], and the tangential components of the inertia forces may be neglected. Let w be the normal deflection, ψ the stress function for the forces in the middle surface, R the radius of the middle surface of the shell, a a characteristic dimension and ω the frequency of vibration. The system of equations for small oscillations of the shell will have the form

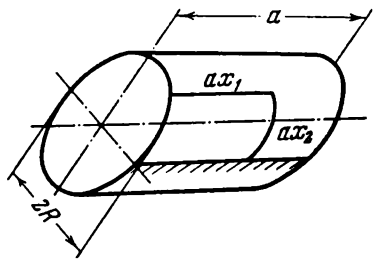

Fig. 6

$$D\Delta\Delta w - \frac{a^2}{R}\frac{\partial^2\psi}{\partial x^2} + \frac{\gamma h a^4}{g}\frac{\partial^2 w}{\partial t^2} = 0,$$

$$\frac{1}{Eh}\Delta\Delta\psi + \frac{a^2}{R}\frac{\partial^2 w}{\partial x_1^2} = 0.$$

By substitution of

$$w = \Delta\Delta\varphi \exp(-i\omega t), \quad \psi = \frac{-Eha^2}{R}\frac{\partial^2\varphi}{\partial x_1^2}\exp(-i\omega t),$$

where φ is some function of the coordinates x_1 and x_2, this system leads to the single equation

$$\Delta\Delta\Delta\Delta\varphi + c^4 \frac{\partial^2 \varphi}{\partial x_1^4} - \frac{\gamma h \omega^2 a^4}{gD} \Delta\Delta\varphi = 0, \tag{4.1}$$

where

$$c^4 = \frac{Eh}{D} \frac{a^4}{R^2} = \frac{12(1-\mu^2)a^4}{R^2 h^2}. \tag{4.2}$$

We obtain for the frequency ω the expression

$$\omega^2 = \frac{gD}{\gamma h a^4} \left[(k_1^2 + k_2^2)^2 + \frac{c^4 k_1^4}{(k_1^2 + k_2^2)^2} \right]. \tag{4.3}$$

The problem is to find asymptotic estimates for k_1 and k_2.

After fairly involved calculations one may obtain the following equations:

$$\Delta_1^*(r^2) = r^6 - (k_1^2 + 4k_2^2) r^4 + k_2^2 r^2 \left[2k_1^2 + 5k_2^2 + \right. \tag{4.4}$$

$$\left. + \frac{c^4(2k_1^2 + k_2^2)}{(k_1^2 + k_2^2)^2} \right] - k_1^4 \left[k_1^2 + 2k_2^2 + \frac{c^4 k_1^2}{(k_1^2 + k_2^2)^2} \right] = 0,$$

$$\Delta_2^*(r^2) = r^6 - (4k_1^2 + k_2^2) r^4 + k_1^2 r^2 \left[5k_1^2 + 2k_2^2 - \right. \tag{4.5}$$

$$\left. - \frac{c^4 k_1^2}{(k_1^2 + k_2^2)^2} \right] - k_1^4 \left[2k_1^2 + k_2^2 - \frac{c^4(2k_1^2 + k_2^2)}{(k_1^2 + k_2^2)^2} \right] = 0.$$

The equation (4.4) corresponds to the dynamic edge effect at the circular edge ($x_1 = 0$), the equation (4.5) to the edge effect at the straight edge ($x_2 = 0$).

Applying the criteria of § 1 for the absence of degeneration, we find that a dynamic edge effect will always be present at the circular edge. At the rectilinear edge, degeneration will take place, if

$$k_1^2 + k_2^2 < c^2, \tag{4.6}$$

i.e., for sufficiently small wave numbers. The thinner the shell, the smaller will be the wave numbers for which degeneration occurs. We recall that the static edge effect at the asymptotic edge always degenerates [1].

It is readily shown that the polynomial $\Delta_2^*(r^2)$ may be factored:

$$\Delta_2^*(r^2) = [r^2 - (2k_1^2 + k_2^2)] \cdot \left\{ r^4 - 2k_1^2 r^2 + k_1^4 \left[1 - \frac{c^4}{(k_1^2 + k_2^2)^2} \right] \right\}.$$

Hence the solution of the type (1.8) for the straight edge takes the form

$$\Phi_2(x_2) = C_{21} \sin k_2 x_2 + C_{22} \cos k_2 x_2 + C_{23} \exp\left[-x_2(2k_1^2 + k_2^2)^{\frac{1}{2}}\right] +$$
$$+ C_{24} \exp\left[-k_1 x_2 \left(1 + \frac{c^2}{k_1^2 + k_2^2}\right)^{\frac{1}{2}}\right] +$$
$$+ C_{25} \exp\left[-k_1 x_2 \left(1 - \frac{c^2}{k_1^2 + k_2^2}\right)^{\frac{1}{2}}\right].$$

By subjecting the functions $\Phi_1(x_1)$ and $\Phi_2(x_2)$ to the boundary conditions, we find the coefficients $C_{11}, C_{12}, \cdots, D_{22}$ after which, using the conditions for the combination (2.2), we obtain the equations for k_1 and k_2. In order to find the frequency ω, it is sufficient to substitute the evaluated wave numbers into the formula (4.3).

The method studied in this paper has been developed with reference to differential equations with constant coefficients. It may be extended to equations with variable coefficients, if the last vary sufficiently slowly in comparison with the corrective solutions. The generating solution will then, of course, have a form differing from (1.3).

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ON THE PROBLEM OF STRESS CONCENTRATIONS IN THE PRESENCE OF MANY HOLES

CHEN LIN-SI
Dairen, China

The school of the theory of elasticity established by N. I. Muskhelishvili, as the result of the application of the apparatus of the theory of functions of a complex variable, attained brilliant successes in the plane problems of the theory of elasticity. Using the method of integral equations, D. I. Sherman, G. N. Savin and others solved a number of important problems which are of great practical and theoretical interest.

The works of N. I. Muskhelishvili's school obtained wide acknowledgement among Chinese students. Interest in Muskhelishvili's method was even more intense and obtained wide dissemination after the Chinese translation of his book *Some basic problems of the mathematical theory of elasticity* in 1958 which became a handbook of Chinese students working in the field of the theory of elasticity.

Relying on Muskhelishvili's method, Chinese students completed a series of scientific investigations. One of these investigations, executed in the Dairen Polytechnic Institute, concerned the study of the stresses near overflow holes in a gravitation dam which was realized with the help of Muskhelishvili's method. This work was organized for the Institute by the group under the direction of Tan Li-ming and it was the result of a cooporative effort involving Chou-Chen-ti, Sung Khuang-chung, and also a group of students specializing in the field of applied mechanics.

§ 1. The first fundamental problem of the plane theory of elasticity in a multiply connected region may be reduced to the search for two functions $\varphi(x)$ and $\psi(x)$, holomorphic in this region including the point at infinity and satisfying on each of its boundaries the equation

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = f(t), \tag{1.1}$$

where t is the coordinate of points of the contour and $f(t)$ is the given boundary function which depends on the applied external forces.

Let the unknown functions $\varphi(z)$ and $\psi(z)$ have the form

$$\begin{aligned} \varphi(z) &= \sum_{s=1}^{n} \sum_{k=1}^{\infty} a_k^s (z - c_s)^{-k}, \\ \psi(z) &= \sum_{s=1}^{n} \sum_{k=1}^{\infty} b_k^s (z - c_s)^{-k}, \end{aligned} \tag{1.2}$$

where n is the number of closed curves bounding the region and c_s is some point inside the corresponding curve.

We first shift the origin of coordinates to some point c_j and execute the transformation $z = z_j + c_j$. Then we have for the above analytic functions in the new coordinate system

$$\begin{aligned}\varphi(z_j) &= \sum_{k=1}^{\infty} \overset{j}{a}_k z_j^{-k} + \sum_{s=1}^{n}{}' \sum_{k=1}^{\infty} \overset{s}{a}_k (z_j - \overset{s}{d}_j)^{-k},\\ \psi(z_j) &= \sum_{k=1}^{\infty} \overset{j}{b}_k z_j^{-k} + \sum_{s=1}^{n}{}' \sum_{k=1}^{\infty} \overset{s}{b}_k (z_j - \overset{s}{d}_j)^{-k},\end{aligned} \tag{1.3}$$

where the stroke attached to the external sum in the second term indicates the omission of the term corresponding to $s = j$ and $\overset{s}{d}_j = c_s - c_j$ is the distance of the new origin of coordinates from the point c_s. The boundary condition (1.1) then assumes on L_j the form

$$F(t_j) = \varphi(t_j) + (t_j + c_j)\overline{\varphi'(t_j)} + \overline{\psi(t_j)} - f(t_j) = 0, \tag{1.4'}$$

where t_j is the value of z_j on L_j.

Proceeding to the conjugate values we obtain

$$\overline{F(t_j)} = \overline{\varphi(t_j)} + (\overline{t_j} + \overline{c_j})\varphi'(t_j) + \psi(t_j) - \overline{f(t_j)} = 0, \tag{1.4''}$$

$$j = 1, 2, \cdots, n.$$

On the basis of Harnack's theorem ([1], p. 354) the boundary condition (1.4) may be replaced by the integral relations

$$\begin{aligned}\frac{1}{2\pi i}\int_{L_j} \frac{F(t_j)}{t_j - z}\,dt = 0,\\ \frac{1}{2\pi i}\int_{L_j} \frac{\overline{F(t_j)}}{t_j - z}\,dt = 0,\end{aligned} \tag{1.5}$$

where the point z lies outside the contour L_j.

Expanding $1/(t_j - z)$ in the series

$$\frac{1}{t_j - z} = -\left(\frac{1}{z} + \frac{t_j}{z^2} + \frac{t_j^2}{z^3} + \cdots\right) = -\sum_{p=0}^{\infty} \frac{t_j^p}{z^{p+1}} \tag{1.6}$$

and substituting this expansion in (1.5) we obtain

$$\begin{gathered}\sum_{p=0}^{\infty} \frac{1}{2\pi i} \int_{L_j} \frac{t_j^p F(t_j)}{z^{p+1}}\, dt = 0, \\ \sum_{p=0}^{\infty} \frac{1}{2\pi i} \int_{L_j} \frac{t_j^p \overline{F(t_j)}}{z^{p+1}}\, dt = 0, \\ j = 1, 2, \cdots, n.\end{gathered} \tag{1.7}$$

Hence there follows

$$\begin{gathered}\frac{1}{2\pi i} \int_{L_j} t_j^p F(t_j)\, dt = 0, \\ \frac{1}{2\pi i} \int_{L_j} t_j^p \overline{F(t_j)}\, dt = 0,\end{gathered} \tag{1.8}$$

$$p = 0, 1, 2, \cdots, \infty, \quad j = 1, 2, \cdots, \infty.$$

Substituting (1.3) and (1.4) in (1.8) and integrating, we obtain two infinite systems of algebraic equations for the determination of the coefficients $\overset{j}{a}_k$ and $\overset{j}{b}_k$ ($k = 1, 2, \cdots, \infty$, $j = 1, 2, \cdots, n$). In the case of circular holes, if one selects c_j at the center of the circles, the system (1.8) converges. With the help of this method a number of problems were solved which were concerned with the stress concentrations near certain circular holes weakening a given medium [2, 3, 4]. For the problem of stress concentrations in the infinite plane weakened by infinite rows of equal circles distributed along orthogonal and rectangular nets, the law of change of the stress concentration factors was obtained in dependence on the distance between the circles in [5]. This result has definite practical value in such problems as that of the design of foundations.

§ 2. In the case of non-circular holes, the expansion of the functions $\varphi(z_j)$ and $\psi(z_j)$ in series of the form (1.3) leads to a divergent process. In order to avoid this we apply expansions of the form

$$\begin{gathered}\varphi(z_j) = \sum_{k=1}^{\infty} \overset{j}{\alpha}_k [f_j(z_j)]^{-k} + \sum_{s=1}^{n}{}' \sum_{k=1}^{\infty} \overset{s}{\alpha}_k [f_s(z_j - \overset{s}{d}_j)]^{-k}, \\ \psi(z_j) = \sum_{k=1}^{\infty} \overset{j}{\beta}_k [f_j(z_j)]^{-k} + \sum_{s=1}^{n}{}' \sum_{k=1}^{\infty} \overset{s}{\beta}_k [f_s(z_j - \overset{s}{d}_j)]^{-k}, \\ j = 1, 2, \cdots, n,\end{gathered} \tag{2.1}$$

where the function

$$f_j(z_j) = \zeta_j = R_j\left(z_j + \frac{\overset{j}{m}_1}{z_j} + \frac{\overset{j}{m}_2}{z_j^2} + \cdots\right) \tag{2.2}$$

maps the outside of the unit circle in the ζ_j plane conformally onto the outside of the contour L_j in the z_j plane. For this process the determination of the $\overset{j}{\alpha}_k$ and $\overset{j}{\beta}_k$ is usually found to be a convergent process. The functions $f_j(z_j)$ are the inverses of the functions of Muskhelishvili for the same basic region

$$\omega(\zeta) = z = R'\left(\zeta + \frac{m_1'}{\zeta} + \frac{m_2'}{\zeta^2} + \cdots\right). \tag{2.3}$$

We substitute (2.1) into the boundary condition (1.4), multiply afterwards by $(1/2\pi i)[f_j(z_j)]^p df_j(z_j)$ and then integrate along the contour L_j; this operation is equivalent to the application of Harnack's theorem. Thus, we obtain a sufficient number of systems of linear algebraic equations for the determination of the $\overset{j}{\alpha}_k$ and $\overset{j}{\beta}_k$.

By the help of this method we solved the problem of the strip of infinite length with one circular hole. Attention must be drawn to the fact that one has the following relations between the coefficients $\overset{j}{a}_k$ and $\overset{j}{b}_k$ of the functions (1.3) and the coefficients $\overset{j}{\alpha}_k$ and $\overset{j}{\beta}_k$ of the functions (2.1):

$$\alpha = TA, \quad \beta = TB, \tag{2.4}$$

where α, β and A, B are column matrices with the elements $\overset{j}{\alpha}_k$, $\overset{j}{\beta}_k$ and $\overset{j}{a}_k$, $\overset{j}{b}_k$ and T is an infinite square matrix determined by (2.3). Therefore, using the functions (1.3), one may determine A and B, and then evaluate α and β from the formulae (2.4). Using this method we solved the problem of the stress concentrations near a quadratic opening in the infinite region. Recently, we also completed the work on the infinite plane with two square holes and, finally, on the region bounded by three orthogonally intersecting contours and weakened by one circular hole.

§ 3. The problem of the stress concentrations in an infinite space with a straight circular funnel of constant cross-section in the case of a homogeneous stress field orientated at some angle to its axis may (with due consideration of the independence of the states of stress and strain of the coordinate along the axis of the funnel) be studied by decomposing

it simply into two two-dimensional problems, namely, a problem of plane strain and a Neumann problem for harmonic functions. One can obtain very simple solutions of these problems with the help of the theory of conformal mapping [8].

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CONJUGATE PROBLEMS OF THE THEORY OF THIN SHELLS

K. F. CHERNYKH
Leningrad, U.S.S.R.

The successes in the solution of problems in the plane theory of elasticity (attained, to a large extent, by the efforts of our scientists in the field of mechanics) have shown appreciable influence on the development of the theory of thin shells. This influence manifests itself also in problems which are not directly connected with the application of functions of a complex variable. In particular, the author's papers [4, 5] dealt with problems of displacement dislocations, multi-valued stress functions connected with non-self-equilibrating edge loadings, of deformations of the edges of shells, etc. In essence, in relation to thin shell problems, these references used results which were set forth in the monograph of N. I. Muskhelishvili [1].

Recently, in the plane theory of elasticity, a great deal of attention has been given to the method of conjugate problems [3] which, on the basis of a known solution of some problem, permits construction of solutions of problems corresponding to certain new boundary conditions by elementary means. The present paper shows that the method of conjugate problems may also be applied in the theory of thin shells. In the study of this problem use will be made of relations of shell theory in complex form, due to V. V. Novozhilov. On the whole, the notation to be used will coincide with that in his book [2].

§ 1. We refer the middle surface of the shell to the lines of curvature α_1, α_2 and denote by A_1, A_2, R_1, R_2 the corresponding Lamé coefficients and principal radii of curvature. For the homogeneous problem (to the study of which we will limit ourselves) the state of stress of the shell is characterized by the specification of the three complex forces [2]

$$\begin{aligned} \tilde{T}_1 &= T_1 - iEhc\kappa_2 = -iEhc\tilde{\kappa}_2, \\ \tilde{T}_2 &= T_2 - iEhc\kappa_1 = -iEhc\tilde{\kappa}_1, \\ \tilde{S} &= S + iEhc\tau = iEhc\tilde{\tau}. \end{aligned} \tag{1.1}$$

Here E is Young's modulus, h is the thickness of the shell, μ is Pois-

son's ratio,

$$c = \frac{h}{\sqrt{12(1-\mu^2)}},$$

$$\kappa_1 = -\frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_1}\frac{\partial w}{\partial\alpha_1} - \frac{u}{R_1}\right) - \frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial w}{\partial\alpha_2} - \frac{v}{R_2}\right),$$

$$\kappa_2 = -\frac{1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial w}{\partial\alpha_2} - \frac{v}{R_2}\right) - \frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\left(\frac{1}{A_1}\frac{\partial w}{\partial\alpha_1} - \frac{u}{R_1}\right),$$

$$\tau = -\frac{1}{A_1A_2}\left(\frac{\partial^2 w}{\partial\alpha_1\partial\alpha_2} - \frac{1}{A_1}\frac{\partial A_1}{\partial\alpha_2}\frac{\partial w}{\partial\alpha_1} - \frac{1}{A_2}\frac{\partial A_2}{\partial\alpha_1}\frac{\partial w}{\partial\alpha_2}\right) +$$

$$+\frac{1}{R_1}\left(\frac{1}{A_2}\frac{\partial u}{\partial\alpha_2} - \frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}u\right) + \frac{1}{R_2}\left(\frac{1}{A_1}\frac{\partial v}{\partial\alpha_1} - \frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}v\right),$$

$\tilde{\kappa}_1$, $\tilde{\kappa}_2$, $\tilde{\tau}$ are the components of the bending deformation in terms of the complex displacements

$$\tilde{u} = u + i\bar{u}, \quad \tilde{v} = v + i\bar{v}, \quad \tilde{w} = w + i\bar{w}, \tag{1.2}$$

the real parts of which are displacements, while the imaginary parts are the corresponding stress functions.

The complex forces correspond to a solution of some problem of shell theory, if and only if the complex displacements satisfy the system of differential equations

$$\begin{aligned} \tilde{\varepsilon}_1 + \mu\tilde{\varepsilon}_2 + ic(1-\mu^2)\tilde{\kappa}_2 &= 0, \\ \tilde{\varepsilon}_2 + \mu\tilde{\varepsilon}_1 + ic(1-\mu^2)\tilde{\kappa}_1 &= 0, \\ \tilde{\omega} - ic2(1+\mu)\tilde{\tau} &= 0. \end{aligned} \tag{1.3}$$

Here $\tilde{\varepsilon}_1$, $\tilde{\varepsilon}_2$, $\tilde{\omega}$ are the components of the tangential deformation

$$\varepsilon_1 = \frac{1}{A_1}\frac{\partial u}{\partial\alpha_1} + \frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}v + \frac{w}{R_1},$$

$$\varepsilon_2 = \frac{1}{A_2}\frac{\partial v}{\partial\alpha_2} + \frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}u + \frac{w}{R_2},$$

$$\omega = \frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{u}{A_2}\right) + \frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{v}{A_1}\right)$$

in terms of the complex displacements (1.2).

The forces and moments are expressed in terms of the complex relations

$$T_1 = \operatorname{Re} \tilde{T}_1, \quad T_2 = \operatorname{Re} \tilde{T}_2, \quad S = \operatorname{Re} \tilde{S},$$

$$M_1 = -c \operatorname{Im}\{\tilde{T}_2 + \mu \tilde{T}_1\}, \quad M_2 = -c \operatorname{Im}\{\tilde{T}_1 + \mu \tilde{T}_2\}, \tag{1.4}$$

$$H = (1 - \mu) c \operatorname{Im} \tilde{S}.$$

We will introduce, in addition, complex boundary quantities, related to the boundary contour of the middle surface Γ (Fig. 1),

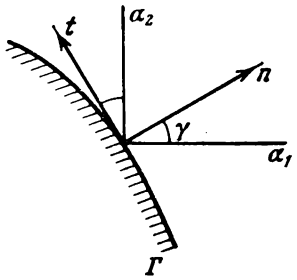

Fig. 1

$$\tilde{T}_n = \tilde{T}_{nn}\mathbf{n} + \tilde{T}_{nt}\mathbf{t} + \tilde{T}_{nm}\mathbf{m}, \quad \tilde{M}_{nn},$$

where

$$\tilde{T}_{nn} = T_{nn} - iEhc\kappa_{tt} = -iEhc\tilde{\kappa}_{tt},$$

$$\tilde{T}_{nt} = T_{nt} + iEhc\kappa_{tn} = iEhc\tilde{\kappa}_{tn},$$

$$\tilde{T}_{nm} = T_{nm} - iEhc\kappa_{tm} = -iEhc\tilde{\kappa}_{tm},$$

$$\tilde{M}_{nn} = M_{nn} + iEhc\varepsilon_{tt} = iEhc\tilde{\varepsilon}_{tt}.$$

Here $\mathbf{m}$ is the unit normal to the middle surface, while $\mathbf{n}$ and $\mathbf{t}$ are unit vectors in the directions of the normal and the tangent to the contour Γ respectively; T_{nn}, T_{nt}, T_{nm} are the components of the force vector, applied to the edge of the shell (and including the components of the distributed twisting moment M_{nt}); M_{nn} is the bending moment at the edge of the shell; κ_{tt}, κ_{tn}, κ_{tm}, ε_{tt} are quantities deduced in [4] which completely characterize the deformation of the edge of the shell. They are expressed in terms of the strain components by the relations

$$\varepsilon_{tt} = \cos^2\gamma \cdot \varepsilon_2 - \sin\gamma \cos\gamma \cdot \omega + \sin^2\gamma \cdot \varepsilon_1,$$

$$\kappa_{tt} = \cos^2\gamma \cdot \kappa_2 - 2\sin\gamma\cos\gamma \cdot \tau + \sin^2\gamma \cdot \kappa_1 + \sin\gamma\cos\gamma\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \cdot \frac{\omega}{2},$$

$$\kappa_{tn} = \sin\gamma\cos\gamma \cdot (\kappa_2 - \kappa_1) + (\cos^2\gamma - \sin^2\gamma) \cdot \tau + \left(\frac{\sin^2\gamma}{R_1} - \frac{\cos^2\gamma}{R_2}\right) \cdot \frac{\omega}{2} -$$

$$- \left(\frac{\sin^2\gamma}{R_1} + \frac{\cos^2\gamma}{R_2}\right)\left[\sin\gamma\cos\gamma \cdot (\varepsilon_2 - \varepsilon_1) + (\cos^2\gamma - \sin^2\gamma) \cdot \frac{\omega}{2}\right],$$

$$\kappa_{tm} = -\left\{\frac{\sin\gamma}{A_1}\frac{\partial}{\partial\alpha_1} + \frac{\cos\gamma}{A_2}\frac{\partial}{\partial\alpha_2}\right\}\cdot$$

$$\cdot\left[\sin\gamma\cos\gamma\,.\,(\varepsilon_2 - \varepsilon_1) + (\cos^2\gamma - \sin^2\gamma)\,.\,\frac{\omega}{2}\right] -$$

$$-\frac{\sin\gamma}{A_1A_2}\left\{\frac{\partial A_1\,.\,\varepsilon_1}{\partial\alpha_2} - \frac{1}{A_2}\frac{\partial}{\partial\alpha_1}\left(A_2^2\,.\,\frac{\omega}{2}\right) - \frac{\partial A_1}{\partial\alpha_2}\,.\,\varepsilon_2\right\} -$$

$$-\frac{\cos\gamma}{A_1A_2}\left\{\frac{\partial A_2\,.\,\varepsilon_2}{\partial\alpha_1} - \frac{1}{A_1}\frac{\partial}{\partial\alpha_2}\left(A_1^2\,.\,\frac{\omega}{2}\right) - \frac{\partial A_2}{\partial\alpha_1}\varepsilon_1\right\}.$$

§ 2. Side by side with a basic, initial solution defined by the complex displacement $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ we will consider a conjugate solution for which

$$\begin{aligned}\tilde{u}^\wedge &= i\beta\tilde{u} = i\beta(u + i\bar{u}),\\ \tilde{v}^\wedge &= i\beta\tilde{v} = i\beta(v + i\bar{v}),\\ \tilde{w}^\wedge &= i\beta\tilde{w} = i\beta(w + i\bar{w}),\end{aligned} \tag{2.1}$$

where β is an arbitrary real constant.

Since the quantities introduced differ from the original complex displacements by a constant multiplier only, they will satisfy the equation (1.3). Hence a solution constructed with their help will solve some homogeneous problem of shell theory.

By separating the real parts in (2.1) we obtain the expressions for the displacements of the conjugate problem

$$u^\wedge = -\bar{u}, \quad v^\wedge = -\bar{v}, \quad w^\wedge = -\bar{w}.$$

Further, substituting the quantities (2.1) into the equalities (1.1) and (1.4), we arrive at the following expressions for the forces and moments of the conjugate problem:

$$T_1^\wedge = \beta\frac{M_2 - \mu M_1}{(1-\mu^2)c}, \quad T_2^\wedge = \beta\frac{M_1 - \mu M_2}{(1-\mu^2)c},$$

$$S^\wedge = -\beta\frac{H}{(1-\mu)c},$$

$$M_1^\wedge = -\beta c(T_2 + \mu T_1), \quad M_2^\wedge = -\beta c(T_1 + \mu T_2),$$

$$H^\wedge = \beta c(1-\mu)S.$$

Using the relations (1.5), we obtain the relationships between the

boundary quantities of the basic and the conjugate problems:

$$T^{\wedge}_{nn} = \beta Ehc\kappa_{tt}, \qquad \kappa^{\wedge}_{tt} = -\beta\frac{T_{nn}}{Ehc},$$

$$T^{\wedge}_{nt} = -\beta Ehc\kappa_{tn}, \quad \kappa^{\wedge}_{tn} = \beta\frac{T_{nt}}{Ehc},$$

$$T^{\wedge}_{nm} = -\beta Ehc\kappa_{tm}, \quad \kappa^{\wedge}_{tm} = -\beta\frac{T_{nm}}{Ehc},$$

$$M^{\wedge}_{nn} = -\beta Ehc\varepsilon_{tt}, \qquad \varepsilon^{\wedge}_{tt} = \beta\frac{M_{nn}}{Ehc}.$$

Therefore it is seen that the static boundary quantities of the basic problem correspond to the deformations of the conjugate problem and vice versa.

As a non-trivial example we will consider the application of the method of conjugate problems to the solution of R. A. Clark's problem [6] of the

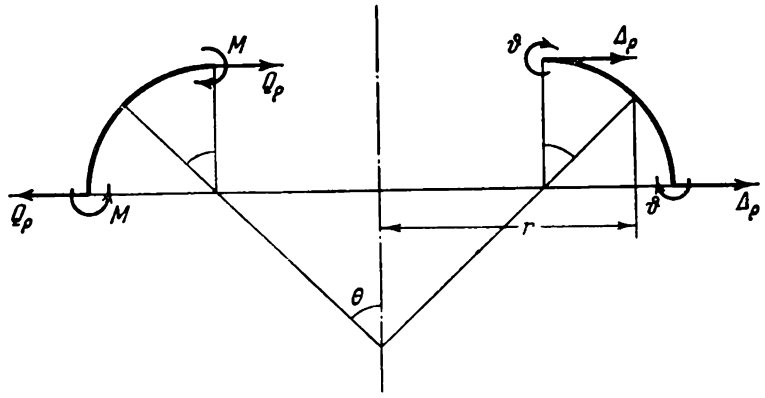

Fig. 2

symmetric bending of a quarter toroidal shell (Fig. 2), bent by moments applied to its edges. The boundary conditions of this problem have the form

$$Q_\rho = 0, \quad M_1 = M_0 \quad \text{for } \theta = 0,$$

$$Q_\rho = M_1 = 0 \quad \text{for } \theta = \pi/2.$$

Using the above method it is not difficult to establish that the forces

and moments of the conjugate problem

$$T_1^{\wedge} = \beta \frac{M_2 - \mu M_2}{(1-\mu^2)c}, \quad M_1^{\wedge} = -\beta c(T_2 + \mu T_1),$$

$$T_2^{\wedge} = \beta \frac{M_1 - \mu M_2}{(1-\mu^2)c}, \quad M_2^{\wedge} = -\beta c(T_1 + \mu T_2)$$

solve the homogeneous problem with the boundary conditions

$$\vartheta^{\wedge} = 0, \quad \Delta_\rho^{\wedge} = \beta r \frac{M}{Ehc} \quad \text{for } \theta = 0,$$

$$\vartheta^{\wedge} = \Delta_\rho^{\wedge} = 0 \qquad \text{for } \theta = \pi/2.$$

The present notation is explained in Fig. 2.

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THE INTERPRETATION OF PLASTIC DEFORMATIONS AS HEREDITY PHENOMENA

GUSTAVO COLONNETTI
Torino, Italy

§ 1. For a long time it has been known that the irreversible phenomena called hysteresis, plasticity, flow, relaxation and creep appear in natural bodies when the latter are deformed by the action of loads which surpass certain limits, but no all-embracing mathematical theory has ever been conceived.

It has only become possible now, as a consequence of the recent development of a general theory of infinitesimal deformations, that such a theory can take its place at the side of the classical theory of elasticity.

In this theory one must abandon the concept of the bi-univocal correspondence between stresses and strains on which the classical theory rests. One must admit that the deformation at any moment is no longer only a function of the present values of the stresses, but also depends on their values during the past. And then one does not only arrive at a rational interpretation of the phenomenon of hysteresis which the theory of elasticity would never have been able to explain, but one can also see the reason for those variations of the elastic constants which the classical theory would never be able to admit without repudiating itself, variations that show some real changes in the state of the material. The fact of assuming that a deformation at any moment is a function not only of the present values of the stresses, but also of those which have preceded them in time, means to place the phenomena under consideration among hereditary phenomena. It has thus become necessary to resort to the mathematical theory of heredity.

This theory arose during the early years of this century, although at that time not everyone was able to agree as to its applicability to problems of mathematical physics. In the chapter of his work "De la méthode dans les sciences", devoted to mechanics, M. Painlévé asserted that there were no real problems of a hereditary nature. Those that were presented as such were only, in his opinion, problems destined to vanish when our knowledge of bodies and of their constitution should become more complete.

But, at the present state of our scientific knowledge, these problems actually do arise and with an ever-increasing frequency. It is therefore necessary to solve them. Already, in 1907 E. Picard, in an article "Sur la

mécanique classique et ses approximations successives", asserted the need to distinguish between two branches of mechanics, that of heredity and that of non-heredity. The last concerns the case where the future of a system only depends on its actual condition at a given instant or, in a more general manner (if one regards the forces as being also able to depend on the speeds), on the present state and on the infinitely adjoining state that has preceded it. Heredity mechanics, on the contrary, relates to the case where each action leaves a heritage in the system and the actual state depends on the entire preceding history. However, this distinction has not been able to lead to a systematic study due to the lack of suitable analytic methods.

We owe to the great Italian mathematician Vito Volterra the idea of placing physical questions which relate to the problems of heredity alongside certain classes of integro-differential equations for which, in 1887, he had obtained a series development analogous to Taylor series. In a very remarkable memoir, published in "Acta Mathematica" in 1910, Volterra proved that, by their nature, problems of mathematical physics and of non-hereditary mechanics can always be made to depend on ordinary differential equations or on equations with partial derivatives. The initial conditions constitute the arbitrary constants or the arbitrary functions that appear in the integration.

For the problems of mathematical physics of heredity, on the contrary, the analysis of differential equations is not enough. In fact, the actual state of a system depends on its history, and the latter is characterized by all the values taken by the parameters during a certain period of time; hence it is necessary to envisage quantities which depend on all the values of these parameters regarded as functions of time.

One is thus brought to the analysis of quantities which depend on all the values of one or several functions — functions of lines or of hyperspaces — and Volterra has shown in his memoir that thanks to the developments in a series analogous to that of Taylor, the question is presented as a completely natural extension of the solution of systems of algebraic equations of the first degree, where the number of equations and of unknowns increases indefinitely.

It thus happened that Volterra was able to deduce, by the study of integro-differential equations of the elliptic type, the general lines of an analytic theory of hereditary phenomena without particularizing the functions that characterize them.

In fact, as in ordinary problems of mathematical physics, it is useful to leave constants undetermined whenever it is possible, and only to fix them numerically when one applies the formulae to concrete problems;

similarly, it is useful to leave undetermined the functions which characterize the hereditary phenomena when one deals with heredity questions in general and to solve the problems under consideration with the greatest possible generality. One is always able to determine the functions when they are unknown by comparing the general solutions obtained with the results of direct observation.

In order to present the magnitudes which depend on all the values that a function α of time is able to take in an interval of time $(0, T)$, Volterra has chosen a function that he writes generically in the form

$$F|[\alpha_0^T(t)]|.$$

We shall adopt it to present any one of the characteristics of deformation at a given instant T, admitting that with it (certain conditions being satisfied) this function can be developed in a series analogous to a Taylor series, and that in this series terms of higher order than the first are negligible. This is what Volterra called *linear heredity.*

If one chooses as the origin of time the instant of the first application of external forces and if one assumes that at this moment each element of the body is in its natural undeformed state, any one of the characteristics of deformation at the instant T can be expressed as the sum of two terms:

the *elastic deformation* which we will denote here generically by the letter ε and which is always, by definition, a linear function of the state of stress,

and the *plastic deformation* which we will always denote by the same greek letter, but with a bar on top, in the expression for which

$$\bar{\varepsilon}(T) = \int_0^T \alpha(t) f(T, t)\, dt$$

the function f represents what Volterra has called, in a general manner, the *coefficient of heredity.* It is easy to give it a physical interpretation. In fact, this coefficient is at each instant equal to the ratio of the rate of increase of the plastic deformation to the value of the function α.

This is what has suggested to us to give the function α the name of *determining function* of the plastic deformations.

Now it is a question of how to formulate some reasonable hypotheses for the form of the two functions α and f.

We will assume α to be linear in time, that is to say

$$\alpha = ct,$$

where c is an arbitrary constant. This is an hypothesis that at first sight

might appear very restrictive; the rest of our discussion will show that it will not prevent us from envisaging and solving a great number of problems which are among the most interesting in the vast domain of technical applications.

As regards the coefficient of heredity, we shall assume that it is related to the determining function — and by that, to time — by a relation of the type

$$f = a_1 + a_2\alpha + a_3\alpha^2 + \cdots,$$

where the coefficients α are constants the values of which can vary with the characteristics of the material and with the rate of deformation imposed on it.

In these hypotheses, the plastic deformation at the instant T can be written in the following manner:

$$\bar{\varepsilon}(T) = \int_0^T ct(a_1 + a_2ct + a_3c^2t^2 + \cdots)\,dt =$$

$$= a_1c\frac{T^2}{2} + a_2c^2\frac{T^3}{3} + a_3c^3\frac{T^4}{4} + \cdots =$$

$$= \frac{a_1}{2c}\alpha^2(T) + \frac{a_2}{3c}\alpha^3(T) + \frac{a_3}{4c}\alpha^4(T) + \cdots.$$

It goes without saying that the hypotheses which we have adopted and from which we have deduced these expressions for the plastic deformation are entirely arbitrary. Their choice will be justified subsequently, if the physical theory which arises from it offers us the possibility of coordinating between them phenomena which so far have presented themselves as isolated and devoid of links to other known phenomena.

It is necessary to state immediately that two cases present themselves in practice which call for different choices of the determining function.

As a matter of fact, if the intervention of the plastic deformations manifests itself under the action of external forces applied to the body and of internal stresses which arise at different points of the body, one must choose as determining function one of the characteristics of elastic deformation. In contrast, when one has to study a state of coaction of imposed deformations, one must choose as determining function one of the characteristics of the total deformation.

As regards the heredity coefficient, one has to make a distinction depending on whether the terms of order higher than the first are or are not negligible compared to the first order term.

In the first case, i.e., when the heredity coefficient is, or may be considered to be constant, the envisaged deformations will be simple changes of shape and of dimensions. In contrast, the case of the variable coefficient allows us to envisage phenomena where the change of shape and of dimensions is accompanied by changes in the physical state of the body.

This distinction is of fundamental importance. It is, to my knowledge, the first time that one has the possibility of analytically characterizing two classes of phenomena by a different form of one function (the heredity coefficient): those which, as in the hysteresis with closed cycles, allows a return of the material to the initial state, and those which, as in relaxation or strain-hardening, are definitely irreversible.

It is obvious that the study of phenomena of the first class is necessarily subordinated to the choice of an initial state which is well defined and easy to reproduce. Such is the neutral state, which can be obtained by the well known method of alternating decreasing stresses. In speaking of such a state, the law of variation of the plastic deformations, for stresses and then for elastic deformations increasing linearly with time, will be represented with a very good approximation by a parabola, tangent at the origin of the axes to the straight line which graphically illustrates Hooke's Law.

The equation of this parabola is deduced from the last of the general equations of which we have come to speak by choosing as determining function the elastic deformation and by neglecting, in the series development, the terms of order higher than the first. One can then write them in the form

$$\bar{\varepsilon} = k\varepsilon^2, \quad k = \frac{a_1}{2c}.$$

Analogously, after inversion of the direction of variation of the stress — and, most generally, all the times when the stress varies periodically with constant speed between well determined limits — the law of variation of the plastic deformations will be represented by other parabolas of the type

$$\bar{\varepsilon}' = k'\varepsilon'^2,$$

where the condition of the closed cycle demands that

$$k' = \tfrac{1}{2}k.$$

In contrast, the most general expression of the heredity coefficient must be used in the study of problems of the second class, i.e., for irreversible phenomena. The law of variation of the plastic deformations can then be

expressed by an equation of the type

$$\bar{\varepsilon} = k_1\varepsilon^2 + k_2\varepsilon^3 + k_3\varepsilon^4 + \cdots.$$

Two cases can present themselves according to whether all the terms of higher order than the first are positive or negative.

In the first case the curve representing the phenomena, after having followed closely for a certain time the parabola of which we have spoken, directs itself more or less rapidly towards the axis of the deformations.

This turn illustrates very well what the technicians call relaxation of the material, i.e., the progressive elimination of resistance — or if you prefer it, the greater deformability which certain materials exhibit when they are plastically deformed.

If, on the contrary, the set of terms of higher order is negative, the curve representing the phenomena moves from the parabola on the opposite side, and points towards the axis of the forces (or of the time). This is the case of strain-hardening or increase of resistance — in the sense of less deformability — which occurs in certain bodies when the plastic deformations exceed a certain limit.

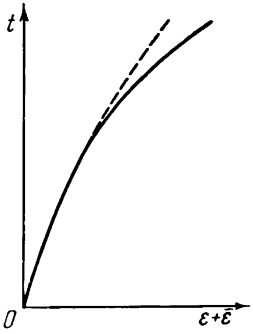

Fig. 1.

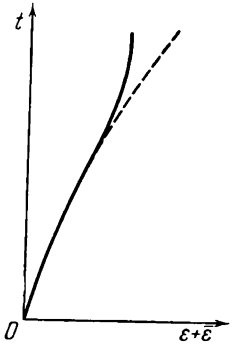

Fig. 2.

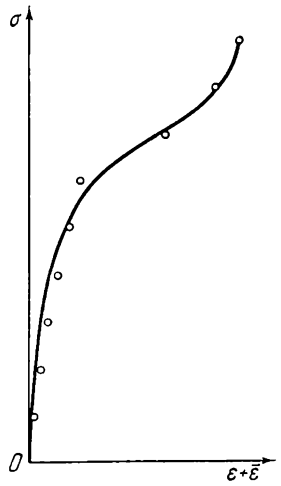

Fig. 3.

Here is an example where the correspondence between the theoretical deductions and the results of experiments is particularly significant.

A steel wire of 1.9 mm. in diameter was submitted to tension increasing in time at the rate of 2 kg. per second. The deformations were measured over a length of 66 mm. by an Amsler strain gauge with an accuracy of one hundredth of a millimetre. The results of these measurements are

shown in Fig. 3 by small circles, and interpreted by the curve that corresponds to the equation

$$\bar{\varepsilon} = k_2\varepsilon^3 + k_4\varepsilon^5 + k_6\varepsilon^7$$

the coefficients of which were determined in such a manner as to minimize the difference between the function and the results of the experiment. It is necessary to say that this operation is rather delicate; it was performed well by the FINAC machine at the National Institute for Applications of Calculus at Rome, with which one can obtain a precision up to eighteen decimals. But the result is so satisfactory that there can be no reasonable doubt regarding the validity of our interpretation of the phenomenon.

I wish to present another example to you which belongs to another chapter on the strength of materials. This time it concerns a bar of concrete, subjected to a deformation increasing slowly with time.

It is now the total deformation (and no longer the elastic deformation) which must be chosen as the determining function. The representative curve, if the heredity coefficient were constant, would still be a parabola, but that shown dotted in Fig. 4. With a variable heredity coefficient,

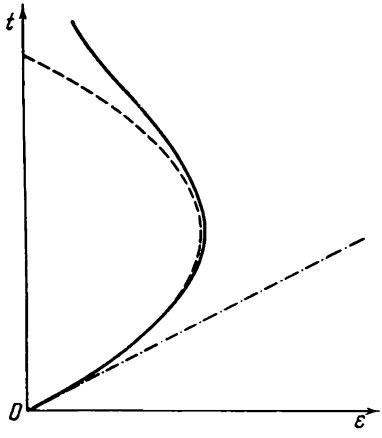

Fig. 4.

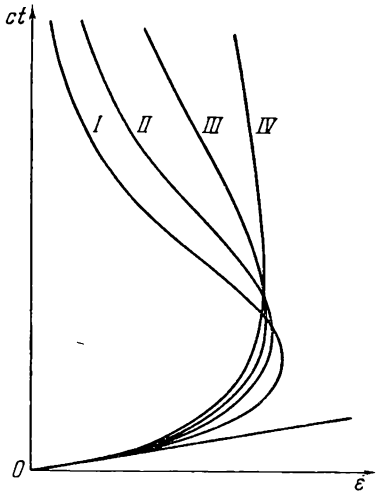

Fig. 5.

its slope will be of the kind shown in the same figure by a solid line and goes away from the parabola more or less depending on the importance of the terms of higher order than the first.

One is thus able to produce curves which approach as closely as one likes those obtained experimentally for different concretes by varying the rate of deformation between the greatest limits. See, for example, Fig. 5

which can be found in a report by von Ruesch and to which I will refer again before long. The four curves that you see there have been obtained with very different rates of deformation:

one in thousand per minute,
" " " per hour,
" " " per day,
" " " per hundred days.

Each of these curves corresponds to a well determined law of variation of the heredity coefficient, about which enough is known to characterize the nature of the phenomenon.

§ 2. I do not wish to take advantage of your time by exposing to you the details of calculations that no longer present any substantial difficulty. I prefer to point out the conclusions to which the theory leads in connection with the stress distribution in the cross-section of a bent beam.

This is one of the fundamental problems of the strength of structures which is of interest to mathematicians, physicists and engineers. As regards this problem, which has been known and studied for a long time, there is, however, one new aspect to be considered.

We will consider the case of a bent beam, assume that its deformation grows linearly in time and attempt to define the state of equilibrium at the instant T. That is to say, we wish to know what would be, at this instant, the distribution either of the deformations, or of the stresses, at each cross-section of the beam.

We shall return to the general equation

$$\bar{\varepsilon}(T) = a_1 c \frac{T^2}{2} + a_2 c^2 \frac{T^3}{3} + a_3 c^3 \frac{T^4}{4} + \cdots,$$

where, for some given value of T, the rate c is a function of the position of the point under consideration. More precisely, the quantity c increases proportionally to the distance of the point from the neutral axis of the cross-section where the point is situated.

We will now assume that $c = \omega y$, ω being an arbitrary constant. Thus, one obtains for the plastic deformation an expression of the type

$$K_1 y + K_2 y^2 + K_3 y^3 + \cdots,$$

where the coefficients depend on the characteristic of the material and on the rate of bending of the beam.

Having chosen as determining function the total deformation

$$\varepsilon + \bar{\varepsilon} = ct,$$

one deduces immediately the expression for the elastic deformation at the instant T

$$\varepsilon = \omega T y - K_1 y - K_2 y^2 - K_3 y^3 - \cdots.$$

Now, different possibilities arise because of the different characteristics of the material and of the different rates of deformation.

We first note that, if one assumes the heredity coefficient to be constant and then K_1 alone among the coefficients K is different from zero, the plastic deformation

$$\bar{\varepsilon} = K_1 y$$

as well as the elastic deformation

$$\varepsilon = (\omega T - K_1) y$$

will be proportional to y; the distribution of the deformations (and the stresses) on the cross-section will be linear as if the beam has retained its perfectly elastic property.

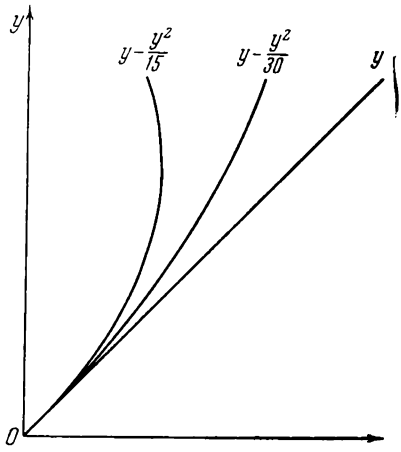

Fig. 6.

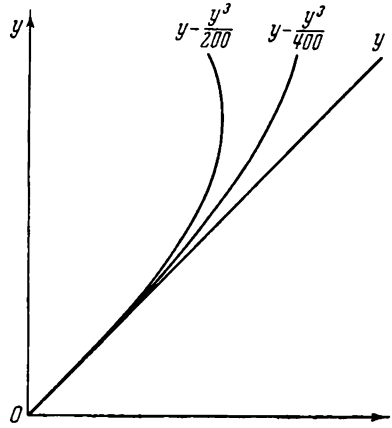

Fig. 7.

Generally such a hypothesis must be dismissed. It suffices, however, to admit that a single one of the constants differs from zero, in order for the theory to offer us the explanation of certain facts which experiment has since revealed to us.

Let us suppose, for example, that the heredity coefficient can be written in the form

$$f = a_1 + a_n(\varepsilon + \bar{\varepsilon})^{n-1}.$$

One has then

$$\bar{\varepsilon} = K_1 y + K_n y^n$$

and

$$\varepsilon = (\omega T - K_1)y - K_n y^n.$$

The intervention of plastic deformations then will determine not only quantitative changes, but qualitative ones in the law of distribution of elastic deformation — and hence of the stresses — on the cross-section of the beam. Such a law will no longer be linear.

In the Figs. 6, 7, and 8, this law of the distribution of elastic deformations and stresses has been shown for certain special cases corresponding respectively to the three different values of $n = 2, 3$, and 4 and for two different values of the ratio between the coefficient K_n and the coefficient $\omega T - K_1$.

Certain conclusions follow now immediately.

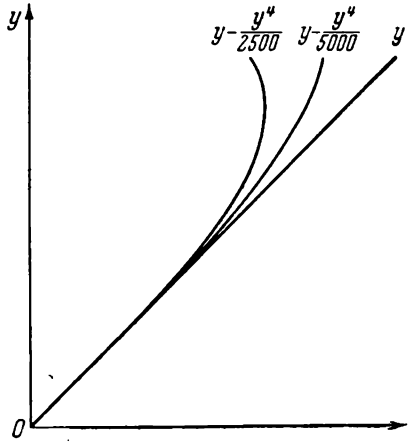

Fig. 8.

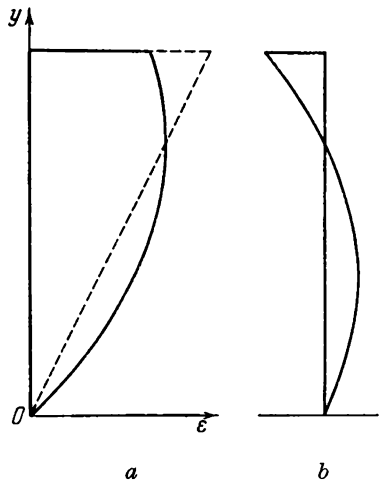

Fig. 9.

Contrary to those admitted usually — and contrary to those which I have admitted myself in my early studies — the plastic deformations can appear in regions of the beam which are closest to the neutral axis and thus subject to the smallest stresses.

Such a fact cannot be translated in any case into a simple increase of the total deformation, i.e., of the curvature of the beam, which could be interpreted as the result of a fictitious increase of the external load due to the plastic deformations. Deformations and stresses at some point of the cross-section will not only depend on the external loading (and on its fictitious increase due to the set of plastic deformations),

but also on the values of the plastic deformations which are achieved at this point.

The new distribution of stresses (Figs. 9*a*, *b*) compared with that which would apply in an elastic regime (shown in the figure by a dotted line) indicates immediately a decrease of the values at points where the material in the elastic region would have been submitted to the largest stresses and an increase of the values where in the elastic regime the strength of the material would have been used least.

One can no longer say that the greatest stresses occur near the edges of the section, and in any case at the points of the section which are furthest away from the neutral axis. This evidently happens for values of K_n below a certain limit; but for some larger values of K_n — such as those that can be encountered by increasing the rate of deformation or of the time — the elastic deformations and the stresses keep increasing only up to a certain value of y, when they begin to decrease more or less rapidly. The greatest stresses will then appear at points situated inside the section.

In conclusion, we are led to recognize that a state of coaction can always occur, even under the action of very small loadings, and that, as a consequence, a pure and simple return of the beam to its initial undeformed state, in general, would not be possible.

This state of coaction can be characterized by referring the diagram of elastic deformations to the dotted line. This diagram [reproduced in Fig. 9*b*] can be used to demonstrate the state of coaction after unloading of the beam.

§ 3. Certain of these conclusions correspond to facts which have been known for a long time.

For example, the fact that, at the time of the first application of even very light loads concrete structures show permanent deformations, has been well known to designers. But this is, I believe, the first time that it finds a rational interpretation and the possibility of being foreseen and evaluated quantitatively.

In fact, no one pays any attention to the states of coaction which will follow, and one neglects this phenomena on account of its smallness which excludes all immediate risk. These states, however, do exercise an influence which is not necessarily negligible in the successive states of the structure, even on those which can endanger the strength.

A systematic set of experiments, conducted with methods and instruments which permit disclosing the presence of permanent curves (and more generally of deformations), even very small ones, would offer engineers the chance to go deeply into the problem. Meanwhile it should

be noted that the need is arising for a revision of current ideas even on the part of those who are studying the problem with a view to practical applications.

I will refer to the subject of more recent works of the "Commission pour la pression excentrique" of the "Comité Européen du Béton", as they have been written by the President of the Commission von Ruesch in the report which he presented in Vienna last April.

In this report the necessity has been recognized, as I have said just now, to replace in the design calculations of elements of reinforced concrete, the stress-strain diagram which assumes a purely elastic behavior of concrete, by another diagram which takes account of its elasto-plastic behavior.

Many times already — von Ruesch tells us — attempts have been made to deduce such a diagram from known results. Of particular interest to this subject is a paper by Scholz who has proposed to verify whether all the experimental results could be explained by a single and unique stress-strain law. The result was negative. The differences with one unique law were only negligible for concrete of high strength, the behavior of which is nearly elastic. In contrast, concrete of low strength, for which the influence of plastic deformations is important, exhibited considerable differences.

Von Ruesch has tried to explain these failures encountered up to now in the search for a unique fundamental law. He estimates that, until now, one has not taken into account sufficiently the rate of loading. When a beam is subjected to bending, the different fibres of the compressed area are found to be deformed at different rates, which are proportional to their distance from the neutral axis. It is necessary to take this into account when determining the distribution of the stresses.

With this goal, von Ruesch has proposed to study the influence of the rate of deformation on the shape of the stress-strain curve for bars in simple compression.

Some identical bars were loaded at different rates, which were held constant at the time of each experiment. According to von Ruesch, one can deduce from these constraint-deformation curves the law of distribution of the stresses at the different points of the compressed zone of a bent beam.

Now the diagrams that von Ruesch obtained in this way are definitely of the same type as those that I have come to present to you; one should now be able to consider the theory as fully confirmed by practice. Nevertheless, it is necessary to remember that it is a question of results obtained by indirect methods. Some doubt then could still remain.

Direct confirmations — that is to say, those executed directly on the fibres of the beam — will evidently be very delicate, but not impossible. Work is being done to solve this problem in the Laboratories of the Istituto Dinamometrico of Turin.

The different fibres of a bent metal beam, assembled in such a way as to make them interdependent when they have to resist external loading, can be rendered independent after the beam has been unloaded. Electric wire strain gauges of high precision have been placed on the different fibres and their deformations registered, before as well as after the destruction of the links between them. A direct measurement of the plastic deformations in the different parts of the beam is thus possible.

It is thanks to experiments of this kind that we should soon finally be able to be clear on the subject of the possibility of interpreting the most varied elasto-plastic phenomena by the theory of heredity about which I have been speaking here.

ON THE THEORY OF ONE-DIMENSIONAL SINGULAR EQUATIONS

I. I. DANILIUK
Novosibirsk, U.S.S.R.

§ 1. Introduction. Basic assumptions

The theory of one-dimensional singular integral equations in which a principal part is played by well-known theorems of F. Noether attained its final form in the works of N. I. Muskhelishvili, I. N. Vekua and a number of their followers. Following Hölder, under the classical assumptions of continuity of the coefficients in the equations with the possible exception of a finite number of first order discontinuities, this theory was constructed for systems of closed as well as open Liapunov curves. A complete account of this theory is contained in N. I. Muskhelishvili's book [1] (cf. likewise [2]).

S. G. Mikhlin has considered singular equations in the Hilbert space L_2 and has shown that Noether's theorems are preserved for only one of the assumptions of continuity of the coefficients of the equations [3]. Later on, in an article by B. V. Khvedelidze [7], these results were obtained also in the space L_p, even when the coefficients have a finite number of points of discontinuity. The operators of Noether have also been studied from an abstract point of view in the paper by F. V. Atkinson [8]. [1]

The objective of the present paper is the proof of the theorems of Noether for less inconvenient assumptions regarding the coefficients of the singular part of the equations. We will consider equations of the type

$$(1.1) \qquad K\varphi(s) \equiv a(s)\varphi(s) + \frac{b(s)}{\pi}\int_0^{2\pi} \frac{\varphi(\sigma)d\sigma}{1 - e^{i(s-\sigma)}} + L\varphi(s) = f(s).$$

We immediately formulate the basic assumptions for which the study of the equation (1.1) will be executed:

1) The given functions $a(s)$, $b(s)$, $0 \leq s \leq 2\pi$, defined almost everywhere on the segment $[0, 2\pi]$, are measurable and satisfy almost every-

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[1] A detailed survey of the work in this field in the U.S.S.R. will be found in the book "Forty Years of Mathematics in the U.S.S.R.", Vol. 1, Moscow, Fizmatgiz, 1959, pp. 656–665.

where on $[0, 2\pi]$ the inequalities

$$|a(s)| \leq M, \ |b(s)| \leq M, \ M = \text{const.}$$

It follows from this assumption, by the strength of a known property of singular operators, that the dominant operator $K^0\varphi(s)$ of the operator $K\varphi(s)$,

$$K^0\varphi(s) \equiv a(s)\varphi(s) + \frac{b(s)}{\pi}\int_0^{2\pi} \frac{\varphi(\sigma)d\sigma}{1 - e^{i(s-\sigma)}},$$

is defined on the entire space $L_p(0, 2\pi)$, $p \geq 1$ and transforms any function $\varphi(s) \in L_p$, $p > 1$ into the function $K^0\varphi(s)$ belonging to the same space.

2) The operator $L\varphi(s)$ transforms any function $\varphi(s)$ of the space L_p into some function of the same space and is completely continuous.

It follows from these two conditions that the operator $K\varphi(s)$ transforms the space $L_p(0, 2\pi)$, $p > 1$ into functions of the same space. Therefore it is natural to impose on the function $f(s)$ the following condition:

3) The function $f(s)$ is defined almost everywhere on the segment $[0, 2\pi]$; it is measurable and belongs to the same space L_p.

The following condition refers to the functions $S_1(s) \equiv a(s) + b(s)$ and $S_2(s) \equiv b(s) - a(s)$:

4) There exists a number $m > 0$ such that the inequalities

$$|S_1(s)| \geq m, \ |S_2(s)| \geq m$$

are fulfilled for almost all s belonging to the segment $[0, 2\pi]$. It is natural to call the operator satisfying the conditions 4), in correspondence with the terminology of the book [1], an operator of the normal type. As has been shown in [4], in the case of continuous coefficients $a(s)$, $b(s)$, the vanishing of the functions $S_1(s)$, $S_2(s)$ leads to the result that Noether's theorems are not true. Obviously such functions cannot satisfy the conditions 4).

The last condition concerns the function

$$\theta(s) = \arg D(s), \ D(s) \equiv S_2(s)/S_1(s).$$

5) There exists a single-valued branch $\theta(s)$ of the function $\arg D(s)$ which satisfies the following conditions:

α) At each point $\sigma \in [0, 2\pi]$ there exist both the limits

$$\lim_{s \to \sigma - 0} \theta(s) = \theta(\sigma - 0), \ \lim_{s \to \sigma + 0} \theta(s) = \theta(\sigma + 0),$$

where the difference $h(\sigma) = \theta(\sigma + 0) - \theta(\sigma - 0)$ vanishes for all

$\sigma \in [0, 2\pi]$ with the exception of a closed, not larger than an enumerable set of points $\{s_k\}$.

β) The series, consisting of the set of numbers $h_k = h(s_k)$, $k = 1, 2, \cdots$ converges absolutely

$$\sum_k |h_k| < \infty.$$

γ) The function $\theta(s)$ permits the representation

$$\theta(s) = \theta_0(s) + \theta_1(s), \tag{1.2}$$

where $\theta_0(s)$ is a continuous function [1] with period 2π and $\theta_1(s)$ is the "jump function" of the function $\theta(s)$.

We note that the representations (1.2) admit, for example, functions of the important class of functions with bounded variation, where in this case the continuous function $\theta_0(s)$ also has bounded variation.

Under these assumptions a certain function $\varphi(s)$ will be called a solution of the equation (1.1), if it is defined almost everywhere, is measurable, belongs to the space L_p and almost everywhere satisfies the equation (1.1).

§ 2. Definition of the index

We will divide the set of numbers $\{h_k\}$, consisting of the jumps of the function $\theta(s)$, into two sets. In the first set, we place all positive numbers h_k and we denote it by $\{h_k^+\}$; in the second set, we place all negative numbers h_k and we denote by $\{h_k^-\}$ the set of their absolute values. Without limitation of generality we may assume that the numbers h_k^+ and h_k^- have been arranged for increasing k in the order in which their values do not increase.

For every given number p, we may now define four positive numbers $\overset{(p)}{\kappa_1}(D)$, $\overset{(p)}{\kappa_2}(D)$, $\overset{(p)}{\kappa_3}(D)$ and $\overset{(p)}{\kappa_4}(D)$ in the following manner. We define the numbers $\overset{(p)}{\kappa_1}(D)$ and $\overset{(p)}{\kappa_2}(D)$ by the conditions

$$\begin{aligned}
&\frac{h_r^+}{2\pi} > \frac{1}{p'}, \quad r = 1, 2, \cdots, \overset{(p)}{\kappa_1}(D); \\
&\frac{h_r^+}{2\pi} = \frac{1}{p'}, \quad r = \overset{(p)}{\kappa_1}(D) + 1, \cdots, \overset{(p)}{\kappa_1}(D) + \overset{(p)}{\kappa_2}(D); \\
&\frac{h_r^+}{2\pi} < \frac{1}{p'}, \quad r > \overset{(p)}{\kappa_1}(D) + \overset{(p)}{\kappa_2}(D); \quad p' = \frac{p}{p-1}, \quad p > 1.
\end{aligned} \tag{2.1}$$

[1] The condition $\theta_0(0) = \theta_0(2\pi)$ has been assumed only for the sake of simplicity. (Cf. [5]).

Obviously, by the strength of the assumption $5\beta)$, such numbers exist and are determined uniquely. It follows from this condition that there also exist uniquely determined integers $\overset{(p)}{\kappa_3}(D)$ and $\overset{(p)}{\kappa_4}(D)$ for which

$$\begin{aligned} &\frac{h_r^-}{2\pi} > \frac{1}{p}, \quad r = 1, 2, \cdots, \overset{(p)}{\kappa_3}(D); \\ &\frac{h_r^-}{2\pi} = \frac{1}{p}, \quad r = \overset{(p)}{\kappa_3}(D) + 1, \cdots, \overset{(p)}{\kappa_3}(D) + \overset{(p)}{\kappa_4}(D); \\ &\frac{h_r^-}{2\pi} < \frac{1}{p}, \quad r > \overset{(p)}{\kappa_3}(D) + \overset{(p)}{\kappa_4}(D). \end{aligned} \tag{2.2}$$

In this case the number $\overset{(p)}{\kappa}(D)$, defined by the formula

$$\overset{(p)}{\kappa}(D) = \overset{(p)}{\kappa_1}(D) - \overset{(p)}{\kappa_3}(D) - \overset{(p)}{\kappa_4}(D), \tag{2.3}$$

will be called [1] the index of the function $D(s)$ in the class $L_p(0, 2\pi)$. It is readily seen that, if the function $\arg D(s)$ is continuous, then the numbers h_k are equal to $\pm 2\pi$ and the formula (2.3) gives the known index κ of the function $D(s)$

$$\kappa = \frac{1}{2\pi} [\arg D(s)]_r,$$

which does not depend on the number p.

In addition,

$$\begin{aligned} &\overset{(p')}{\kappa_1}\left(\frac{1}{D}\right) = \overset{(p)}{\kappa_3}(D), \quad \overset{(p')}{\kappa_2}\left(\frac{1}{D}\right) = \overset{(p)}{\kappa_4}(D), \\ &\overset{(p')}{\kappa_3}\left(\frac{1}{D}\right) = \overset{(p)}{\kappa_1}(D), \quad \overset{(p')}{\kappa_4}\left(\frac{1}{D}\right) = \overset{(p)}{\kappa_2}(D), \end{aligned}$$

and since

$$\overset{(p')}{\kappa}\left(\frac{1}{D}\right) = \overset{(p')}{\kappa_1}\left(\frac{1}{D}\right) - \overset{(p')}{\kappa_3}\left(\frac{1}{D}\right) - \overset{(p')}{\kappa_4}\left(\frac{1}{D}\right),$$

the equality

$$\overset{(p')}{\kappa}\left(\frac{1}{D}\right) = -\overset{(p)}{\kappa}(D) \tag{2.4}$$

will be fulfilled if and only if $\overset{(p)}{\kappa_2}(D) = \overset{(p)}{\kappa_4}(D) = 0$, in particular, when the function $\arg D(s)$ is continuous.

[1] In the present paper we will adhere to a notation which differs somewhat from that in Ref. [5].

§ 3. Solution of the homogeneous characteristic equation

We will consider the following homogeneous equation containing the dominant operator K^0:

$$K^0\varphi(s) \equiv a(s)\varphi(s) + \frac{b(s)}{\pi}\int_0^{2\pi} \frac{\varphi(\sigma)d\sigma}{1 - e^{i(s-\sigma)}} = 0. \tag{3.1}$$

We assume that some function $\varphi(s)$ of the class L_p is a solution of the equation (3.1). We construct with the help of a Cauchy type integral the piecewise holomorphic function

$$\Phi(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\varphi(\sigma)d\sigma}{1 - ze^{-i\sigma}}, \tag{3.2}$$

vanishing at infinity. As is known, the function $\Phi(z)$ belongs to the Riesz class $H^{\pm}_{\delta}$ with some positive δ ([6], Chap. II). Applying to the integral (3.2) the known Sokhotskii-Plemelj formulae, proved by I. I. Privalov for the case of an integrable $\varphi(s)$ ([6], Chap. III, § 3), we obtain

$$\Phi^{\pm}(e^{is}) = \pm \tfrac{1}{2}\varphi(s) + \frac{1}{2\pi}\int_0^{2\pi} \frac{\varphi(\sigma)d\sigma}{1 - e^{i(s-\sigma)}}, \tag{3.3}$$

where $\Phi^{\pm}(e^{is})$ denotes the limiting value of $\Phi(z)$ as the point z tends to the point e^{is} along a completely non-tangential path from within and from without the unit circle, respectively. Consequently, $\Phi^{\pm}(e^{is})$ belongs to the space L_p, and then, by the strength of a theorem of V. I. Smirnov ([6], Chap. II, § 6) the function $\Phi(z)$ belongs to the class $H^{\pm}{}_p$.

Starting from the formula (3.3), it is easy to establish that the function (3.2) fulfills almost everywhere on the segment $[0, 2\pi]$ the boundary condition of the homogeneous Hilbert problem

$$\Phi^{+}(e^{is}) + D(s)\Phi^{-}(e^{is}) = 0, \tag{3.4}$$

where $\Phi(z) \in H^{\pm}{}_p$ and $\Phi(\infty) = 0$. Conversely, if the function $\Phi(z)$ belongs to the class $H^{\pm}{}_p$, if it vanishes at infinity and satisfies almost everywhere the boundary condition (3.4), then the function $\varphi(s) = \Phi^{+}(e^{is}) - \Phi^{-}(e^{is})$ satisfies the equation (3.1).

Using the results of [5] with respect to the problem (3.4), we may formulate the following proposition:

LEMMA 1. *If the index* $\overset{(p)}{\kappa}(D)$ *of the function* $D(s)$ *is non-negative*

$(\overset{(p)}{\kappa}(D) \geq 0)$, *then the homogeneous equation* (3.1) *has exactly* $\overset{(p)}{\kappa}(D)$ *linearly independent (within the field of complex numbers) solutions of the class* $L_p(0, 2\pi)$; *if the same index* $\overset{(p)}{\kappa}(D)$ *is negative* $(\overset{(p)}{\kappa}(D) < 0)$, *the equation* (3.1) *has no non-trivial solution in the class* L_p.

§ 4. The homogeneous equation, adjoint to the characteristic equation

We will now consider the following homogeneous equation, containing the operator $K^{0\prime}$, adjoint to the characteristic operator,

$$K^{0\prime}\psi(s) \equiv a(s)\psi(s) - \frac{1}{\pi}\int_0^{2\pi} \frac{b(\sigma)\psi(\sigma)}{1 - e^{i(s-\sigma)}}\, d\sigma = 0, \tag{4.1}$$

and we assume that $\psi(s)$ is a solution of this equation, belonging to the adjoint class $L_{p'}(0, 2\pi)$. Writing, as before,

$$\Psi(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{b(\sigma)\psi(\sigma)}{1 - ze^{-i\sigma}}\, d\sigma,$$

we readily deduce that under the assumption 1) of § 1 the function $\Psi(z)$ belongs to the class $H^{\pm}{}_{p'}$, vanishes at infinity and solves almost everywhere on the segment $[0, 2\pi]$ the following Hilbert boundary value problem:

$$\Psi^{+}(e^{is}) + \frac{1}{D(s)}\,\Psi^{-}(e^{is}) = 0. \tag{4.2}$$

Using the terminology of the book [1] we will call the problem (4.2) adjoint to the problem (3.4). It is easy to show that conversely, if $\Psi(z)$ belongs to the class $H^{\pm}{}_{p'}$, vanishes at infinity and solves the problem (4.2), then the function $\psi(s) \equiv 2\Psi^{+}(e^{is})/[b(s) + a(s)]$ gives a solution of the equation (4.1), belonging to the class $L_{p'}(0, 2\pi)$.

We will show now that, if the index $\overset{(p)}{\kappa}(D)$ of the function $D(s)$ is non-negative, then the homogeneous problem (4.2) does not have non-trivial solutions of the class $H^{\pm}{}_{p'}$, vanishing at infinity, so that the homogeneous equation (4.1) does not have non-trivial solutions of the class $L_{p'}$.

In fact, let the function $\Psi \in H^{\pm}{}_{p'}$ vanish at infinity and solve the problem (4.2) and let Φ denote a function of the class $H^{\pm}{}_{p}$ which solves

the problem (3.4) and is bounded at infinity. By the strength of the assumption $\overset{(p)}{\kappa}(D) \geq 0$ such a solution $\Phi(z)$ always exists [5].

We will now consider the function Ω, defined by the formula

$$\Omega(z) = \Phi(z)\Psi(z).$$

This function belongs to the class $H^{\pm}_1$, it vanishes at infinity and, as follows from the conditions (3.4) and (4.2), it solves the simplest Hilbert problem

$$\Omega^{+}(e^{is}) - \Omega^{-}(e^{is}) = 0.$$

However, as follows from a theorem of G. M. Fikhtengol'ts (cf., for example, [6], Chap. II, § 5), this function vanishes identically, whence there follows the identity $\Psi \equiv 0$.

We formulate the obtained result in the form of the proposition:

LEMMA 2: *If the index* $\overset{(p)}{\kappa}(D)$ *of the function* $D(s)$ *is non-negative* $(\overset{(p)}{\kappa}(D) \geq 0)$, *then the homogeneous equation* (4.1) *does not have non-trivial solutions of the class* $L_{p'}(0, 2\pi)$.

§ 5. On the normal solubility of the operator K°

As follows from the arguments of § 3 the non-homogeneous characteristic equation

$$(5.1) \qquad K^0\varphi(s) \equiv a(s)\varphi(s) + \frac{b(s)}{\pi}\int_0^{2\pi} \frac{\varphi(\sigma)d\sigma}{1 - e^{i(s-\sigma)}} = f(s), \quad (\varphi, f \in L_p)$$

is equivalent to the non-homogeneous Hilbert problem

$$(5.2) \qquad \Phi^{+}(e^{is}) + D(s)\Phi^{-}(e^{is}) = g(s), \quad g(s) = \frac{f(s)}{b(s) + a(s)},$$

in the class $H^{\pm}_p$ of functions, vanishing at infinity, where by the strength of the assumption 4) of § 1 the function $g(s)$ belongs to the space L_p together with the function $f(s)$.

Let $Z_1(s)$ be the fundamental function, constructed in § 4 of Ref. [5]. It belongs to some class $H^{\pm}_{\delta}$, $\delta > 0$, satisfies almost everywhere the condition (3.4) and does not vanish at any point inside as well as outside the unit circle. As has been shown in § 6 of [5], the absolute value

of the function $Z_1^+(e^{is})$ is given by the formula

$$|Z_1^+(e^{is})| =$$

$$= |D(s)|^{\frac{1}{2}} \exp\left\{-\frac{1}{4\pi}\int_0^{2\pi} \theta_0(\sigma) \operatorname{cotan}\frac{s-\sigma}{2}\, d\sigma\right\} \frac{\prod_r \left\{\sin\left|\frac{s-s_r^+}{2}\right|\right\}^{\varkappa_r^+/2\pi}}{\prod_r \left\{\sin\left|\frac{s-s_r^-}{2}\right|\right\}^{\varkappa_r^-/2\pi}}.$$

We consider now the two polynomials

$$R_1(z) = \prod_{r=1}^{\overset{(p)}{\varkappa_1}(D)+\overset{(p)}{\varkappa_2}(D)} (z - e^{is_r^+}),\quad R_2(z) = \prod_{r=1}^{\overset{(p)}{\varkappa_3}(D)} (z - e^{is_r^-}),$$

and construct the new function

$$Z(z) = Z_1(z)\frac{R_2(z)}{R_1(z)}.$$

As before, this function satisfies almost everywhere the condition (3.4), does not have a zero inside and within a finite distance outside the unit circle, and, as follows from the assumption 3) of § 1 and the lemmas proved in § 3 of Ref. [5], it satisfies the condition

$$[Z^{\pm}(e^{is})]^{-1} \in L_{p'+0}. \tag{5.3}$$

In addition, it has at infinity the order $\overset{(p')}{\varkappa}(1/D)$, since by the relations of § 2

$$\overset{(p)}{\varkappa_3}(D) - \overset{(p)}{\varkappa_1}(D) - \overset{(p)}{\varkappa_2}(D) = \overset{(p')}{\varkappa_1}\left(\frac{1}{D}\right) - \overset{(p')}{\varkappa_3}\left(\frac{1}{D}\right) - \overset{(p')}{\varkappa_4}\left(\frac{1}{D}\right) = \overset{(p')}{\varkappa}\left(\frac{1}{D}\right).$$

We consider now the piecewise holomorphic function

$$\Phi(z) = \frac{Z(z)}{2\pi}\int_0^{2\pi} \frac{g(\sigma)}{Z^+(e^{i\sigma})}\,\frac{d\sigma}{1 - ze^{-i\sigma}}. \tag{5.4}$$

It follows from the fact that $g(\sigma) \in L_p$ and from the condition (5.3) that $g(\sigma)[Z^+(e^{i\sigma})]^{-1} \in L_{1+0}$. Therefore we readily obtain from formulae of the form (3.3) that the limiting values $\Phi^{\pm}$ exist almost everywhere and that these values satisfy the condition (5.2). Consequently, we still have to investigate when the function (5.4) vanishes at infinity and when its limiting values $\Phi^{\pm}(e^{is})$ belong to the space L_p.

If $\overset{(p')}{\varkappa}(1/D) \leq 0$, the function (5.4) vanishes always at the point at infinity, whatever may be the function $g(\sigma)$. However, if $\overset{(p')}{\varkappa}(1/D) > 0$,

it is necessary and sufficient for the vanishing of $\Phi(z)$ at the point $z = \infty$ that $g(\sigma)$ satisfy the conditions

$$\int_0^{2\pi} \frac{g(s)e^{iks}ds}{Z^+(e^{is})} = 0, \quad k = 1, 2, \cdots, \overset{(p')}{\kappa}\left(\frac{1}{D}\right). \tag{5.5}$$

We consider the system of functions

$$\Psi_k(z) = \frac{z^k}{Z(z)}, \quad k = 0, 1, \cdots, \overset{(p')}{\kappa}\left(\frac{1}{D}\right) - 1. \tag{5.6}$$

It follows from the condition (5.3) that all these functions belong to the class $H^{\pm}_{p'}$, vanish at infinity and solve the associate problem (4.2). On the other hand, it follows from lemma 1, by applying it to the problem (4.2), that this problem has exactly $\overset{(p')}{\kappa}(1/D)$ solutions in the class of functions vanishing at infinity (if, as is the case here, $\overset{(p')}{\kappa}(1/D) > 0$). Therefore the system (5.6) represents a complete system of solutions of the problem (4.2) in the above class and, consequently, the system of functions

$$\psi_k(s) = \frac{2\Psi_k^+(e^{is})}{b(s) + a(s)}, \quad k = 0, 1, \cdots, \overset{(p')}{\kappa}\left(\frac{1}{D}\right) - 1,$$

represents a complete system of solutions of the equation (4.1) in the space $L_{p'}$. The conditions (5.5) are equivalent to the system of equalities

$$\int_0^{2\pi} f(s)\psi_k(s)de^{is} = 0, \quad k = 0, 1, \cdots, \overset{(p')}{\kappa}\left(\frac{1}{D}\right) - 1. \tag{5.7}$$

Thus, for the function $\Phi(z)$, constructed by the use of the formula (5.4), to vanish at the point at infinity it is necessary and sufficient that the right hand side $f(s)$ of the equation (5.1) satisfy the conditions (5.7).

The boundary values $\Phi^{\pm}$ of the function (5.4) are given by the formulae

$$\Phi^+(e^{is}) = \frac{g(s)}{2} + \frac{Z^+(e^{is})}{2\pi}\int_0^{2\pi} \frac{g(\sigma)}{Z^+(e^{i\sigma})} \frac{d\sigma}{1 - e^{i(s-\sigma)}},$$

$$\Phi^-(e^{is}) = \tfrac{1}{2}\frac{g(s)}{D(s)} - \frac{1}{D(s)} \frac{Z^+(e^{is})}{2\pi}\int_0^{2\pi} \frac{g(\sigma)}{Z^+(e^{i\sigma})} \frac{d\sigma}{1 - e^{i(s-\sigma)}}.$$

Therefore, on the basis of the assumption 4) of § 1, the function $\Phi^{\pm}(e^{is})$

belongs together with the function $g(s)$ to the space L_p if and only if the following condition is fulfilled:

6) The function

$$Zg(s) \equiv \frac{Z^+(e^{is})}{2\pi} \int_0^{2\pi} \frac{g(\sigma)}{Z^+(e^{i\sigma})} \frac{d\sigma}{1 - e^{i(s-\sigma)}} \in L_p.$$

In particular, we assume that $\overset{(p)}{\kappa_2}(D) = \overset{(p)}{\kappa_4}(D) = 0$. Then it is readily shown, again using a lemma of § 3 of [5], that the function $Z(z)$ satisfies apart from the condition (5.3) also the condition

$$Z^+(e^{is}) \in L_p. \tag{5.8}$$

If the sets $\{h_r^+\}$ and $\{h_r^-\}$ are finite, then, starting from the conditions (5.3) and (5.8) and using a result of [7] (p. 24, Theorem 5) it may be shown that the operator $Zg(s)$ is bounded in the space L_p.

Thus, we have:

LEMMA 3. *In the assumptions* 1)—5) *of* § 1 *and* 6) *of* § 5 *the operator* K^0 *is normally soluble in the space* L_p.

NOTE. In order that $Zg(s) \in L_p$ for every $g \in L_p$, consequently, for normal solubility of the operator K^0 the conditions $\overset{(p)}{\kappa_2}(D) = \overset{(p)}{\kappa_4}(D) = 0$ or the equivalent conditions (2.4) are necessary, since the set of Noether operators with one and the same index of the defect is open ([8], Theorem IV).

§ 6. Basic results. Concluding remarks

Let k denote the number of linearly independent (in the field of complex numbers) solutions of the homogeneous equation $K\varphi(s) = 0$ belonging to the space L_p; analogously, we denote by k' the number of linearly independent (in the same field) solutions of the associate homogeneous equation $K'\psi(s) = 0$ belonging to the space $L_{p'}$. We assume that the assumptions 1)—5) of § 1 are fulfilled. If, in addition, we assume that $\overset{(p)}{\kappa_2}(D) = \overset{(p)}{\kappa_4}(D) = 0$ and the condition 6) of § 5 is fulfilled and, in particular, if the set $\{h_k\}$ is finite, then the following three theorems of Noether are true:

THEOREM 1. *The numbers* k *and* k' *are finite.*

THEOREM 2. *There exists between the numbers* k *and* k' *and* $\overset{(p)}{\kappa}(D)$ *the relationship* $k - k' = \overset{(p)}{\kappa}(D)$.

THEOREM 3. *The operator* K *is normally soluble in the space* L_p.

For the dominant operator K^0, these theorems follow from the Lemmas 1—3, proved above. For this Theorems 1 and 2 are obviously true only under the assumptions 1)—5) of § 1. Theorem 2 in this case is replaced by the following more general proposition:

$$k - k' = \begin{cases} \overset{(p)}{\kappa}(D), \text{ if } \overset{(p)}{\kappa}(D) \geq 0, \\ \min\left\{-\overset{(p')}{\kappa}\left(\frac{1}{D}\right), 0\right\}, \text{ if } \overset{(p)}{\kappa}(D) < 0. \end{cases}$$

For a general operator K these theorems now follow under the above assumptions from the results of [8].

The analysis of the equation (1.1) relies on the fact that the integration is executed essentially along the unit circle which offered the possibility of using the results of. [5], which deals with a study of the corresponding Hilbert problem. However, this limitation is not essential, and it is not difficult to formulate the requirement for an arbitrary contour Γ, for which the above propositions are maintained. The equation reduces in this case to the form (1.1) at the expense of a change of the completely continuous operator L. As regards an equation involving integration along a system of contours, an additional investigation is required, since we are then dealing, in essence, with a system of equations. We note likewise that the above results apply also to the case of open contours.

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THE TWO-DIMENSIONAL PROBLEMS OF THE NONHOMOGENEOUS ISOTROPIC MEDIUM

DU CHING-HUA
Peking, China

In this paper the equation of the stress function for the two-dimensional problems of the nonhomogeneous isotropic medium is established. Then the inter-boundary conditions of the stress function for the cases with sudden change of elastic properties between adjacent regions are discussed. The conditions of the boundary with known displacements expressed in stress functions are also presented. For a straight inter-boundary line between two regions with sudden change of elastic properties, the equations on the interboundary can be put into finite difference form. Then together with the equation of the stress function, relaxation techniques can be used.

§ 1. In the plane problem of elasticity, equilibrium requires

$$\begin{aligned} &\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0, \\ &\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y = 0. \end{aligned} \tag{1.1}$$

The equation of compatibility is

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}. \tag{1.2}$$

The stress-strain relations are
for the case of plane strain:

$$\begin{aligned} \varepsilon_x &= c_{11}(\sigma_x - \nu\sigma_y) + (1+\mu)\alpha T, \\ \varepsilon_y &= c_{11}(\sigma_y - \nu\sigma_x) + (1+\mu)\alpha T, \\ \gamma_{xy} &= 2(1+\nu)c_{11}\tau_{xy}; \end{aligned} \tag{1.3'}$$

for the case of plane stress:

$$\begin{aligned} \varepsilon_x &= c_{11}(\sigma_x - \nu\sigma_y) + \alpha T, \\ \varepsilon_y &= c_{11}(\sigma_y - \nu\sigma_x) + \alpha T, \\ \gamma_{xy} &= 2(1+\nu)c_{11}\tau_{xy}, \end{aligned} \tag{1.3''}$$

where $c_{11} = (1 - \mu^2)/E$, $\nu = \mu/(1 - \mu)$ for the case of plane strain, $c_{11} = 1/E, \nu = \mu$ for the case of plane stress. The functions $E(x, y), \mu(x, y)$ are elastic moduli and Poisson's ratio and αT denotes the effect of a temperature change.

When the body forces have a potential V, introduced by

$$\sigma_x = \frac{\partial^2\varphi}{\partial y^2} + V, \quad \sigma_y = \frac{\partial^2\varphi}{\partial x^2} + V, \quad \tau_{xy} = -\frac{\partial^2\varphi}{\partial x \partial y},$$

we obtain,

$$\nabla^2[c_{11}\nabla^2\varphi] -$$

$$-\left[\frac{\partial^2(1+\nu)c_{11}}{\partial y^2}\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2(1+\nu)c_{11}}{\partial x^2}\frac{\partial^2\varphi}{\partial y^2} - 2\frac{\partial^2(1+\nu)c_{11}}{\partial x\partial y}\frac{\partial^2\varphi}{\partial x\partial y}\right] =$$

$$(1.4') \quad = -\nabla^2(1+\mu)\alpha T - \nabla^2[(1+\nu)c_{11}V] \quad \text{for the case of plane strain,}$$

$$(1.4'') \quad = -\nabla^2\alpha T - \nabla^2[(1+\nu)c_{11}V] \quad \text{for the case of plane stress.}$$

For $0 \leq \mu(x, y) \leq \frac{1}{2}$ in plane strain: $0 \leq \nu \leq 1$, in plane stress: $0 \leq \nu \leq \frac{1}{2}$. If the effect of changes of μ can be neglected, introducing $e = E_0/E$, the equations (1.4) can be reduced to

$$\nabla^2[e\nabla^2\varphi] - \frac{1}{1-\mu}\left[\frac{\partial^2 e}{\partial y^2}\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2 e}{\partial x^2}\frac{\partial^2\varphi}{\partial y^2} - 2\frac{\partial^2 e}{\partial x\partial y}\frac{\partial^2\varphi}{\partial x\partial y}\right] =$$

$$(1.5') \quad = -\frac{E_0}{1-\mu}\nabla^2(\alpha T) - \frac{1-2\mu}{1-\mu}\nabla^2[eV] \quad \text{for the case of plane strain,}$$

$$(1.5'') \quad = -E_0\nabla^2(\alpha T) - (1-\mu)\nabla^2[eV] \quad \text{for the case of plane stress.}^1$$

For a simply connected region, the equations (1.4) or (1.5) together with the boundary values of φ and $\partial\varphi/\partial n$ (i.e., the boundary forces are known) completely determine the stresses in the regions. For a multiply connected region additional conditions of dislocations should be imposed on φ. Such properties have been discussed thoroughly in [2].

The solution of the equations (1.4) or (1.5) is, in general, very tedious. But it can be effected by treating $\nabla^2[c_{11}\nabla^2\varphi]$ as $\nabla^2\psi = f(x, y)$ and $\psi = c_{11}\nabla^2\varphi$. As $c_{11}(x, y)$ is known, this can be done in a Poisson type electrical analogue network. The terms of the form

$$\alpha(x, y)\frac{\partial^2\varphi}{\partial x^2} + \beta(x, y)\frac{\partial^2\varphi}{\partial y^2} + \gamma(x, y)\frac{\partial^2\varphi}{\partial x\partial y}$$

can be taken as iteration terms.

[1] We note that this equation is of similar form to the bending equation of a plate with variable thickness [1], but we have put the equation (1.5) into symmetric form with additional terms.

§ 2. In the case of two adjacent regions with different moduli E_1, E_2 and μ_1, μ_2, the continuity requirements require on the interface

$$u_1(s) = u_2(s), \quad v_1(s) = v_2(s),$$

$$\sigma_{n1} = \sigma_{n2} = \sigma_n, \quad \tau_{ns1} = \tau_{ns2} = \tau_{ns}, \tag{2.1}$$

$$\text{but } \sigma_{s1} \neq \sigma_{s2},$$

where u,v are the displacements normal and tangential to the boundary curve.

By introducing $\varepsilon_n = \partial u/\partial n$, $\varepsilon_s = \partial v/\partial s + u/\rho$, $\gamma_{ns} = \partial u/\partial s + \partial v/\partial n - v/\rho$, where ρ is the radius of curvature of the boundary curve, the boundary strains there are related by,

$$\varepsilon_{s1} = \varepsilon_{s2},$$

$$\frac{\partial \gamma_{ns1}}{\partial s} + \frac{\varepsilon_{n1}}{\rho} - \frac{\partial \varepsilon_{s1}}{\partial n} = \frac{\partial \gamma_{ns2}}{\partial s} + \frac{\varepsilon_{n2}}{\rho} - \frac{\partial \varepsilon_{s2}}{\partial n}. \tag{2.2}$$

In using the stress strain relations, the equations (2.2) can be changed into expressions in terms of stresses

$$c_1(\sigma_{s1} - \nu_1\sigma_n) = c_2(\sigma_{s2} - \nu_2\sigma_n),$$

$$2c_1(1+\nu_1)\frac{\partial \tau_{ns}}{\partial s} + \frac{c_1}{\rho}(\sigma_n - \nu_1\sigma_{s1}) - c_1\left(\frac{\partial \sigma_{s1}}{\partial n} - \nu_1\frac{\partial \sigma_n}{\partial n}\right) = \tag{2.3}$$

$$= 2c_2(1+\nu_2)\frac{\partial \tau_{ns}}{\partial s} + \frac{c_2}{\rho}(\sigma_n - \nu_2\sigma_{s2}) - c_2\left(\frac{\partial \sigma_{s2}}{\partial n} - \nu_2\frac{\partial \sigma_n}{\partial n}\right),$$

where $c = 1/E$, $\nu = \mu$ for plane stress and $c = (1-\mu^2)/E$, $\nu = \mu/(1-\mu)$ for plane strain.

By introducing the stress function φ and its derivatives

$$\sigma_s = \frac{\partial^2 \varphi}{\partial n^2}, \quad \sigma_n = \frac{\partial^2 \varphi}{\partial s^2} + \frac{1}{\rho}\frac{\partial \varphi}{\partial n}, \quad \tau_{ns} = -\frac{\partial}{\partial n}\left(\frac{\partial \varphi}{\partial s}\right);$$

one obtains

$$\frac{\partial^2 \varphi_1}{\partial n^2} - \nu_1\left(\frac{\partial^2 \varphi_1}{\partial s^2} + \frac{1}{\rho}\frac{\partial \varphi_1}{\partial n}\right) = \frac{c_2}{c_1}\left[\frac{\partial^2 \varphi_2}{\partial n^2} - \nu_2\left(\frac{\partial^2 \varphi_2}{\partial s^2} + \frac{1}{\rho}\frac{\partial \varphi_2}{\partial n}\right)\right],$$

$$(2+\nu_1)\frac{\partial}{\partial n}\left[\frac{\partial^2 \varphi_1}{\partial s^2}\right] + \frac{1+2\nu_1}{\rho}\left[\frac{\partial^2 \varphi_1}{\partial s^2} + \frac{1}{\rho}\frac{\partial \varphi_1}{\partial n}\right] + \frac{\partial^3 \varphi_1}{\partial n^3} = \tag{2.4}$$

$$= \frac{c_2}{c_1}\left\{(2+\nu_2)\frac{\partial}{\partial n}\left[\frac{\partial^2 \varphi_2}{\partial s^2}\right] + \frac{1+2\nu_2}{\rho}\left[\frac{\partial^2 \varphi_2}{\partial s^2} + \frac{1}{\rho}\frac{\partial \varphi_2}{\partial n}\right] + \frac{\partial^3 \varphi_2}{\partial n^3}\right\}$$

and

$$\varphi_1 = \varphi_2, \quad \frac{\partial \varphi_1}{\partial s} = \frac{\partial \varphi_2}{\partial s}, \quad \frac{\partial \varphi_1}{\partial n} = \frac{\partial \varphi_2}{\partial n},$$

$$\frac{\partial^2 \varphi_1}{\partial s^2} = \frac{\partial^2 \varphi_2}{\partial s^2}, \quad \frac{\partial^2 \varphi_1}{\partial n \partial s} = \frac{\partial^2 \varphi_2}{\partial n \partial s}, \text{ but } \frac{\partial^2 \varphi_1}{\partial n^2} \neq \frac{\partial^2 \varphi_2}{\partial n^2}. \tag{2.5}$$

It should be noted that in the equation (2.4) the orders of the derivatives of the bracket terms are not reversible; but by using orthogonal curvilinear coordinates it is easy to see that

$$\frac{\partial}{\partial n}\left(\frac{\partial \varphi}{\partial s}\right)=\frac{\partial^2 \varphi}{\partial n \partial s}-\frac{1}{\rho}\frac{\partial \varphi}{\partial s},$$

$$F(\varphi)\equiv(2+\nu)\frac{\partial}{\partial n}\left[\frac{\partial^2 \varphi}{\partial s^2}\right]+\frac{(1+2\nu)}{\rho}\left[\frac{\partial^2 \varphi}{\partial s^2}+\frac{1}{\rho}\frac{\partial \varphi}{\partial n}\right]+\frac{\partial^3 \varphi}{\partial n^3}=$$

$$(2.6)\qquad =(2+\nu)\frac{\partial}{\partial s}\left[\frac{\partial}{\partial n}\left(\frac{\partial \varphi}{\partial s}\right)\right]-\frac{1-\nu}{\rho}\left[\frac{\partial^2 \varphi}{\partial s^2}+\frac{1}{\rho}\frac{\partial \varphi}{\partial n}\right]+\frac{\partial^3 \varphi}{\partial n^3}=$$

$$=(2+\nu)\frac{\partial^3 \varphi}{\partial s^2 \partial n}-\frac{1-\nu}{\rho}\left[\frac{\partial^2 \varphi}{\partial s^2}+\frac{1}{\rho}\frac{\partial \varphi}{\partial n}\right]-$$

$$-\frac{2+\nu}{\rho}\left[\frac{\partial^2 \varphi}{\partial s^2}-\frac{1}{\rho}\frac{d\rho}{ds}\frac{\partial \varphi}{\partial s}\right]+\frac{\partial^3 \varphi}{\partial n^3}.$$

This will make the expression of the second equation of (2.4) somewhat longer.

If the boundary is a straight line in the x direction, we have, using $c=1/E$, $\nu=\mu$,

$$(2.7)\qquad \frac{\partial^2 \varphi_1}{\partial y^2}-\mu_1\frac{\partial^2 \varphi_1}{\partial x^2}=\frac{E_1}{E_2}\left(\frac{\partial^2 \varphi_2}{\partial y^2}-\mu_2\frac{\partial^2 \varphi_2}{\partial x^2}\right),$$

$$\frac{\partial^3 \varphi_1}{\partial y^3}+(2+\mu_1)\frac{\partial^3 \varphi_1}{\partial y \partial x^2}=\frac{E_1}{E_2}\left[\frac{\partial^3 \varphi_2}{\partial y^3}+(2+\mu_2)\frac{\partial^3 \varphi_2}{\partial y \partial x^2}\right].$$

§ 3. In the case of problems with mixed boundary conditions for the isotropic medium, if on one part of the boundary the displacements u and v are specified, then for the case of plane stress on that part of the boundary

$$(3.1)\qquad \frac{1}{E}(\sigma_s-\mu\sigma_n)=\frac{\partial v}{\partial s}+\frac{u}{\rho},$$

$$\frac{2(1+\mu)}{E}\frac{\partial \tau_{ns}}{\partial s}+\frac{1}{\rho}\frac{1}{E}(\sigma_n-\mu\sigma_s)-\frac{1}{E}\frac{\partial}{\partial n}(\sigma_s-\mu\sigma_n)=$$

$$=\frac{\partial^2 u}{\partial s^2}-\frac{1}{\rho}\frac{\partial v}{\partial s}+\frac{v}{\rho}\frac{d\rho}{ds}.$$

In terms of the stress function this becomes

$$\frac{\partial^2\varphi}{\partial n^2} - \mu\left(\frac{\partial^2\varphi}{\partial s^2} + \frac{1}{\rho}\frac{\partial\varphi}{\partial n}\right) = E\left(\frac{\partial v}{\partial s} + \frac{u}{\rho}\right),$$

$$\frac{\partial}{\partial n}\left[\frac{\partial^2\varphi}{\partial s^2}\right] + \frac{(1+2\mu)}{\rho}\left[\frac{\partial^2\varphi}{\partial s^2} + \frac{1}{\rho}\frac{\partial\varphi}{\partial n}\right] + \frac{\partial^3\varphi}{\partial n^3} = \tag{3.2}$$

$$= E\left(\frac{1}{\rho}\frac{\partial v}{\partial s} - \frac{v}{\rho}\frac{d\rho}{ds} - \frac{\partial^2 u}{\partial s^2}\right).$$

The equations (3.2) are the displacement boundary conditions for the stress function. For the case of a fixed straight boundary $u = 0$ and $v = 0$,

$$\frac{\partial^2\varphi}{\partial y^2} - \mu\frac{\partial^2\varphi}{\partial x^2} = 0,$$

$$\frac{\partial^3\varphi}{\partial y^3} + (2+\mu)\frac{\partial^3\varphi}{\partial x^2\partial y} = 0. \tag{3.3}$$

We note here that for a straight edge there is a correspondence between the plane problem of elasticity with a fixed edge and the plate problem with a free edge, except that μ changes sign. But for the case of a curved boundary, this correspondence is not fulfilled, because the second equation of (3.2) includes additional terms which do not appear in the Kirchhoff conditions for the free edge of a bent plate.

The equations (3.3) can be put into finite difference form, together with the governing equation $\nabla^4\varphi = 0$ and the other parts of the boundary conditions of the type specifying φ, $\partial\varphi/\partial n$, and the problem can be solved, say, by relaxation techniques [3].

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PROBLEMS OF THE THEORY OF ELASTICITY OF ANISOTROPIC MEDIA, RELATING TO THE PLANE

G. Iu. Dzhanelidze
Leningrad, U.S.S.R.

Solutions of the equations of the theory of elasticity in the form of series or of sums expanded in powers of one of the Cartesian coordinates apparently were found first by Cauchy when considering the problem of the equilibrium of shells [2]. Subsequently they attracted the attention of many investigators engaged in the study of problems of equilibrium of rods, plates and shells, in this strict order (B. Saint-Venant, A. Clebsch, M. Lévy, [9], J. H. Michell, et al.).

In 1901, E. Almansi [1] gave a scheme for the solution in terms of displacements of the problem of equilibrium of an isotropic elastic cylinder, loaded along the side surface by forces depending polynomially on the axial coordinate, and he showed that it reduced to a series of plane problems of the theory of elasticity and a problem of torsion.

Studying the question of solutions of the general problem of the theory of elasticity of an isotropic body, expanded in powers of one of the coordinates, G. V. Kolosov [6] gave an outline of a recurrence method for their construction and obtained, as particular cases, the results of A. Clebsch, M. Lévy and V. A. Steklov [16–17].

Without pausing to enumerate other important studies, we note the work of A. I. Lur'e [12–13] in which an operational method for the construction of solutions in the form of series in powers of one of the coordinates is proposed.

The solution of the problem of Almansi for a rod made of an arbitrarily anisotropic material will be considered below as an example of the question of problems of the theory of elasticity of anisotropic media relating to the plane (in the sense of the dependence of the stress on only two coordinates).

Let the components of the loading on the side surface of a prismatic rod be given in the form of a polynomial in powers of the axial coordinate z

$$\mathbf{f}(z, s) = \sum_{k=1}^{n} \mathbf{a}_k(s) \frac{z^k}{k!}, \tag{1}$$

where s is the arc length of the boundary of the cross-section.

The unknown stresses must satisfy the equilibrium equations for con-

tinuous media, the compatibility equations in terms of stresses obtained by substitution of the expressions for the strains in terms of the stresses in the compatibility conditions of Saint-Venant, the boundary conditions on the side surface and the conditions at an end, expressing the static equivalence of the stresses τ_{zx}, τ_{zy}, and σ_z to the projections of the resultant vector R_x, R_y, R_z and of the resultant moment M_x, M_y, M_z of the forces applied to the end.[1]

We will first differentiate the equilibrium equations, the compatibility conditions and the boundary conditions on the side surface $(n+1)$ times with respect to the coordinate z. Then the $(n+1)$-th derivative of the stresses will satisfy the same equations as the stresses, and the boundary conditions on the side surface will correspond to the case $\mathbf{f} = 0$.

Comparison of the relations satisfied by the $(n+1)$-th derivatives of the stresses with the equations in terms of stresses of the problem of Saint-Venant for anisotropic media shows that they coincide. Therefore the stresses are determined by $(n+1)$ successive integrations of the solution of the problem of Saint-Venant, found for anisotropic media by S. G. Lekhnitskii [10].

After simple manipulations we obtain the expressions for the unknown functions:

$$
\begin{aligned}
\sigma_x &= \sum_{k=0}^{n} \sigma_x^{(k)}(x, y)\,\frac{z^k}{k!} + \frac{\partial^2 \Phi}{\partial y^2}\,\frac{z^{n+1}}{(n+1)!}, \\
\sigma_y &= \sum_{k=0}^{n} \sigma_y^{(k)}(x, y)\,\frac{z^k}{k!} + \frac{\partial^2 \Phi}{\partial x^2}\,\frac{z^{n+1}}{(n+1)!}, \\
\tau_{xy} &= \sum_{k=0}^{n} \tau_{xy}^{(k)}(x, y)\,\frac{z^k}{k!} - \frac{\partial^2 \Phi}{\partial x \partial y}\,\frac{z^{n+1}}{(n+1)!}, \\
\tau_{zx} &= \sum_{k=0}^{n} \tau_{zx}^{(k)}(x, y)\,\frac{z^k}{k!} + \left(\frac{\partial \Psi}{\partial y} - \frac{C_1}{2}\,x^2\right)\frac{z^{n+1}}{(n+1)!}, \\
\tau_{zy} &= \sum_{k=0}^{n} \tau_{zy}^{(k)}(x, y)\,\frac{z^k}{k!} - \left(\frac{\partial \Psi}{\partial x} + \frac{C_2}{2}\,y^2\right)\frac{z^{n+1}}{(n+1)!}, \\
\sigma_z &= \sum_{k=0}^{n} \sigma_z^{(k)}(x, y)\,\frac{z^k}{k!} + \sigma_z^*(x, y)\,\frac{z^{n+1}}{(n+1)!} + \\
&\qquad + (C_1 x + C_2 y)\,\frac{z^{n+2}}{(n+2)!}.
\end{aligned}
\tag{2}
\tag{3}
$$

[1] The loading on the other end may not be given arbitrarily; its static equivalent is determined by the conditions of equilibrium of the rod as a whole.

Here $\sigma_x^{(k)}(x, y) \cdots, \tau_{zx}^{(k)}(x, y)$ are functions subject to determination;

$$
\begin{aligned}
a_{33}\sigma_z^*(x, y) &= \tfrac{1}{2}a_{35}C_1x^2 + \tfrac{1}{2}(a_{34}C_1 + a_{35}C_2)xy + \tfrac{1}{2}a_{34}C_2y^2 - \\
&- a_{31}\frac{\partial^2\Phi}{\partial y^2} - a_{32}\frac{\partial^2\Phi}{\partial x^2} + a_{36}\frac{\partial^2\Phi}{\partial x\partial y} + a_{34}\left(\frac{\partial\Psi}{\partial x} + \frac{C_2}{2}y^2\right) - \\
&- a_{35}\left(\frac{\partial\Psi}{\partial y} - \frac{C_1}{2}x^2\right) + A + B_1x + B_2y.
\end{aligned} \tag{4}
$$

The functions $\Phi(x, y)$ and $\Psi(x, y)$ are solutions of the system

$$
\begin{aligned}
L_2\Psi + L_3\Phi &= -D + (2a_{31}C_2 - a_{36}C_1 - \beta_{45}C_1)x + \\
&+ (a_{36}C_2 - 2a_{32}C_1 + \beta_{45}C_2)y, \\
L_3\Psi + L_4\Phi &= \beta_{14}C_2 + \beta_{25}C_1,
\end{aligned} \tag{5}
$$

for the boundary conditions on the contour Γ of the simply connected cross-section

$$
\Phi = 0, \quad \frac{\partial\Psi}{\partial s} = \frac{C_1}{2}x^2\frac{dy}{ds} - \frac{C_2}{2}y^2\frac{dx}{ds}; \tag{6}
$$

L_2, L_3 and L_4 are differential operators, determined by the formulae

$$
\begin{aligned}
L_2 &= \beta_{44}\frac{\partial^2}{\partial x^2} - 2\beta_{45}\frac{\partial^2}{\partial x\partial y} + \beta_{55}\frac{\partial^2}{\partial y^2}, \\
L_3 &= -\beta_{24}\frac{\partial^3}{\partial x^3} + (\beta_{25} + \beta_{46})\frac{\partial^3}{\partial x^2\partial y} - \\
&\qquad - (\beta_{14} + \beta_{56})\frac{\partial^3}{\partial x\partial y^2} + \beta_{15}\frac{\partial^3}{\partial y^3}, \\
L_4 &= \beta_{22}\frac{\partial^4}{\partial x^4} - 2\beta_{26}\frac{\partial^4}{\partial x^3\partial y} + \\
&\qquad + (2\beta_{12} + \beta_{66})\frac{\partial^4}{\partial x^2\partial y^2} 2\beta_{16}\frac{\partial^4}{\partial x\partial y^3} + \beta_{11}\frac{\partial^4}{\partial y^4},
\end{aligned} \tag{7}
$$

$\beta_{ik} = a_{ik} - a_{3i}\dfrac{a_{3k}}{a_{33}}$, a_{ik} are the elastic constants of the anisotropic medium, entering into the stress displacement relations; A, B_1, B_2, C_1, C_2, D are arbitrary constants.

Substituting the series (2)–(3) into the equilibrium equations, the compatibility equations and the boundary conditions (1), we arrive after

comparison of the coefficients of equal powers of z at a recurrent system of equations.

The structure of this system is such that its solution must begin with equations corresponding to $k = n$, and then consecutively pass on to equations corresponding to smaller values of k. We will show, how by knowing the functions $\sigma_x^{(k)}, \cdots, \tau_{zx}^{(k)}$ for $k = n+2, n+1, n, \cdots, \nu+1$, we may find them for $k = \nu$.

From the equations corresponding to the second, third and sixth compatibility equations in terms of displacements (if we admit them in the capacity of intermediate formulae)

$$\frac{\partial^2 \varepsilon_z^{(\nu)}}{\partial x^2} + \varepsilon_x^{(\nu+2)} = \frac{\partial \gamma_{zx}^{(\nu+1)}}{\partial x}, \quad \frac{\partial^2 \varepsilon_z^{(\nu)}}{\partial y^2} + \varepsilon_y^{(\nu+2)} = \frac{\partial \gamma_{zy}^{(\nu+1)}}{\partial y},$$

$$\frac{\partial^2 \varepsilon_z^{(\nu)}}{\partial x \partial y} = \frac{1}{2}\left[\frac{\partial \gamma_{yz}^{(\nu+1)}}{\partial x} + \frac{\partial \gamma_{zx}^{(\nu+1)}}{\partial y} - \gamma_{xy}^{(\nu+2)}\right],$$

we find by integration the function

$$\varepsilon_z^{(\nu)} = \int_{(x_0, y_0)}^{(x, y)} \chi_1^{(\nu)} dx + \chi_2^{(\nu)} dy + E^{(\nu)} + E_1^{(\nu)} x + E_2^{(\nu)} y, \tag{8}$$

where $E^{(\nu)}$, $E_1^{(\nu)}$, $E_2^{(\nu)}$ are arbitrary constants, $\chi_1^{(\nu)}$ and $\chi_2^{(\nu)}$ are the known functions of the solutions of the preceding systems

$$\chi_1^{(\nu)} = \int_{(x_0, y_0)}^{(x, y)} N_1^{(\nu)} dx + N_2^{(\nu)} dy, \quad \chi_2^{(\nu)} = \int_{(x_0, y_0)}^{(x, y)} N_2^{(\nu)} dx + N_3^{(\nu)} dy, \tag{9}$$

with

$$N_1^{(\nu)} = \frac{\partial \gamma_{zy}^{(\nu+1)}}{\partial x} - \varepsilon_x^{(\nu+2)},$$

$$N_2^{(\nu)} = \tfrac{1}{2}\left[\frac{\partial \gamma_{yz}^{(\nu+1)}}{\partial x} + \frac{\partial \gamma_{zx}^{(\nu+1)}}{\partial y} - \gamma_{xy}^{(\nu+2)}\right], \tag{10}$$

$$N_3^{(\nu)} = \frac{\partial \gamma_{zy}^{(\nu+1)}}{\partial y} - \varepsilon_y^{(\nu+2)}.$$

Then

$$a_{33}\sigma_z^{(\nu)} = \int_{(x_0, y_0)}^{(x, y)} \chi_1^{(\nu)} dx + \chi_2^{(\nu)} dy - a_{13}\sigma_x^{(\nu)} - a_{23}\sigma_y^{(\nu)} - a_{34}\tau_{zy}^{(\nu)} - \tag{11}$$

$$- a_{35}\tau_{zx}^{(\nu)} - a_{36}\tau_{xy}^{(\nu)} + E^{(\nu)} + E_1^{(\nu)} x + E_2^{(\nu)} y.$$

For the determination of the remaining stresses one has to solve the equations

$$\frac{\partial \sigma_x^{(\nu)}}{\partial x} + \frac{\partial \tau_{xy}^{(\nu)}}{\partial y} + \tau_{zx}^{(\nu+1)} = 0, \tag{12}$$

$$\frac{\partial \tau_{xy}^{(\nu)}}{\partial x} + \frac{\partial \sigma_y^{(\nu)}}{\partial y} + \tau_{zy}^{(\nu+1)} = 0, \tag{13}$$

$$\frac{\partial \tau_{zx}^{(\nu)}}{\partial x} + \frac{\partial \tau_{zy}^{(\nu)}}{\partial y} + \sigma_z^{(\nu+1)} = 0, \tag{14}$$

and the remaining compatibility conditions (the first, fourth and fifth) for the boundary conditions

$$\begin{aligned} \sigma_x^{(\nu)} \cos(n, x) + \tau_{xy}^{(\nu)} \cos(n, y) &= a_\nu(s), \\ \tau_{xy}^{(\nu)} \cos(n, x) + \sigma_y^{(\nu)} \cos(n, y) &= b_\nu(s), \\ \tau_{xz}^{(\nu)} \cos(n, x) + \tau_{yz}^{(\nu)} \cos(n, y) &= c_\nu(s), \end{aligned} \tag{15}$$

where $a_k(s)$, $b_k(s)$, $c_k(s)$ are the projections of the vector $\mathbf{a}_k(s)$.

The equilibrium equations (12)–(14) are satisfied identically, if one introduces functions $\Phi^{(\nu)}(x, y)$ and $\Psi^{(\nu)}(x, y)$ with the help of the expressions

$$\sigma_x^{(\nu)} = \frac{\partial^2 \Phi^{(\nu)}}{\partial y^2} - \eta_1, \quad \sigma_y^{(\nu)} = \frac{\partial^2 \Phi^{(\nu)}}{\partial x^2} - \eta_2, \quad \tau_{xy}^{(\nu)} = -\frac{\partial^2 \Phi^{(\nu)}}{\partial x \partial y} - \zeta, \tag{16}$$

$$\tau_{zx}^{(\nu)} = \frac{\partial \Psi^{(\nu)}}{\partial y} - \xi_1, \quad \tau_{zy}^{(\nu)} = -\frac{\partial \Psi^{(\nu)}}{\partial x} - \xi_2, \tag{17}$$

where ξ_1, ξ_2, η_1, η_2, ζ with opposite signs are particular solutions of the non-homogeneous equations (12)–(14).

Substitution of the expressions (16), (17) into Hooke's Law for utilization of the formula (11) and of the remaining compatibility conditions leads to the system

$$\begin{aligned} L_2\Psi^{(\nu)} + L_3\Phi^{(\nu)} = {} & \frac{a_{34}}{a_{33}} \chi_1^{(\nu)} - \frac{a_{35}}{a_{33}} \chi_2^{(\nu)} + \frac{a_{34}}{a_{33}} E_1^{(\nu)} - \frac{a_{35}}{a_{33}} E_2^{(\nu)} - \\ & - \int_{(x_0, y_0)}^{(x, y)} (M_1^{(\nu)} dx + M_2^{(\nu)} dy) - D^{(\nu)}, \end{aligned} \tag{18}$$

$$L_3\Psi^{(\nu)} + L_4\Phi^{(\nu)} = \frac{a_{36}}{a_{33}} N_2^{(\nu)} - \frac{a_{13}}{a_{33}} N_3^{(\nu)} - \frac{a_{23}}{a_{33}} N_1^{(\nu)},$$

where

$$M_1^{(\nu)} = \frac{\partial \gamma_{xy}^{(\nu+1)}}{\partial x} - 2\frac{\partial \varepsilon_x^{(\nu+1)}}{\partial y}, \qquad M_2^{(\nu)} = 2\frac{\partial \varepsilon_y^{(\nu+1)}}{\partial x} - \frac{\partial \gamma_{xy}^{(\nu+1)}}{\partial y}. \tag{19}$$

Introducing (16) and (17) into the boundary conditions (15), we have

$$\begin{aligned} \frac{d}{ds}\left(\frac{\partial \Phi^{(\nu)}}{\partial y}\right) &= a_\nu(s) + \eta_1 \cos(n, x) + \zeta \cos(n, y), \\ -\frac{d}{ds}\left(\frac{\partial \Phi^{(\nu)}}{\partial x}\right) &= b_\nu(s) + \zeta \cos(n, x) + \eta_2 \cos(n, y), \\ \frac{\partial \Psi}{\partial s} &= c_\nu(s) + \xi_1 \cos(n, x) + \xi_2 \cos(n, y). \end{aligned} \tag{20}$$

The conditions of single-valuedness of the functions $\Phi^{(\nu)}$ and $\Psi^{(\nu)}$ and of their derivatives in the case of a simply connected cross-section reduce to the relations [4]

$$\begin{aligned} &\iint \tau_{zx}^{(\nu+1)}\, dx\, dy + \oint a_\nu(s)\, ds = 0, \\ &\iint \tau_{zy}^{(\nu+1)}\, dx\, dy + \oint b_\nu(s)\, ds = 0, \\ &\iint [x\tau_{yz}^{(\nu+1)} - y\tau_{zx}^{(\nu+1)}]\, dx\, dy + \oint [xb_\nu(s) - ya_\nu(s)]\, ds = 0, \\ &\iint \sigma_z^{(\nu+1)}\, dx\, dy + \oint c_\nu(s)\, ds = 0, \end{aligned} \tag{21}$$

which may also be readily obtained by construction of equilibrium conditions of the part of the cylinder contained between the two sections $z = 0$ and z.

At the stage under consideration the solution contains the four new arbitrary constants $E^{(\nu)}$, $E_1^{(\nu)}$, $E_2^{(\nu)}$, $D^{(\nu)}$; simultaneously, the four conditions (21) are obtained for the determination of the arbitrary constants entering into the solution corresponding to the preceding stage.

We will now present the general evaluation of the constants. The multipliers of z^{n+2} and z^{n+1} contain six arbitrary constants C_1, C_2, A, B_1, B_2, D.

From the conditions of solubility of the boundary value problems for the functions $\Phi^{(n)}$ and $\Psi^{(n)}$ are found C_1, C_2, D (the last constant enters into the expressions for the functions Φ and Ψ) and A.

The formulae for the functions $\sigma_x^{(n)}, \cdots, \tau_{zx}^{(n)}$ contain the constants B_1, B_2 and four new constants $E^{(n)}$, $E_1^{(n)}$, $E_2^{(n)}$, $D^{(n)}$, i.e., again six constants. The constants $E^{(n)}$, $E_1^{(n)}$, $E_2^{(n)}$, $D^{(n)}$ are found from the conditions of solubility of the boundary problems for the functions $\Phi^{(n-1)}$ and $\Psi^{(n-1)}$. Therefore there enter into the formulae for the functions $\sigma_x^{(n-1)}, \cdots, \tau_{zx}^{(n-1)}$ the constants B_1, B_2 and four new constants $E^{(n-1)}$, $E_1^{(n-1)}$, $E_2^{(n-1)}$, $D^{(n-1)}$, etc.

At the last stage one has $n = 0$. The constants B_1, B_2 and four new constants $E^{(0)}$, $E_1^{(0)}$, $E_2^{(0)}$, $D^{(0)}$ enter into the formulae for the functions $\Phi^{(0)}$ and $\Psi^{(0)}$. These six constants corresponding to the usual solution of the problem of Saint-Venant must be determined from the end conditions for the stress resultants.

For the consideration of concrete problems the above procedure can often be essentially simplified. One may state several general considerations regarding the form of the loading for which such a simplification becomes possible.

The unwieldiness of the solutions of the boundary problems (18), (20) is not rarely due to the presence of terms $a_\nu(s)$, $b_\nu(s)$, $c_\nu(s)$ which do not contain explicitly the factors $\cos(n, x)$ and $\cos(n, y)$.

In fact, published solutions of particular cases of the problem studied for isotropic [3, 5, 15] and anisotropic [11, 7, 8] media as a rule are concerned with loadings which have the general form

$$
\begin{aligned}
a_k(s) &= g_k(s)\cos(n, x) - f_k(s)\cos(n, y),\\
b_k(s) &= f_k(s)\cos(n, x) + g_k(s)\cos(n, y),\\
c_k(s) &= l_k(s)\cos(n, x) + m_k(s)\cos(n, y).
\end{aligned}
\tag{22}
$$

For $g_k(s) = l_k(s) = m_k(s) = 0$ the loading (22) reduces to a force directed at each point of the side surface along the tangent to the boundary which lies in the cross-section (the case $g_0(s) = g_0$, $g_1(s) = g_1$ has been studied in [5]).

If $f_k(s), \cdots, m_k(s)$ represent on the contours the values of several polynomials in terms of the coordinates x, y, one may sometimes construct a solution in which the functions $\Psi^{(\nu)}$ and $\Phi^{(\nu)}$ are polynomials. For example, this takes place in the case of the elliptic cylinder, since for the ellipse the square of the direction cosines of the normal and of their products are special second order polynomials, and therefore the

conditions (20), after multiplication by $\cos(n, x)$ or $\cos(n, y)$, may be satisfied by polynomial expressions [7].

In the general case, however, one has to use the methods of N. I. Muskhelishvili [14] for the solution of the plane problems obtained.

We note that those problems for which the loading is characterized by coefficients $c_k(s)$, not containing explicitly the direction cosines of the normal, turn out to be more complicated.

Even the simplest problem of equilibrium of an elliptic cylinder, on the side surface of which a uniformly distributed load c_0 acts parallel to the axis, while the end is subject to the static equivalent of the axial loading, leads to stress functions represented in elliptic coordinates by infinite series with fairly complicated coefficients.

The present method of reduction of the space problem to the determination of a certain succession of functions of two coordinates may also be applied to other problems, for example, the solution of problems of equilibrium of thick anisotropic plates.

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ON NEW TYPES OF INTEGRAL EQUATIONS, SOLUBLE IN CLOSED FORM

F. D. GAKHOV

Rostov-on-the-Don, U.S.S.R.

The number of types of integral equations, exact solutions of which may be expressed in closed form, i.e., by quadratures in terms of functions characterizing the original equation without application of infinite processes, is not large.

For over a century the solutions of the classical equations of Abel

$$\int_a^x \frac{\varphi(t)}{(x-t)^\alpha}\,dt = f(x) \tag{1}$$

and Schlömilch

$$\frac{2}{\pi}\int_0^{\pi/2} \varphi(x \sin\theta)\,d\theta = f(x)$$

have been known.

The number of classical equations also includes those of the large group where the kernels represent the basis for different types of integral transforms [Fourier, Laplace, Mellin, Hankel, cosine-, sine-transforms, etc.]. The development of the theory of these equations, based on applications of the corresponding transforms, has been continued right up to the present day. In what follows the methods of integral transforms will not be used by us, and therefore we do not dwell on this wide class of equations which is important due to its applications.

Another idea forms the basis of the solution of the so-called characteristic singular integral equations with Cauchy type kernels and integrals, understood in the sense of the principal value,

$$a(t)\varphi(t) + \frac{b(t)}{\pi i}\int_L \frac{\varphi(\tau)}{\tau - t}\,d\tau = f(t), \tag{2}$$

first given by T. Carleman [1] for a particular case and developed later in all details by I. N. Vekua [2—3]. The idea of the solution consists of the analytic continuation of the relation (2) given on the contour L into the complex plane and of the transformation of the last into a Riemann boundary value problem. For this purpose use is made of the

Cauchy type integral

$$\Phi(z) = \frac{1}{2\pi i} \int_L \frac{\varphi(\tau)}{\tau - z} d\tau, \tag{3}$$

which has the unknown solution of the equation (2) as density.

Using the formulae of Sokhotskii

$$\Phi^{\pm}(t) = \pm \tfrac{1}{2}\varphi(t) + \frac{1}{2\pi i} \int_L \frac{\varphi(\tau)}{\tau - t} d\tau, \tag{A}$$

$$\varphi(t) = \Phi^{+}(t) - \Phi^{-}(t), \quad \frac{1}{\pi i} \int_L \frac{\varphi(\tau)}{\tau - t} d\tau = \Phi^{+}(t) + \Phi^{-}(t) \tag{A'}$$

for the boundary values, we obtain from (2) for the function $\Phi(z)$, piecewise analytic outside L, the boundary condition of the Riemann problem

$$\Phi^{+}(t) = \frac{a(t) - b(t)}{a(t) + b(t)} \Phi^{-}(t) + \frac{f(t)}{a(t) + b(t)} \tag{4}$$

with the supplementary condition $\Phi(\infty) = 0$. We obtain from the known solution of this problem the solution of the equation (2) in closed form.

In the same way, S. Gellerstedt [4], A. V. Bitsadze [5] and S. G. Mikhlin [6] solved the following integral equation, which is somewhat more general than (2):

$$a(t)\varphi(t) + \frac{b(t)}{\pi i} \int_L \left[\frac{1}{\tau - t} + k(t, \tau)\right] \varphi(\tau)\, d\tau = f(t), \tag{5}$$

where k is some given function encountered in the boundary value problems for mixed (elliptic-hyperbolic) differential equations.

F. D. Gakhov and L. I. Chibrikova [7] noted that the success in solving the equation [5] was ensured by the fact that its kernel

$$K(t, \tau) = \frac{1}{\tau - t} + k(t, \tau)$$

belongs to the class of automorphic functions, i.e., it remains invariant for a certain group of linear substitutions

$$\omega_0(t) \equiv t, \ \omega_1(t), \cdots, \ \omega_{n-1}(t),$$

$$K[\omega_k(t), \tau] = K(t, \tau),$$

or— more generally — it acquires for these substitutions a certain multiplier. Carleman's method of continuation into the complex plane remains applicable here. Instead of the Cauchy type integral (3) one uses here the integral with the automorphic kernel

$$\Phi(z) = \frac{1}{2\pi i} \int \left[\frac{1}{\tau - z} + k(z, \tau) \right] \varphi(\tau)\, d\tau, \tag{6}$$

where one uses for the reduction to a Riemann boundary value problem formulae which are quite analogous to Sokhotskii's formulae (A), (A').

Quite recently new types of integral equations have been discovered, the solutions of which may be given in closed form. They differ sharply only by the analytic character of the kernels from the singular integral equations with Cauchy type kernels considered above and from the equations of the convolution type [1], and above all they lie close to the classical equations of Abel [1]. The method of solution — utilization of the idea of Carleman of analytic continuation into the complex plane and of reduction to a Riemann boundary value problem — relates them to the singular equations with Cauchy type kernels. However, there is an essential difference; we will say more about this below.

In 1958, A. E. Heins and R. C. MacCamy [9] studied the integral equation

$$\int_0^\infty k(|x - t|)\varphi(t)\, dt = f(x) \tag{7}$$

with an analytic right hand side and a kernel of the form

$$k(w) = P(w) \log w + Q(w),$$

where P, Q are even integral functions.[2]

The equation (7) is an equation of the convolution type and it may be solved by use of Fourier (or Laplace) transforms. The degree of effectiveness of this method is determined on the basis of the degree of difficulty experienced in the evaluation of the integral transforms encountered. The authors proposed another method of solution. Using Carleman's

[1] Although the methods of solution of the two last types of equations differ essentially, they are very close to each other by their analytic nature.

[2] In [9], the kernel was given without modulus sign, but this was an obvious misprint, since all the succeeding reasoning was presented for an equation of the form (7).

idea they introduced the function

$$\Phi(z) = \int_0^\infty k(t - z)\varphi(t)dt, \tag{8}$$

analytic in the complex plane with cuts along the real positive axis.

The behavior of its limiting values $\Phi^{\pm}(x)$ on the sides of the cut when $z \to x \pm i0$ is determined by the presence in the kernel of the multi-valued function $\log(t - z) = \log|t - z| + i \arg(t - z)$. Defining the branch of the argument in the corresponding manner we will have

$$\log(t - z) \to \log|x - t| \mp \pi i, \quad z \to x \pm i0 \text{ for } t < x, \tag{9}$$

$$\log(t - z) \to \log|x - t|, \quad z \to x \pm i0 \text{ for } t > x. \tag{10}$$

Hence we obtain without difficulty

$$\Phi^{\pm}(x) = \int_0^\infty k(|x - t|)\varphi(t)\,dt \mp \pi i \int_0^x P(x - t)\varphi(t)\,dt, \tag{B}$$

or

$$\begin{aligned} \Phi^{+}(x) - \Phi^{-}(x) &= 2\pi i \int_0^x P(x - t)\varphi(t)\,dt, \\ \Phi^{+}(x) + \Phi^{-}(x) &= 2\int_0^\infty k(|x - t|)\varphi(t)\,dt. \end{aligned} \tag{B'}$$

These formulae (B) and (B′) are the analogues to Sokhotskii's formulae (A) and (A′) for the Cauchy type integral.[1]

Using the second of the formulae (B′) we reduce (7) to the Riemann boundary value problem

$$\Phi^{+}(t) = -\Phi^{-}(t) + 2f(t). \tag{11}$$

In order to make this problem completely definite, it is still necessary to give the behavior of the unknown function at infinity where, generally speaking, the function $\Phi(z)$ will have an essential singularity. Generally speaking, the last will be a singular point for $\Phi(z)$. If the representation (8) makes it possible to determine the asymptotic behavior of $\Phi(z)$ at infinity, and its behavior at the origin of coordinates is given (it is bounded or admits an integrable singularity), then one will know exactly

[1] In the simplest case $P \equiv Q \equiv 1$, these formulae may be obtained by integration of Sokhotskii's formulae.

the class of functions in which the solution of the problem (11) must be sought, and the solution of the last may be obtained in accordance with the general theory [10, 11].

Once $\Phi(z)$ has been determined as the solution of the boundary value problem, we obtain for the unknown solution of the integral equation (7) from the first formula (B′) the simple integral equation of the convolution type

$$\int_0^x P(x-t)\varphi(t)\,dt = \frac{1}{2\pi i}[\Phi^-(x) - \Phi^+(x)], \tag{12}$$

soluble with the help of the one-sided Laplace transform.

In [9] the idea of solution explained here has not been expressed sufficiently clearly; the entire attention of the authors was concentrated on the solution of an example (the diffraction equation the kernel of which is the Hankel function $H_0^{(1)}$), while the general reasoning is given schematically. The equation (7) under consideration leads to the very special type of boundary value problem (11) which makes it possible to obtain solutions of particular artificial examples, independently of the general theory; the authors did not even use the term boundary value problem and it is not quite easy for the reader to understand the essence of the method used. The absence of generality in the reasoning also prevents the authors themselves from exploiting with sufficient completeness the idea advanced by them.

Certain generalizations suggest themselves. For example, all the reasoning of the authors remaining unaltered, one may free oneself of the limitation to even integral functions P, Q. It is sufficient for this, if the absolute signs do not refer to the entire kernel, but only to the single logarithmic term, i.e., one can write the equation in the form

$$\int_0^\infty [P(x-t)\log|x-t| + Q(x-t)]\varphi(t)\,dt = f(x).$$

Careful study of the formulae (B′) and their comparison with the analogous Sokhotskii formulae (A′) leads immediately to the conclusion that the same method may be applied to the more general equation

$$a(x)\int_0^\infty [P(x-t)\log|x-t| + Q(x-t)]\varphi(t)\,dt - \pi i b(x)\int_0^x P(x-t)\varphi(t)\,dt = \tfrac{1}{2}f(x). \tag{13}$$

Utilization of the same analytic function (8) as before leads us by means of the formulae (10) to the Riemann boundary value problem in its general form

$$\Phi^+(x) = -\frac{a(x) - b(x)}{a(x) + b(x)}\Phi^-(x) + \frac{f(x)}{a(x) + b(x)}. \tag{14}$$

In order to be able to apply to the last the methods of the general theory, one has to demand that the coefficients $a(x)$, $b(x)$ satisfy a Hölder condition and the condition $a(x) \pm b(x) \neq 0$. The unknown function $\varphi(x)$ will then be determined again, as before, by the equation (12).

We will now consider the possibility of further generalizations for the preservation of the method. The analytic continuation into the complex plane necessarily assumes analyticity of the kernel in the original equation with respect to x. Further, the deduction of the formulae (B), (B') relies essentially on the presence of the logarithmic term in the kernel. The assumption of the difference character of the kernel is evidently not useful for the determination of the analytic character of the kernel; in fact, for such a character the kernel determined by the formula (8) as an analytic function may not have a singularity anywhere except on the contour of integration.[1] The assumption that P and Q are integral functions entails the boundedness of $\Phi(z)$ in the entire finite plane. In addition, the difference character of the function P is used for the determination of $\varphi(t)$ from the equation (12). Repudiation of the requirement of the difference character of Q and transition to the general case $Q(x, t)$ will be possible when the character of all singularities of the function

$$\int_0^\infty Q(z, t)\varphi(t)\, dt.$$

can be determined.

If this should be possible, all the reasoning would remain unaltered with the exception that the Riemann boundary value problem (14) must be solved in the class of functions with singularities of a given character at given points. The repudiation of the assumption of the difference character of P entails more serious consequences. First of all, as in the preceding work, it is necessary to determine the distribution of singularities of the function

$$\int_0^\infty P(z, t) \log (t - z)\varphi(t)\, dt$$

[1] Or on a contour, parallel to it.

and correspondingly to change the class of unknown solutions of the boundary value problem (14). However, in addition, the unknown solution $\varphi(t)$ of the equation must now be determined by us not from an equation with a difference kernel, but from an equation of the Volterra type of the first kind

$$\int_0^x P(x, t)\varphi(t)\,dt = \frac{1}{2\pi i}[\Phi^-(x) - \Phi^+(x)]. \tag{15}$$

Consequently, we may not assert in this case that the original equation is soluble in closed form; however, it may be reduced to the simpler form (15). For this, since the equation (15) is soluble unconditionally and uniquely, the qualitative side of the question (solubility or insolubility, evaluation of the number of linearly independent solutions or conditions of solubility) will be completely determined by the boundary value problem (14), and, in fact, by the magnitude of the index of the function

$$\frac{a(x) - b(x)}{a(x) + b(x)}.$$

We consider next the question of the possibility of a generalization in relation to the contour of integration. Can one deviate from the requirement that the contour is the real positive axis and proceed to an arbitrary curve? We limit ourselves only to the simplest question regarding the possibility of replacing the half-axis by any finite segment $[\alpha, \beta]$ of the real axis.

It is easily seen that all the reasoning leading to the formulae for the limiting values (9), (10) remains completely unaltered. However, one difficulty arises: the function represented by the formula

$$\Phi(z) = \int_\alpha^\beta k(t - z)\varphi(t)\,dt$$

will have a logarithmic branch point at infinity, and, consequently, it will be multi-valued in the plane cut along the segment $[\alpha, \beta]$. Without dwelling on the study of the general case, we will show how this difficulty may be removed in the case $P \equiv 1$. Let there be given the equation

$$a(x)\int_\alpha^\beta [\log|t - x| + Q(x, t)]\varphi(t)\,dt - \pi i b(x)\int_\alpha^x \varphi(t)\,dt = \tfrac{1}{2}f(x). \tag{16}$$

In order to remove the multi-valuedness, we introduce an auxiliary analytic function in the following form:

$$\Phi(z) = \int_\alpha^\beta \left[\log\left(1 - \frac{\alpha - t}{\alpha - z}\right) + Q(z,t)\right]\varphi(t)\,dt = \tag{17}$$

$$= \int_\alpha^\beta [\log(t - z) + Q(z,t)]\varphi(t)\,dt - \log(\alpha - z)\int_\alpha^\beta \varphi(t)\,dt.$$

It is readily seen that $\Phi(z)$ will be a single-valued function (if Q is single-valued). Letting

$$\int_\alpha^\beta \varphi(t)\,dt = A, \tag{18}$$

we obtain for the limiting values on the contour the formulae

$$\Phi^+(x) - \Phi^-(x) - 2\pi i A = -2\pi i \int_\alpha^x \varphi(t)\,dt,$$

$$\Phi^+(x) + \Phi^-(x) + 2A\log(x - \alpha) = \int_\alpha^\beta [\log|t - x| + Q(x,t)]\varphi(t)\,dt. \tag{B''}$$

The given integral equation leads to the boundary value problem

$$\Phi^+(x) = -\frac{a(x) - b(x)}{a(x) + b(x)}\Phi^-(x) + \frac{f(x) + 2\pi i A\,b(x) - 2A\,a(x)\log(x - \alpha)}{a(x) + b(x)}.$$

Solving the problem obtained and assuming A to be a known constant, we will, in the end, determine this constant from the condition (18). For this purpose one needs in each case a study of the question whether the equality (18) may not degenerate into an identity. In the last case A enters into the solution as an additional arbitrary constant.

Certain other possible generalizations will be considered later.

Quite recently a paper has appeared by K. D. Sakaliuk [12], which is basic for ideas related to those discussed here. It gives a closed form solution of the new type of integral equation

$$a(x)\int_\alpha^x \frac{\varphi(t)dt}{(x - t)^\mu} + b(x)\int_x^\beta \frac{\varphi(t)dt}{(t - x)^\mu} = f(x), \tag{19}$$

$$0 < \mu < 1,$$

a generalized classical Abel equation (1).[1] Here a, b, f are certain given real functions. As basis for the presented process of solution it will be sufficient to assume that the coefficients $a(x)$, $b(x)$ have derivatives which satisfy a Hölder condition and that $f(x)$ is required to have a derivative satisfying a Hölder condition everywhere except at the ends α, β; at the ends, $f(x)$ must vanish with an order less than unity.

The fulfillment of these conditions ensures the possibility of finding solutions in the class of functions

$$\varphi(x) = \frac{\varphi^*(x)}{(x-\alpha)^{1-\mu-\varepsilon_1}(\beta-x)^{1-\mu-\varepsilon_2}}, \tag{20}$$

where ε_1, $\varepsilon_2 > 0$ and $\varphi^*(x)$ satisfies a Hölder condition.

Continuation into the complex plane is achieved by means of the function

$$\Phi(z) = [(z-\alpha)(\beta-z)]^{(\mu-1)/2} \int_\alpha^\beta \frac{\varphi(t)}{(t-z)^\mu}\,dt, \tag{21}$$

analytic in the plane with a cut along the segment $[\alpha, \beta]$; the function

$$[(z-a)(b-z)]^{(\mu-1)/2}(t-z)^{-\mu}$$

denotes here a branch of this function in the plane, cut along $[\alpha, \beta]$.

It is readily seen from the expansion of $\Phi(z)$ about corresponding points, that it exhibits at infinity and at the ends α, β respectively the orders

$$\Phi(z) = O\left(\frac{1}{|z|}\right), \quad O\left(\frac{1}{|z-\alpha|^{(1-\mu)/2}}\right), \quad O\left(\frac{1}{|z-\beta|^{(1-\mu)/2}}\right). \tag{22}$$

For the sake of brevity we introduce the notation

$$[(z-\alpha)(\beta-z)]^{(1-\mu)/2} = R(z).$$

On the upper side of the cut we attribute the argument zero to the numbers $z-a$ and $b-z$ in the neighborhood of the point x; then their arguments on the lower side must be assumed to be equal to 2π. We define the argument of $t-z$ on the sides of the cut in the same manner as was done in the deduction of the formulae (9). Then we obtain

[1] Reference should still be made to an idea developed in the paper [13] by T. Carleman.

from the expression

$$\Phi(z) = \frac{1}{R(z)} \left[\int_{\alpha}^{x} \frac{\varphi(t)}{(t-z)^{\mu}} dt + \int_{x}^{\beta} \frac{\varphi(t)}{(t-z)^{\mu}} dt \right]$$

the following formulae for the limiting values of $\Phi(z)$ on the sides of the cut:

$$\text{(C)} \qquad \begin{aligned} \Phi^{+}(x) &= \frac{1}{R(x)} \left[e^{\mu\pi i} \int_{\alpha}^{x} \frac{\varphi(t)}{(x-t)^{\mu}} dt + \int_{x}^{\beta} \frac{\varphi(t)}{(t-x)^{\mu}} dt \right], \\ \Phi^{-}(x) &= -\frac{1}{R(x)} \left[\int_{\alpha}^{x} \frac{\varphi(t)}{(x-t)^{\mu}} dt + e^{\mu\pi i} \int_{x}^{\beta} \frac{\varphi(t)}{(t-x)^{\mu}} dt \right]. \end{aligned}$$

The formula (21) and those following from it,

$$\text{(C')} \qquad \begin{aligned} \int_{\alpha}^{x} \frac{\varphi(t)dt}{(x-t)^{\mu}} &= R(x) \frac{e^{\mu\pi i}\Phi^{+}(x) + \Phi^{-}(x)}{e^{2\mu\pi i} - 1}, \\ \int_{x}^{\beta} \frac{\varphi(t)}{(t-x)^{\mu}} dt &= -R(x) \frac{\Phi^{+}(x) + e^{\mu\pi i}\Phi^{-}(x)}{e^{2\mu\pi i} - 1}, \end{aligned}$$

play here the same role as Sokhotskii's formulae (A), (A') for the solution of the singular integral equation with a Cauchy type kernel (2) or the formulae (B), (B') for the solution of the integral equations with logarithmic kernels (7).

With the help of the formulae (C') the equation (19) is reduced to the Riemann boundary value problem

$$\text{(23)} \qquad \Phi^{+}(x) = \frac{e^{\mu\pi i}b(x) - a(x)}{e^{\mu\pi i}a(x) - b(x)} \Phi^{-}(x) + \frac{(e^{2\mu\pi i} - 1)f(x)}{R(x)[e^{\mu\pi i}a(x) - b(x)]}.$$

Solving the last we obtain the solution of the original equation (18) from one of the Abel equations (C') which are known to be soluble unconditionally and uniquely. Consequently, the equation (19) is soluble in closed form.

It may be shown without serious difficulties that a solution of the boundary problem (23) in the class of functions satisfying the conditions (21) is equivalent to a solution of the equation (19) in the class of functions (20). Thus, the questions of the solubility of the equation (18) — number

of linearly independent solutions or conditions of solubility — are determined by the magnitude of the index of the function

$$\frac{e^{\mu\pi i}b(x) - a(x)}{e^{\mu\pi i}a(x) - b(x)}.$$

We will study the question of the extension of the method to more general classes of equations. A natural generalization is represented by the introduction into the kernel of some integral single-valued function of the difference of the arguments as a factor. As a result two new difficulties arise. The first is the determination of the character of the singularity of the auxiliary analytic function at infinity representing the continuation into the complex plane. The second is the final stage of the solution, the determination of the unknown functions from equations analogous to (C'). The last will not be Abel equations, but equations of the convolution type, soluble with the help of the one-sided Laplace transform. The degree of the effectiveness of the solution of this equation will be determined by the character of the complications encountered in performing the corresponding Laplace transforms (and inversions).

Unfortunately, we do not have at the present time sufficiently simple completed examples which we could present as illustrations of the above reasoning. We consider the question of a further expansion of the classes of integral functions, soluble in closed form. In what follows, for the sake of brevity, we will call a kernel of the form $1/(t - x)$ polar, a kernel of the form $\log|t - x|$ logarithmic, and a kernel of the form $|t - x|^{-\alpha}$ $(0 < \alpha < 1)$ fractional. We have presented above in closed form the solutions of the equations (2), (13), (18) with polar, logarithmic or fractional kernels respectively.[1] It is now natural to raise the question regarding the search for solutions in closed form for types of equations the kernels of which contain different combinations of singularities of the above three types. For this purpose singularities may be combined by additions as well as by multiplications. Work in this direction is only in its early stages and therefore we can report here only a few facts and have to limit ourselves to formal statements regarding possible means of research.

Here the starting point must be formed by the study of formulae for the limiting values of functions, continuing analytically into the complex plane equations given on contours. In the case of sums of singularities analogous formulae may again be obtained by addition

[1] We refer here to the dominant part of the kernel on which depends the behavior of the limiting values of the functions representing the continuation into the complex plane.

of the corresponding formulae (A), (A'), (B), (B'), (C), (C'). For the superposition of singularities in the form of products the corresponding formulae must be deduced anew.

We will demonstrate these considerations by use of an example of a very simple combination when the kernel is the sum of a polar and a logarithmic singularity without the complication of additional terms, while segments of the real axis are to serve as contour of integration.

In this case the continuation into the complex plane is carried out, of course, by the function

$$\Phi(z) = \frac{p}{2\pi i}\int_{\alpha}^{\beta}\frac{\varphi(t)}{t-z}\,dt + \frac{1}{2\pi i}\int_{\alpha}^{\beta}\varphi(t)\log\left(1-\frac{\alpha-t}{\alpha-z}\right)dt,$$

where p is some constant.

Reasoning in the same way as in the deduction of the formulae (A), (A'), (B), (B'), (B''), we obtain the two pairs of formulae

(AB)
$$\Phi^{\pm}(x) = \pm\frac{p}{2}\varphi(x) + \frac{p}{2\pi i}\int_{\alpha}^{\beta}\frac{\varphi(x)}{t-x}\,dt \mp \frac{1}{2}\int_{\alpha}^{x}\varphi(t)\,dt +$$

$$+\frac{1}{2\pi i}\int_{\alpha}^{\beta}\log|t-x|\varphi(t)\,dt - \frac{1}{2\pi i}[\log(x-\alpha) \mp \pi i]\int_{\alpha}^{\beta}\varphi(t)\,dt,$$

(A'B')
$$\Phi^{+}(x) - \Phi^{-}(x) - A = p\varphi(x) - \int_{\alpha}^{x}\varphi(t)\,dt,$$

$$\Phi^{+}(x) + \Phi^{-}(x) + \frac{1}{\pi i}A\log(x-\alpha) =$$

$$= \frac{p}{\pi i}\int_{\alpha}^{\beta}\frac{\varphi(t)dt}{t-x} + \frac{1}{\pi i}\int_{\alpha}^{\beta}\log|t-x|\varphi(t)\,dt,$$

where

(24)
$$A = \int_{\alpha}^{\beta}\varphi(t)\,dt.$$

It is easily seen from a study of the last pair of formulae that the

integral equations of the form

$$(25)\quad a(x)\left[[p\varphi(x)-\int_{\alpha}^{x}\varphi(t)\,dt\right]+\frac{b(x)}{\pi i}\int_{\alpha}^{\beta}\left[\frac{p}{t-x}+\log|t-x|\right]\varphi(t)dt=f(x),$$

where $a(x)$, $b(x)$, $f(x)$ are certain given functions,[1] may be reduced to the Riemann boundary value problem

$$(26)\quad \Phi^{+}(x)=\frac{a(x)-b(x)}{a(x)+b(x)}\Phi^{-}(x)+\frac{f(x)+Aa(x)-\frac{1}{\pi i}Ab(x)\log(x-\alpha)}{a(x)+b(x)}.$$

We will now assume that the functions $\Phi^{\pm}(z)$ have been found from the solutions of the boundary problem. The solution of the original equation may then be obtained by use of the first of the formulae (A', B').

Introducing there the notation

$$\int_{\alpha}^{x}\varphi(t)\,dt=u,\quad \frac{1}{p}[\Phi^{+}(x)-\Phi^{-}(x)-A]=g(x),$$

we obtain for the determination of u the linear first order differential equation

$$u'-\frac{1}{p}u=g(x);$$

from its solution

$$(27)\qquad u=e^{x/p}\left[\int_{\alpha}^{x}g(x)e^{-(x/p)}dx+C\right]$$

we deduce the solution of the original equation

$$\varphi(x)=u'=\frac{1}{p}e^{x/p}\left[\int_{\alpha}^{x}g(x)e^{-(p/x)}\,dx+C\right]+g(x).$$

Taking into consideration that, by (24), one has $u(\beta)=A$, we find from (27) the value of the arbitrary constant

$$C=e^{-(\beta/p)}A-\int_{\alpha}^{\beta}g(x)e^{-(x/p)}\,dx.$$

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[1] It is sufficient as a basis for all the following operations on these functions to require that they satisfy a Hölder condition and the conditions $a(x)\pm b(x)\neq 0$.

This equation renders the condition for the determination of A. The question whether this equation may be satisfied identically requires further study. Thus, the equation (26) may be solved in closed form. The qualitative side of the problem is determined by the magnitude of the index of the coefficient $\{a(x) - b(x)\}/\{a(x) + b(x)\}$ of the Riemann problem.

We will try to draw a few conclusions of a general character.

In the above reasoning two circumstances have had essential significance. First, the possibility of expressing all terms of the integral equation, containing the unknown function, by the limiting values of an auxiliary analytic function (reduction of the equation to a boundary value problem). Second, the presence of a link between the unknown function $\varphi(x)$ and the limiting values $\Phi^{\pm}(x)$ which permits obtaining the solution for φ (for conditions that $\Phi^{\pm}(x)$ are already known) in closed form. [1] In the case considered both these conditions were fulfilled and a successful solution was achieved.

For the study of other combinations (fractional-logarithmic, polar-fractional, etc.) it is not difficult to write down types of integral equations which lead to boundary value problems. Due to the lack of symmetry of the formulae (C), (C'), one may write down in the presence of a fractional singularity several different types of such equations. However, in the second part, namely, the search for relations usable for the solution of the equations for $\varphi(x)$ in closed form, serious difficulties are encountered. We are not here in a position to present any final results.

In the preceding reasoning we have not been concerned with the solution of given equations, but with the search for types of equations which could be solved by a certain definite method. Such an effort bears a somewhat artificial character and this reduces the value of the investigation. However, among the rather varied types of equations yielding solutions, without doubt one will also find equations which are encountered in practical problems. The existence of even one such case will be sufficient as justification of this investigation. It is known that the integrals of the linear differential equations of the Fuchs class involve only polar, fractional and logarithmic singularities. Since these equations find wide application in problems of mathematical physics, it is natural to anticipate that also integral equations with such kernels will find applications. This circumstance should greatly raise the interest in their study.

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[1] In the case of a kernel which contains additional terms besides polar-logarithmic terms, further difficulties would be encountered in resolving the question of possible singularities of the analytic function $\Phi(z)$.

Much of what has been said here is not concerned with the character of solutions of problems, but only their formulation.

This explains the novelty and the range of the problems affected. We hope that the fundamental reasoning discussed by us may serve the cause of a very rapid development of this new section of the theory of integral equations.

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GEOMETRIC CRITERIA FOR THE MEMBRANE STATE OF STRESS IN ELASTIC THIN SHELLS

A. L. GOL'DENVEIZER
Moscow, U.S.S.R.

V. E. Novodvorskii has suggested [1] that the state of stress of a thin elastic shell will be a membrane state far away from lines on which one has boundary conditions or conditions of conjugateness, if the middle surface of the shell is rigid, i.e., one may not encounter infinitesimal bending for given support conditions of the edges. This hypothesis in its simplest formulation does not stand verification by means of examples and it has been established that it requires effectively a number of restrictions which will be discussed in the following.

In the present paper, the generalized concept of infinitesimal bending is introduced and with the help of this concept new geometric criteria for the membrane state of stress are formulated which remain in force also in those cases when V. E. Novodvorskii's hypothesis fails to apply.

§ 1. Let the surface S be referred to lines of curvature α, β for which the first and second quadratic forms of S may be written in the form

$$\mathrm{I} = A^2 d\alpha^2 + B^2 d\beta^2,$$

$$\mathrm{II} = L d\alpha^2 + N d\beta^2.$$

Then the deformation of the surface S may be given with the help of six strain components for which we obtain, as usual in shell theory [2, 3],

$$\varepsilon_1 = \frac{A' - A}{A}, \quad \omega = \frac{\pi}{2} - \chi, \quad \varepsilon_2 = \frac{B' - B}{B},$$

$$\kappa_1 = -\frac{L'}{A^2} + \frac{L}{A^2} + \frac{L}{A^2}\varepsilon_1, \quad \kappa_2 = -\frac{N'}{B^2} + \frac{N}{B^2} + \frac{N}{B^2}\varepsilon_2,$$

$$\tau = -\frac{M'}{AB}.$$

In these formulae, primes (') denote quantities referring to the deformed surface, χ is the angle between the coordinate lines on the middle surface and M' is the mean coefficient of the second quadratic form of the deformed surface.

We construct the expressions

$$V_1 = \varepsilon_1^2 + \varepsilon_2^2 + 2\sigma\varepsilon_1\varepsilon_2 + 2(1-\sigma)\frac{\omega^2}{4}, \tag{1.1}$$

$$V_2 = \lambda^2[\kappa_1^2 + \kappa_2^2 + 2\sigma\kappa_1\kappa_2 + 2(1-\sigma)\tau^2],$$

where λ is some characteristic linear dimension of the surface and σ is a number, satisfying the inequality $0 \leq \sigma \leq \frac{1}{2}$, which later on will be identified with Poisson's ratio.

If for a given deformation one has the equality

$$\iint_G V_1 AB d\alpha d\beta = \frac{1}{\eta}\iint_G V_2 AB d\alpha d\beta,$$

where G is the region corresponding to all points of S, then the non-dimensional number η will be called the absolute characteristic of the deformation.

Non-trivial infinitesimal bending is a deformation for which ε_1, ε_2, ω are equal to zero, while simultaneously κ_1, κ_2, τ do not vanish identically. Then and only then is this a deformation whose absolute characteristic is equal to infinity.

For trivial bending deformation, i.e., displacements as a rigid body, η remains undetermined. We will assume that in this case $\eta = \infty$, as any bending state, including the trivial one, may be considered to be a deformation with an infinitely large absolute characteristic.

The case when $\kappa_1 \equiv \kappa_2 \equiv \tau \equiv 0$, while ε_1, ε_2, ω are not simultaneously identically zero, i.e., the deformation which may occur in a surface held between two rigid forms, will not be taken into consideration.

The concept of the absolute characteristic of deformation η remains to some degree indefinite, since λ depends on our arbitrary rule. This is inessential, since we are only interested in the order of magnitude of η.

§ 2. We will construct below examples of the deformation of surfaces which are important for the theory of shells and which have arbitrarily large absolute characteristics η. For this purpose we will rely on the following formulae relating the strain components to the projections of the displacements for a surface S referred to lines of curvature [2, 3]:

$$\varepsilon_1 = \frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta}v - \frac{w}{R_1}, \quad \varepsilon_2 = \frac{1}{B}\frac{\partial v}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha}u - \frac{w}{R_2}, \tag{2.1}$$

$$\omega = \frac{A}{B}\frac{\partial}{\partial \beta}\left(\frac{u}{A}\right) + \frac{B}{A}\frac{\partial}{\partial \alpha}\left(\frac{v}{B}\right),$$

$$\kappa_1 = \frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial w}{\partial\alpha} + \frac{u}{R_1}\right) + \frac{1}{AB}\frac{\partial A}{\partial\beta}\left(\frac{1}{B}\frac{\partial w}{\partial\beta} + \frac{v}{R_2}\right),$$

$$\kappa_2 = \frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial w}{\partial\beta} + \frac{v}{R_2}\right) + \frac{1}{AB}\frac{\partial B}{\partial\alpha}\left(\frac{1}{A}\frac{\partial w}{\partial\alpha} + \frac{u}{R_1}\right),$$

$$\tau = \frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{1}{B}\frac{\partial w}{\partial\beta} + \frac{v}{R_2}\right) - \frac{1}{AB}\frac{\partial A}{\partial\beta}\left(\frac{1}{A}\frac{\partial w}{\partial\alpha} + \frac{u}{R_1}\right) + \frac{1}{R_1}\left(\frac{1}{B}\frac{\partial u}{\partial\beta} - \frac{1}{AB}\frac{\partial B}{\partial\alpha}v\right),$$

where R_1, R_2 are principal radii of curvature.

§ 3. We consider an arbitrary surface S and give its points displacements, determined by the formulae

$$u = 0,\ v = 0,\ w = w_0 \operatorname{Re}\{e^{nf}\}, \tag{3.1}$$

where w_0 and f are certain functions of α, β the first of which is real and the second, generally speaking, complex, while n is a constant.

In studying (2.1) we satisfy ourselves that the formulae for κ_1, κ_2, τ contain second order derivatives of w, while the formulae for ε_1, ε_2, ω do not involve derivatives of w.

It follows therefore that for unbounded growth of n the quantities $\lambda\kappa_1$, $\lambda\kappa_2$, $\lambda\tau$ will become essentially larger than ε_1, ε_2, ω, and we will have a deformation with an arbitrarily large absolute characteristic. The functions w_0 and f remain quite arbitrary in this case; the first of them may be seen to be such that it will comprise boundary conditions of a very wide class, the second may be seen to be such that w remains bounded for $n \to \infty$. For this purpose, it is sufficient to require that in the region of interest here $\operatorname{Re}\{f\} \leq 0$. As a result, we obtain either a quickly oscillating deformation, if $\operatorname{Re}\{f\} \equiv 0$, or a deformation localized near a certain line on which $\operatorname{Re}\{f\} = 0$.

We will call this a deformation with a large variation.

Thus, a surface of arbitrary shape and for very general support conditions of its edges (including also supports which induce bending) may have deformations with an arbitrarily large absolute characteristic; this will be deformations with large variation.

§ 4. We consider an arbitrary surface S, the displacements of the edges of which, are subject to definite limitations. In other words, there must be fulfilled the boundary conditions included in the requirement that

the displacements in a definite direction are to vanish. Further, we assume that the boundary conditions may be decomposed into tangential conditions (the formulation of which involves only u and v) and non-tangential conditions (involving w) and, finally, we assume that together the boundary conditions ensure the rigidity of S, but single, purely tangential conditions will be insufficient for this purpose.

We reject the non-tangential boundary conditions and construct the bending deformation $\{u_1, v_1, w_1\}$ which, by assumption, will remain possible for this. We superimpose on $\{u_1, v_1, w_1\}$ the deformation $\{0, 0, w_2\}$. Then the deformation $\{u_1, v_1, w_1 + w_2\}$ will satisfy the former tangential boundary conditions, because w does not enter into the last. We choose w_2 such that the neglected non-tangential boundary conditions are fulfilled and for this purpose the absolute value of w_2 must fade rapidly with the inward distance from the edge of the region. The formulae (3.1) give such a deformation, if it is assumed in them that on the edge $\operatorname{Re}\{f\} = 0$, in the region G the quantity $\operatorname{Re}\{f\} > 0$, and that it grows monotonically in absolute value with the inward distance from the edge of G. In the deformation $\{u_1, v_1, w_1 + w_2\}$, the components κ_1, κ_2, τ will everywhere dominate the components ε_1, ε_2, ω. Far away from the edges this will be so because there one considers a deformation which differs little from bending. Near the edges, this will be so because there the second derivatives of w in the expressions for κ_1, κ_2, τ will attain large values.

Thus, if the rigidity of the surface is not guaranteed by one of the tangential boundary conditions, the surface may undergo a deformation with arbitrarily large absolute characteristic; these deformations will have large variations only near the edges of the surface.

§ 5. We consider an arbitrary surface S which differs little from a plane. We assume that one may establish on S a coordinate system in which the α and β lines are lines of curvature, the parameters α and β are non-dimensional and the coefficients of the first quadratic form A, B are small compared with the principal radii of curvature R_1, R_2. Then the displacements

$$u = 0, \quad v = 0, \quad w = w(\alpha, \beta),$$

where $w(\alpha, \beta)$ is an arbitrary function, will, generally speaking, correspond to a deformation with an arbitrarily large absolute characteristic, since in (2.1), in the formulae for ε_1, ε_2, ω, the displacement w either does not enter at all or is divided by R_1 or R_2, and it enters into the formulae for κ_1, κ_2, τ without these denominators. An exception is

presented by the case when on a large part of S the moduli of the derivatives of w with respect to α and β are significantly less than w. The arbitrariness of w may be used to satisfy the geometric boundary conditions, to avoid the above mentioned exceptional case and to obtain a deformation with not large variation.

Thus, an arbitrary sufficiently shallow surface S may for any geometric conditions on the edges experience a deformation with an arbitrarily large absolute characteristic and with not large variation.

§ 6. Let S be some part of a second order surface. Then, as is known [2, 4, 5], the problem of the construction of infinitesimal bending of S leads to the integration of the Cauchy-Riemann equations

$$\frac{\partial \xi}{\partial \alpha} - \frac{\partial \eta}{\partial \beta} = 0, \quad \frac{\partial \xi}{\partial \beta} + \frac{\partial \eta}{\partial \alpha} = 0 \tag{6.1}$$

in the region G corresponding to S.

If G does not contain points at which the selected system of curvilinear coordinates has on S a singularity, then there exists a mutually single-valued correspondence between the set of all functions of the complex argument $z = \alpha + i\beta$, analytic in the closed region $\bar{G}$, and the set of all infinitesimal bending deformations of S. The limitation of free displacements of the edge of the surface S corresponds to certain boundary conditions which must be imposed on the contour γ of the region G for the real and imaginary parts of the analytic function $f(z)$. In particular, if at some point of the boundary γ the projection of the displacement on the tangential plane must vanish (complete fixing of the point against tangential displacements), then the analytic function $f(z)$ must vanish at the corresponding point of γ.

It follows from the uniqueness theorem for analytic functions that, if over any arbitrarily small part of the edge complete clamping against tangential displacements is realized, then $f(z)$ must be equal to zero at all points of G, i.e., infinitesimal bending is impossible. The surface S will be rigid. We will show that in this case, for certain supplementary conditions, the surfaces S may experience deformation with arbitrarily large absolute characteristics and this will not be a deformation with large variation. This statement will be proved after the proof of the following lemma.

LEMMA. Let there be given a region G with boundary γ. Let there be separated from γ a part γ_0 all points of which lie inside some circle with radius ρ. In the region G there are points the smallest distance of each of which from the points of γ_0 is larger than some sufficiently large

number R. Then, for arbitrarily small ρ, one may construct in the region G functions $a(\alpha, \beta)$, $b(\alpha, \beta)$ with arbitrarily small absolute values such that the equations

$$\frac{\partial \xi}{\partial \alpha} - \frac{\partial \eta}{\partial \beta} = a(\alpha, \beta), \quad \frac{\partial \xi}{\partial \beta} + \frac{\partial \eta}{\partial \alpha} = b(\alpha, \beta)$$

will have in the region G a solution in which $\xi = \eta = 0$ on γ_0 and $|\xi + i\eta|$ will attain in G sufficiently large values.

PROOF. We denote by z_1 and z_2 the coordinates of the ends of the segment γ_0 and construct the function

$$\xi + i\eta = f(z) = (z - z_1)(z - z_2).$$

For any point on γ_0, we have

$$|f(z)| = |z - z_1|\,|z - z_2| < 4\rho^2,$$

and for any of the points of the region G, the distances of which from the points of γ_0 is larger than a number R, one may write

$$|\xi + i\eta| > R^2.$$

Further, the following relations apply on γ_0:

$$\left|\frac{\partial \xi}{\partial \alpha} + i\frac{\partial \eta}{\partial \alpha}\right| = \left|\frac{\partial}{\partial z} f(z)\right| = |(z - z_1) + (z - z_2)| \leq 2\rho,$$

$$\left|\frac{\partial \xi}{\partial \beta} + i\frac{\partial \eta}{\partial \beta}\right| = \left|i\frac{\partial}{\partial z} f(z)\right| = |i(z - z_1) + i(z - z_2)| \leq 2\rho.$$

We construct in G the function $f_0(z) = \xi_0 + i\eta_0$ which coincides on γ_0 with $f(z)$ and everywhere in G has an arbitrarily small modulus together with its first order derivatives (this is possible because on γ_0 the function $f(z)$ has these properties). The function $f(z) - f_0(z)$ will have for sufficiently large R and sufficiently small ρ all the required properties, if one understands under the conditions of the lemma by $a(\alpha, \beta)$ and $b(\alpha, \beta)$

$$a(\alpha, \beta) = +\frac{\partial \xi_0}{\partial \alpha} - \frac{\partial \eta_0}{\partial \beta}, \quad b(\alpha, \beta) = \frac{\partial \xi_0}{\partial \beta} + \frac{\partial \eta_0}{\partial \alpha}.$$

No difficulties are experienced in extending this result with the help of the theory of generalized functions [5] to a very wide class of surfaces of positive curvature. Therefore one may state the proposition that in general, if the rigidity of a surface is achieved at the expense of clamping over a sufficiently small part of the edge, then this surface may experience deformations with a small variation and with arbitrarily large absolute characteristic.

§ 7. We consider a cylinder of arbitrary cross-section and refer it to the longitudinal coordinate x (along the generators) and s (along the cross-section). Then $A = B = 1$, $R_1 = \infty$, $R_2 = R(s)$ and the formulae (2.1) assume the form

$$\varepsilon_1 = \frac{\partial u}{\partial x}, \quad \varepsilon_2 = \frac{\partial v}{\partial s} - \frac{w}{R}, \quad \omega = \frac{\partial u}{\partial s} + \frac{\partial v}{\partial x},$$
$$\kappa_1 = \frac{\partial^2 w}{\partial x^2}, \quad \kappa_2 = \frac{\partial}{\partial s}\left(\frac{\partial w}{\partial s} + \frac{v}{R}\right), \quad \tau = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial s} + \frac{v}{R}\right). \tag{7.1}$$

Let the cylinder be bounded by two rectilinear generators $s = 0$ and $s = s_0$ and the curvilinear edges $x = 0$ and $x = l$, so that the region G is determined by the inequalities

$$0 \leq x \leq l, \quad 0 \leq s \leq s_0.$$

If all points of the edge $x = 0$ are fixed against tangential displacements u, v, then bending, i.e., deformation for which

$$\varepsilon_1 = \varepsilon_2 = \omega = 0,$$

is impossible. This is well known in the theory of surfaces and may be readily verified with the help of the formulae (7.1).

At the same time, for sufficiently large l, the cylinder may have for such clamping conditions of the edge $x = 0$ a deformation with arbitrarily large absolute characteristic.

Consider as an example the deformation characterized by the displacements

$$u = 0, \quad v = \bar{\omega}x, \quad w = R\frac{d\bar{\omega}}{ds}x, \tag{7.2}$$

where $\bar{\omega} = \bar{\omega}(s)$ is an arbitrary function. In this case

$$\varepsilon_1 = 0, \quad \varepsilon_2 = 0, \quad \omega = \bar{\omega}(s),$$
$$\kappa_1 = 0, \quad \kappa_2 = x\frac{d}{ds}\left[\frac{d}{ds}R\frac{d}{ds} + \frac{1}{R}\right]\bar{\omega}, \quad \tau = \left[\frac{d}{ds}R\frac{d}{ds} + \frac{1}{R}\right]\bar{\omega},$$
$$\iint_G V_1 AB d\alpha d\beta = l\frac{1-\sigma}{4}\int_0^{s_0} \bar{\omega}^2 ds,$$
$$\iint_G V_2 AB d\alpha d\beta = \lambda^2\left\{\frac{l^3}{3}\int_0^{s_0}\left[\frac{d}{ds}\left(\frac{d}{ds}R\frac{d}{ds} + \frac{1}{R}\right)\bar{\omega}\right]^2 ds + \right.$$
$$\left. + 2(1-\sigma)l\int_0^{s_0}\left[\left(\frac{d}{ds}R\frac{d}{ds} + \frac{1}{R}\right)\bar{\omega}\right]^2 ds\right\},$$

where one may understand by λ, for example, the mean value of

$$R(s).$$

Hence it is seen that, generally speaking, (7.2) is a deformation with arbitrarily large (for $l \to \infty$) absolute characteristic

$$\frac{1}{\eta} = O(l^{-2}).$$

An exception is presented by the case when the equation

$$\frac{d}{ds}\left(\frac{d}{ds} R \frac{d}{ds} + \frac{1}{R}\right)\bar{\omega} = 0$$

is satisfied, i.e., the case when $\bar{\omega}$ is subject to the law of sectorial areas.

If $\bar{\omega}(s)$ does not vary too quickly with s, then (7.2) will be a deformation with not large variation.

It may likewise be shown that an arbitrary cone, clamped against tangential displacements at all points of a transverse cross-section, will have a deformation with arbitrarily large absolute characteristic, if it contains points sufficiently close to the vertex.

§ 8. The principle of minimum potential energy says that the problem of the construction of a state of stress and strain of a shell is equivalent to the problem of minimization of the functional (assuming that the system of Gaussian coordinates is orthogonal)

$$A = \tfrac{1}{2}\iint_G VAB d\alpha d\beta - \iint_G \mathbf{U}\cdot\mathbf{P} AB d\alpha\beta,$$

where $\mathbf{P}$ is the vector of the surface loading and $\mathbf{U}$ is the displacement vector of the points of the middle surface of the shell.

Within the bounds of the accuracy of the shell theory, constructed on the Kirchhoff-Love hypothesis, it may be assumed that [3]

$$V_1 = \frac{2Eh}{1-\sigma^2}\left\{\varepsilon_1^2 + \varepsilon_2^2 + 2\sigma\varepsilon_1\varepsilon_2 + 2(1-\sigma)\frac{\omega^2}{4} + \right.$$
$$\left. + \frac{h^2}{3}\left[\kappa_1^2 + \kappa_2^2 + 2\sigma\kappa_1\kappa_2 + 2(1-\sigma)\tau^2\right]\right\},$$

whence, using the notation (1.1), we obtain

$$V = \frac{2Eh}{1-\sigma^2}\left\{V_1 + \frac{h^2}{3\lambda^2} V_2\right\}. \tag{8.1}$$

If the edges of the shell are clamped, so that the boundary forces do not perform work, the vector **U**, minimizing the functional A, must satisfy the relation

$$\iint_G VAB d\alpha d\beta = \iint_G \mathbf{U}\cdot\mathbf{P} AB d\alpha d\beta, \tag{8.2}$$

and therefore

$$A_{min} = -\tfrac{1}{2}\iint_G \mathbf{U}\cdot\mathbf{P} AB d\alpha d\beta. \tag{8.3}$$

Let the problem consist of the construction of the state of stress and strain of a shell subject to the action of a surface loading **P**. We assume with regard to the boundary conditions that they are homogeneous and may be decomposed into two groups: a) geometric conditions consisting of the requirement of the absence of linear or angular displacements in one or the other direction; b) static conditions, consisting of the requirement of the absence of force or moment reactions in one or the other direction.

We will call permissible any deformation D of the middle surface of the shell whose linear or angular displacements satisfy the geometric conditions of the problem. By assumption, the geometric boundary conditions are homogeneous. Therefore, if D is a permissible deformation, then also μD will be a permissible deformation (where μ is an arbitrary constant). Using this fact we introduce the concept of normalized permissible deformation D_n, understanding by this a permissible deformation for which

$$\frac{2Eh}{1-\sigma^2}\iint_G V_2 AB d\alpha d\beta = \overline{V}_2,$$

where $\overline{V}_2$ is some fixed quantity which is identical for all permissible deformations, with the exception of those which are obtained from D_n by multiplication by a constant factor (as was mentioned in § 1, deformations for which $\overline{V}_2 = 0$ are not being considered).

For a certain D_n let

$$\frac{2Eh}{1-\sigma^2}\iint_G V_1 AB d\alpha d\beta = \left(\frac{h}{\lambda}\right)^{2k}\overline{V}_2, \tag{8.4}$$

$$\iint_G \mathbf{U}\cdot\mathbf{P} AB d\alpha d\beta = \left(\frac{h}{\lambda}\right)^{s}\rho\overline{V}_2. \tag{8.5}$$

Here the number k, connected with the absolute characteristic of a deformation D by the relation

$$\eta = \left(\frac{h}{\lambda}\right)^{-2k},$$

and the number s, characterizing the amount of work done by the external forces $\mathbf{P}$ for the displacements $\mathbf{U}$, are for the present undetermined quantities and p is a quantity comparable with unity. The number k increases with η. We will call it the relative characteristic deformation or simply the characteristic.

Since V_1 and V_2 are homogeneous quadratic forms of the displacements and $\mathbf{U}$ depends linearly on the displacements, the relation (8.4) is preserved for transition from D_n to μD_n, and we obtain, instead of (8.5),

$$\iint_G \mathbf{U}\cdot\mathbf{P}AB\,d\alpha\,d\beta = \mu\left(\frac{h}{\lambda}\right)^{s} p\,\overline{V}_2. \tag{8.6}$$

With the help of (8.1) one readily finds that if a shell undergoes a deformation μD_n, then

$$\iint_G VAB\,d\alpha\,d\beta = \mu^2\left[\left(\frac{h}{\lambda}\right)^{2k} + \frac{1}{3}\left(\frac{h}{\lambda}\right)^{2}\right]\overline{V}_2. \tag{8.7}$$

We compare now (8.7) with (8.6) and pose the question with regard to what must be the order of μ, so that the equality (8.2) for which it is possible to attain a minimum is fulfilled. It is readily seen that one must have for this

$$\mu \approx \left(\frac{h}{\lambda}\right)^{s-2k}, \text{ if } k \leq 1,$$

$$\mu \approx \left(\frac{h}{\lambda}\right)^{s-2}, \text{ if } k > 1.$$

When these conditions are fulfilled the equality (8.3) assumes the form

$$A_{min} = -\bar{A}\overline{V}_2 \begin{cases} \bar{A} \approx \left(\frac{h}{\lambda}\right)^{2s-2k} & \text{for } k \leq 1, \\ \bar{A} \approx \left(\frac{h}{\lambda}\right)^{2s-2} & \text{for } k > 1. \end{cases} \tag{8.8}$$

It follows from the principle of minimum potential energy that in actual fact among all permissible deformations that deformation will

occur for which the index of h/λ in the expression for $\bar{A}$ has the (algebraically) smallest value. This means that, generally speaking, there arises in the shell, among all the deformations satisfying the geometric boundary conditions, that deformation which has the largest characteristic. However, for this one must take into consideration two circumstances: first one must give consideration only to such permissible deformations for which the given surface loading performs sufficiently large work (the number s must not be too large); second, the index of h/λ in the expression for $\bar{A}$ decreases as k increases for $k < 1$. Therefore, from the point of view of achieving the minimum of the potential energy, bending, i.e., the deformation with infinite characteristic, does not have an advantage over a deformation with sufficiently large ($k > 1$) characteristic. (In this case, the decisive role may be played by the amount of work which is performed by the given loading over that form of deformation.)

§ 9. We will develop a more detailed form of the equality (8.4) which determines the number k, the characteristic of a deformation under consideration. Substituting for $\overline{V}_2$ its value we obtain after some simplifications

$$\iint_G V_1 AB d\alpha d\beta = \left(\frac{h}{\lambda}\right)^{2k} \iint_G V_2 AB d\alpha d\beta. \tag{9.1}$$

The equality (9.1) may be fulfilled, if, for example, there applies at all points of the region G the relation

$$V_1 \approx \left(\frac{h}{\lambda}\right)^{2k} V_2. \tag{9.2}$$

However, the expressions V_1 and V_2, determined by the formulae (1.1), may be transformed to read

$$V_1 = \frac{1+\sigma}{2}(\varepsilon_1 + \varepsilon_2)^2 + \frac{1-\sigma}{2}(\varepsilon_1 - \varepsilon_2)^2 + 2(1-\sigma)\frac{\omega^2}{4},$$

$$V_2 = \lambda^2\left[\frac{1+\sigma}{2}(\kappa_1 + \kappa_2)^2 + \frac{1-\sigma}{2}(\kappa_1 - \kappa_2)^2 + 2(1-\sigma)\tau^2\right],$$

and since $\sigma < \frac{1}{2}$, we have on the right hand sides of these equalities sums of non-negative terms, and it may be asserted that from (9.2) follows the relation

$$\{\varepsilon_1, \varepsilon_2, \omega\} \approx \{h^k\lambda^{1-k}\kappa_1, h^k\lambda^{1-k}\kappa_2, h^k\lambda^{1-k}\tau\} \tag{9.3}$$

(where it is understood that the largest of the quantities in the braces must be commensurable).

There applies now the approximate formula

$$\boldsymbol{\sigma} = \frac{\mathbf{R}}{2h} - \frac{3}{2h^2}\frac{z}{h}\,\mathbf{Qxn} \tag{9.4}$$

from which the stress vector $\boldsymbol{\sigma}$ may be determined in terms of the force vector $\mathbf{R}$ and the moment vector $\mathbf{Q}$. On the other hand, we have the stress strain relations

$$T_1 = \frac{2Eh}{1-\sigma^2}(\varepsilon_1 + \sigma\varepsilon_2), \quad T_2 = \frac{2Eh}{1-\sigma^2}(\varepsilon_2 + \sigma\varepsilon_1),$$

$$S_1 = -S_2 = \frac{Eh}{1+\sigma}\omega,$$

$$\frac{1}{\lambda}G_1 = -\frac{2Eh}{1-\sigma^2}\frac{h^2}{3\lambda^2}(\lambda\kappa_1 + \sigma\lambda\kappa_2),$$

$$\frac{1}{\lambda}G_2 = -\frac{2Eh}{1-\sigma^2}\frac{h^2}{3\lambda^2}(\lambda\kappa_2 + \sigma\lambda\kappa_1),$$

$$\frac{1}{\lambda}H_1 = -\frac{1}{\lambda}H_2 = \frac{2Eh}{1+\sigma}\frac{h^2}{3\lambda^2}\lambda\tau.$$

Hence it is seen that, if the relations (9.3) are fulfilled, for $k < 1$ the first term on the right hand side of (9.4) will predominate, i.e., the largest stress will result from the forces, while for $k \geq 1$ the second term will predominate, i.e., the largest stresses will be due to the moments.

The relation (9.3) may, of course, also be violated without violation of the basic relation (9.1). Therefore, for $k < 1$, the stresses due to moments may also predominate in different regions (in particular, this will be true, as a rule, in the zone of influence of an edge effect), but in order that the integral relation (9.1) will remain true, the zones with dominating stresses and moments must be relatively small.

Starting from this it may be asserted that for $k < 1$ the state of stress of a shell will be on the whole a membrane state (i.e., momentless everywhere except perhaps in some relatively small zones).

Geometric criteria for an overall membrane state may now be formulated as follows: the state of stress of a thin elastic shell will be a membrane state, if among the deformations which admit the geometric boundary conditions and over the displacements of which the given external forces perform not small work, there are not deformations with a characteristic which is larger or equal to unity.

If $k \geq 1$, the zones of a bending state of stress will either spread over a large section of the shell and the state of stress will be an overall bending state, or these zones, remaining relatively small, will be zones of very intense forces and moments, i.e., we will, for example, have an especially large concentration of edge effect stresses.

§ 10. We will now turn to the hypothesis of V. E. Novodvorskii. It may be assumed to apply only under a certain number of reservations [2] the meanings of which have become clear in the light of the results obtained.

It has been shown in § 3 that an arbitrary surface may have for any edge support deformations with arbitrarily large characteristic and with large variation. If the variation of the loading on the shell is not large, the existence of such a deformation will not play a role, i.e., the load will perform little work over the corresponding displacements. However, if the variation of the external loading is large, there occurs in the shell exactly that one of the permissible deformations with large variation over which the loading performs sufficiently large work. Hence there arises the reservation that the hypothesis of V. E. Novodvorskii refers only to cases for which the variation of the state of stress of a shell is not too large.

A state of stress of a sufficiently shallow shell, loaded by transverse forces, will not be a membrane state for any support conditions of the edges. This is in complete agreement with the fact that, as was shown in § 5, a sufficiently shallow surface may have deformations with arbitrarily large characteristics.

The rigidity of the surfaces referred to in the hypothesis of V. E. Novodvorskii must be secured by single tangential boundary conditions [2]. This reservation has become clear in the light of the results of § 4. If there exists infinitesimal bending satisfying the tangential boundary conditions, there will exist a deformation with an arbitrarily large characteristic, satisfying all geometric boundary conditions and the loading performing sufficiently large work over the displacements of this state; in the shell there arises then either an overall bending state of stress or a state of stress with edge effects of very high intensity (these results were obtained by other means in [6]).

There exists a certain class of surfaces which are peculiar from the point of view of the applicability of the membrane theory to shells, described by these surfaces. In particular, this class includes sufficiently long cylinders and cones containing points sufficiently close to the vertex.

This influence has found its explanation by the results of § 7.

Special care is required in the application of the membrane theory to shells of non-positive curvature, if the edge of the shell passes along an asymptotic line of the middle surface or touches it at separate points. This is explained by the fact that in such cases there arises in the shell instead of a simple edge effect a generalized edge effect with special properties and a more substantially distorted stress distribution than is obtained in the analysis based on the membrane theory. It seems to us that the difference between simple and generalized edge effects may be related to the geometric considerations studied here, but sufficiently complete results of this kind are not yet at the author's disposal. We will only note that, for example, the influence of the degeneration of the edge effect in a long cylindrical shell correlates with the fact that in a long cylinder deformations with arbitrarily large characteristics are possible.

In conclusion, we dwell on the deductions which may be drawn from the results of § 6. Let there be given a problem for which all the conditions for the applicability of V. E. Novodvorskii's hypothesis are fulfilled: the middle surface has no peculiarities, the boundary conditions, securing rigidity, are tangential, the loading has small variation and the edge passes nowhere along an asymptotic line. We will assume that the edge is clamped over some section γ_1 (with the help of tangential connections) and on the remaining part γ_2 it is free. Let the connection be such that it secures rigidity for any arbitrarily small γ_1. Then, by the hypothesis of V. E. Novodvorskii, there will arise in the shell a membrane state of stress, while the shell with a free edge will actually be in a bending state (of course, for conditions of self-equilibrating loading). This is not in conformity with physical reality: it is impossible to radically change the character of the state of stress of a shell by clamping its edge over an arbitrarily small section. The paradox is removed, if one takes into consideration that with the decrease of the length of the clamped section there will arise deformations with a growing characteristic and, correspondingly, the role played by bending will gain in importance, so that there does not occur a spasmodic transition of the state of stress.

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TORSIONAL VIBRATIONS OF AN INITIALLY STRESSED CIRCULAR CYLINDER

A. E. GREEN
Newcastle upon Tyne, England

§ 1. Introduction

In this paper we consider small torsional vibrations of an initially stressed solid circular cylinder which is isotropic and with a centre of symmetry in its undeformed state. The basic equations are derived from large elastic deformation theory, using a general strain energy function, in which thermal stresses are taken into account. It is found that small torsional vibrations are possible under isothermal conditions.

§ 2. Notation and formulae

We use here the notation and formulae as given by Green and Adkins [2] and, for brevity, we quote the main results. Points in the body are defined by a general curvilinear system of coordinates θ_i which moves with the body as it deforms. The initial metric tensors are g_{ij}, g^{ij} and the corresponding values in the deformed body at time t are G_{ij}, G^{ij}. The strain tensor γ_{ij} is defined by

$$\gamma_{ij} = \tfrac{1}{2}(G_{ij} - g_{ij}) \tag{2.1}$$

and we define γ_j^i by the formula

$$\gamma_j^i = g^{ir}\gamma_{rj} = \tfrac{1}{2}(g^{ir}G_{rj} - \delta_j^i). \tag{2.2}$$

Latin indices take the values 1, 2, 3 and repeated indices are summed.

The contravariant stress tensor τ^{ij}, measured per unit area of the deformed body, and referred to θ_i—coordinates in the deformed body, is given by

$$\tau^{ij}\sqrt{I_3} = 2g^{ij}\frac{\partial A}{\partial I_1} + 2B^{ij}\frac{\partial A}{\partial I_2} + 2G^{ij}I_3\frac{\partial A}{\partial I_3}, \tag{2.3}$$

if the body is initially homogeneous and isotropic, where

$$B^{ij} = (g^{ij}g^{rs} - g^{ir}g^{js})G_{rs}, \tag{2.4}$$

and

$$A = A(I_1, I_2, I_3, T) \tag{2.5}$$

is the Helmholtz free energy function. In (2.5) T is temperature, I_1, I_2 and I_3 are the invariants

$$I_1 = g^{ij}G_{ij},\quad I_2 = G^{ij}g_{ij}I_3,\quad I_3 = G/g, \tag{2.6}$$

and

$$G = |G_{ij}|,\quad g = |g_{ij}|. \tag{2.7}$$

The equations of motion, if body forces are zero, are

$$\tau^{ij}\|_i = \rho f^j, \tag{2.8}$$

where f^j are the contravariant components of acceleration and ρ is the density of the deformed body. In (2.8) the double line denotes covariant differentiation with respect to the deformed body using the metric tensors G_{ij}, G^{ij} in forming the Christoffel symbols. Also, if ρ_0 is the (constant) density of the undeformed body,

$$\rho = \rho_0/\sqrt{I_3}. \tag{2.9}$$

If Q^i are the contravariant components of the heat conduction vector, then

$$-T\frac{DS}{Dt} = Q^i\|_i\sqrt{I_3} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial\theta^i}(Q^i\sqrt{G}), \tag{2.10}$$

where

$$S = -\frac{\partial A}{\partial T} \tag{2.11}$$

and D/Dt denotes differentiation with respect to time t holding θ_i fixed. Finally, when the body is initially isotropic, and has a center of symmetry [1, 2],

$$-Q^i = (M_1\delta^i_j + M_2\gamma^i_j + M_3\gamma^i_r\gamma^r_j)T\|^j, \tag{2.12}$$

where M_1, M_2, M_3 are polynomials in the invariants

$$I_1, I_2, I_3, T\|^iT\|_i,\quad T\|_iT\|^j\gamma^i_j,\quad T\|_iT\|^j\gamma^i_k\gamma^k_j \tag{2.13}$$

with coefficients which are continuous functions of T. Also

$$-Q^iT\|_i \geq 0,\quad T\|^i = G^{ij}T\|_j = G^{ij}\frac{\partial T}{\partial\theta^j}. \tag{2.14}$$

§ 3. Small torsional oscillation superposed on uniform extension

We consider a uniform solid circular cylinder of initial radius a_0 and suppose that it is extended along its length, with extension ratio λ, under a

uniform tension s over its plane ends and zero tractions over its curved surface, the temperature T being everywhere constant. The radius of the cylinder becomes a where

$$a = a_0\mu, \tag{3.1}$$

μ being the extension ratio in direction perpendicular to the axis of the cylinder. We take the coordinate θ_i to coincide with cylindrical polar coordinates (r, θ, z) in the deformed cylinder, so that

$$\theta_1 = r, \quad \theta_2 = \theta, \quad \theta_3 = z \tag{3.2}$$

and therefore the initial position of the point (r, θ, z) is $(r/\mu, \theta, z/\lambda)$, and

$$g_{ij} = \begin{bmatrix} \frac{1}{\mu^2} & 0 & 0 \\ 0 & \frac{r^2}{\mu^2} & 0 \\ 0 & 0 & \frac{1}{\lambda^2} \end{bmatrix}, \quad g^{ij} = \begin{bmatrix} \mu^2 & 0 & 0 \\ 0 & \frac{\mu^2}{r^2} & 0 \\ 0 & 0 & \lambda^2 \end{bmatrix}, \quad g = \frac{r^2}{\mu^4\lambda^2}. \tag{3.3}$$

Finally, we assume that the stressed cylinder performs small torsional oscillations about its axis so that at any time t the point (r, θ, z) moves to $(r, \theta + \varepsilon\psi, z)$ and the temperature becomes $T + \varepsilon T'(r, z, t)$, where ε is a positive constant which is small enough for powers of ε above the first to be neglected, and ψ and T' are functions of r, z and t.

It follows that the metric tensor G_{ij} at time t has components

$$G_{ij} = \begin{bmatrix} 1 & \varepsilon r^2 \frac{\partial\psi}{\partial r} & 0 \\ \varepsilon r^2 \frac{\partial\psi}{\partial r} & r^2 & \varepsilon r^2 \frac{\partial\psi}{\partial z} \\ 0 & \varepsilon r^2 \frac{\partial\psi}{\partial z} & 1 \end{bmatrix}, \tag{3.4}$$

if we neglect ε^2. To this order $G = r^2$ and

$$G^{ij} = \begin{bmatrix} 1 & -\varepsilon \frac{\partial\psi}{\partial r} & 0 \\ -\varepsilon \frac{\partial\psi}{\partial r} & \frac{1}{r^2} & -\varepsilon \frac{\partial\psi}{\partial z} \\ 0 & -\varepsilon \frac{\partial\psi}{\partial z} & 1 \end{bmatrix}. \tag{3.5}$$

Hence, from (2.4), (2.6), (3.3), (3.4) and (3.5) we have

$$
\begin{aligned}
&I_1 = 2\mu^2 + \lambda^2, \quad I_2 = 2\mu^2\lambda^2 + \mu^4, \quad I_3 = \mu^4\lambda^2, \\
&B^{11} = r^2B^{22} = \mu^2(\mu^2 + \lambda^2), \quad B^{33} = 2\mu^2\lambda^2, \\
&B^{12} = -\varepsilon\mu^4 \frac{\partial\psi}{\partial r}, \quad B^{23} = -\varepsilon\mu^2\lambda^2 \frac{\partial\psi}{\partial z}, \quad B^{13} = 0.
\end{aligned} \tag{3.6}
$$

Also, from (2.3), (2.5), (3.3), (3.5) and (3.6) we see that

$$
\begin{aligned}
\tau^{11} = r^2\tau^{22} = {} & \frac{2}{\lambda}\frac{\partial A}{\partial I_1} + \frac{2(\mu^2+\lambda^2)}{\lambda}\frac{\partial A}{\partial I_2} + 2\mu^2\lambda\frac{\partial A}{\partial I_3} + \\
& + 2\varepsilon T'\left[\frac{1}{\lambda}\frac{\partial^2 A}{\partial I_1 \partial T} + \frac{\mu^2+\lambda^2}{\lambda}\frac{\partial^2 A}{\partial I_2 \partial T} + \mu^2\lambda\frac{\partial^2 A}{\partial I_3 \partial T}\right],
\end{aligned} \tag{3.7}
$$

$$
\begin{aligned}
\tau^{33} = {} & \frac{2\lambda}{\mu^2}\frac{\partial A}{\partial I_1} + 4\lambda\frac{\partial A}{\partial I_2} + 2\mu^2\lambda\frac{\partial A}{\partial I_3} + \\
& + 2\varepsilon T'\left[\frac{\lambda}{\mu^2}\frac{\partial^2 A}{\partial I_1 \partial T} + 2\lambda\frac{\partial^2 A}{\partial I_2 \partial T} + \mu^2\lambda\frac{\partial^2 A}{\partial I_3 \partial T}\right],
\end{aligned} \tag{3.8}
$$

$$
\begin{aligned}
\tau^{12} &= -2\varepsilon\left(\frac{\mu^2}{\lambda}\frac{\partial A}{\partial I_2} + \mu^2\lambda\frac{\partial A}{\partial I_3}\right)\frac{\partial\psi}{\partial r}, \\
\tau^{23} &= -2\varepsilon\left(\lambda\frac{\partial A}{\partial I_2} + \mu^2\lambda\frac{\partial A}{\partial I_3}\right)\frac{\partial\psi}{\partial z}, \quad \tau^{13} = 0,
\end{aligned} \tag{3.9}
$$

where the derivatives of A are evaluated from (2.5) at the constant values I_1, I_2, I_3 given in (3.6), and for constant T. These derivatives are therefore constants.

If ε^2 and higher powers are neglected the invariants (2.13) reduce to I_1, I_2, I_3 given in (3.6), and, from (2.12), we obtain

$$
\begin{aligned}
-Q^1 &= \varepsilon\left[M_1 + \frac{\mu^2-1}{2}M_2 + \frac{(\mu^2-1)^2}{4}M_3\right]\frac{\partial T'}{\partial r} = \varepsilon k_1\frac{\partial T'}{\partial r}, \\
-Q^3 &= \varepsilon\left[M_1 + \frac{\lambda^2-1}{2}M_2 + \frac{(\lambda^2-1)^2}{4}M_3\right]\frac{\partial T'}{\partial z} = \varepsilon k_3\frac{\partial T'}{\partial z}
\end{aligned} \tag{3.10}
$$

and $Q^2 = 0$, where M_1, M_2, M_3 are polynomials in I_1, I_2, I_3 with coefficients which are single-valued functions of T, so k_1 and k_3 are constants. In view of (2.14) it follows that $k_1 \geq 0$, $k_3 \geq 0$. Also, from (2.10) and (2.11)

$$
\varepsilon T\frac{\partial^2 A}{\partial T^2}\frac{DT'}{Dt} = \mu^2\lambda\left[\frac{1}{r}\frac{\partial}{\partial r}(rQ^1) + \frac{\partial Q^3}{\partial z}\right], \tag{3.11}
$$

where $\partial^2 A/\partial T^2$ is evaluated from (2.5) and is constant. Hence

$$\mu^2\lambda k_1\left(\frac{\partial^2 T'}{\partial r^2}+\frac{1}{r}\frac{\partial T'}{\partial r}\right)+\mu^2\lambda k_3\frac{\partial^2 T'}{\partial z^2}+T\frac{\partial^2 A}{\partial T^2}\frac{DT'}{Dt}=0. \tag{3.12}$$

It is reasonable to assume that, in equation (3.12), $T(\partial^2 A/\partial T^2)\leq 0$ for a physically acceptable equation. If the cylinder is first deformed under constant temperature T, and subsequently the surface of the cylinder is maintained at this temperature, then the temperature throughout the cylinder remains at the constant value T, so that $T'\equiv 0$. Even if T' is zero initially only on the surface of the cylinder it rapidly dies away to zero everywhere. We then see from (3.7) that τ^{11} is a constant and is everywhere zero if

$$\frac{1}{\lambda}\frac{\partial A}{\partial I_1}+\frac{\mu^2+\lambda^2}{\lambda}\frac{\partial A}{\partial I_2}+\mu^2\lambda\frac{\partial A}{\partial I_3}=0, \tag{3.13}$$

an equation giving μ in terms of λ. The non-zero stresses in (3.8) and (3.9) then become

$$\begin{aligned}
\tau^{33}=s&=\frac{2(\lambda^2-\mu^2)}{\lambda\mu^2}\left(\frac{\partial A}{\partial I_1}+\mu^2\frac{\partial A}{\partial I_2}\right),\\
\tau^{12}&=\frac{2\varepsilon}{\lambda}\left(\frac{\partial A}{\partial I_1}+\lambda^2\frac{\partial A}{\partial I_2}\right)\frac{\partial\psi}{\partial r},\\
\tau^{23}&=\frac{2\varepsilon}{\lambda}\left(\frac{\partial A}{\partial I_1}+\mu^2\frac{\partial A}{\partial I_2}\right)\frac{\partial\psi}{\partial z}.
\end{aligned} \tag{3.14}$$

Since (3.14) are the only non-zero components of stress and these are independent of θ, the equations of motion, with the help of (2.9) and (3.5), reduce to

$$\frac{\partial\tau^{12}}{\partial r}+\frac{\partial\tau^{23}}{\partial z}+\frac{3\tau^{12}}{r}+\varepsilon s\frac{\partial^2\psi}{\partial z^2}=\frac{\varepsilon\rho_0}{\mu^2\lambda}\frac{D^2\psi}{Dt^2}. \tag{3.15}$$

Substituting from (3.14) into (3.15) now gives the following differential equation for ψ:

$$\begin{aligned}
2\left(\frac{\partial A}{\partial I_1}+\lambda^2\frac{\partial A}{\partial I_2}\right)\left(\frac{\partial^2\psi}{\partial r^2}+\frac{3}{r}\frac{\partial\psi}{\partial r}\right)+\\
+2\frac{\lambda^2}{\mu^2}\left(\frac{\partial A}{\partial I_1}+\mu^2\frac{\partial A}{\partial I_2}\right)\frac{\partial^2\psi}{\partial z^2}=\frac{\rho_0}{\mu^2}\frac{D^2\psi}{Dt^2}.
\end{aligned} \tag{3.16}$$

If the surface of the cylinder $r=a$ is free from applied stress, then

τ^{1i} $(i = 1, 2, 3)$ vanishes there so that, from (3.14),

$$\frac{\partial \psi}{\partial r} = 0, \quad r = a. \tag{3.17}$$

If we remember that the cylinder is solid so that stresses and displacements are zero when $r = 0$, and if we use the boundary condition (3.17), then a simple harmonic wave

$$\psi = \frac{L}{r} J_1(\alpha r) e^{i(nz+\omega t)} \tag{3.18}$$

exists, where L is a constant,

$$\alpha^2 = \frac{\rho_0\omega^2 - 2n^2\lambda^2\left(\dfrac{\partial A}{\partial I_1} + \mu^2 \dfrac{\partial A}{\partial I_2}\right)}{2\mu^2\left(\dfrac{\partial A}{\partial I_1} + \lambda^2 \dfrac{\partial A}{\partial I_2}\right)}, \tag{3.19}$$

$$\frac{\partial}{\partial a}\left[\frac{J_1(\alpha a)}{a}\right] = 0, \tag{3.20}$$

and $J_1(x)$ is a Bessel function of unit order of the first kind. One solution of (3.20) is $\alpha = 0$, with the corresponding ratio

$$\frac{\omega^2}{n^2} = \frac{2\lambda^2}{\rho_0}\left(\frac{\partial A}{\partial I_1} + \mu^2 \frac{\partial A}{\partial I_2}\right) = \frac{s\lambda^3\mu^2}{(\lambda^2 - \mu^2)\rho_0} \tag{3.21}$$

which represents a wave with speed V where

$$V^2 = \frac{2\lambda^2}{\rho_0}\left(\frac{\partial A}{\partial I_1} + \mu^2 \frac{\partial A}{\partial I_2}\right) = \frac{s\lambda^3\mu^2}{(\lambda^2 - \mu^2)\rho_0}. \tag{3.22}$$

When the body is incompressible, $I_3 = 1$ and $\mu^2 = 1/\lambda$, so that (3.22) becomes

$$V^2 = \frac{2\lambda^2}{\rho_0}\left(\frac{\partial A}{\partial I_1} + \frac{1}{\lambda} \frac{\partial A}{\partial I_2}\right) = \frac{s\lambda^3}{(\lambda^3 - 1)\rho_0}. \tag{3.23}$$

The special case of the Mooney material for which

$$A = C_1(I_1 - 3) + C_2(I_2 - 3) \tag{3.24}$$

is of some interest. The value of V^2 is then

$$V^2 = \frac{2\lambda^2}{\rho_0}(C_1 + C_2/\lambda), \tag{3.25}$$

so that

$$(V/U)^2 = \frac{\lambda^2(C_1 + C_2/\lambda)}{C_1 + C_2}, \tag{3.26}$$

where U is the corresponding value for a cylinder under zero initial stress. For a neo-Hookean solid $C_2 = 0$ and (3.26) reduces to

$$V^2 = \lambda^2 U^2. \tag{3.27}$$

If $C_1 \geq 0$, $C_2 \geq 0$, then $V^2 > U^2$ if $\lambda > 1$ and $V^2 < U^2$ if $\lambda < 1$, so that the speed of propagation is increased when the rod is under tension and decreased when under compression.

If the traction across normal sections $z = 0, l$ is only that due to the initial stress, then we can have torsional vibrations in which

$$\psi = L_m \cos \frac{mxz}{l} \cos\left(\frac{mxVt}{l} + \beta_m\right),$$

m being any integer and where V is given by (3.21).

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BIFURCATION AND UNIQUENESS IN NON-LINEAR MECHANICS OF CONTINUA

R. HILL
Nottingham, England

A synoptic account is given of the author's work during recent years on bifurcation and uniqueness in the general boundary-value problems for elastic and inelastic solids.

§ 1. Definition of solids under consideration

We are concerned here with solids whose mechanical properties in isothermal deformation do not depend on time. Among such solids we consider particularly those characterized by 'rate equations' of type

$$\dot{\tau}^{\alpha\beta} = \frac{\partial W}{\partial \varepsilon_{\alpha\beta}}. \tag{1.1}$$

Here $\tau^{\alpha\beta}$ are the contravariant components of the symmetric Kirchhoff stress tensor in curvilinear coordinates deforming with the material element ($\alpha, \beta = 1, 2, 3$), and the dot denotes the instantaneous convected derivative. W, hereafter referred to as the strain-rate potential, is a single-valued symmetrized homogeneous function of degree two in the strain-rate components $\varepsilon_{\alpha\beta}$, with continuous first derivatives and at least piecewise continuous second derivatives. W may also depend explicitly on the current stress and on the existing anisotropy (which itself is determined by the entire previous strain-history); consequently, in a body undergoing non-uniform distortion, W depends on position. Since W is homogeneous of degree two, equations (1.1) are homogeneous of degree one and so are independent of the time scale. Also, by Euler's theorem,

$$2W = \dot{\tau}^{\alpha\beta}\varepsilon_{\alpha\beta}. \tag{1.2}$$

The Kirchhoff stress-rate has been chosen in (1.1), rather than some other stress-rate independent of rigid-body spin, for the sake of theorems proved later and also because the analysis then appears in its most elegant and concise form. In terms of the more familiar convected derivative of the so-called 'true' stress $\sigma^{\alpha\beta}$ we have

$$\dot{\tau}^{\alpha\beta} = \dot{\sigma}^{\alpha\beta} + \sigma^{\alpha\beta}\varepsilon^{\gamma}_{\gamma}.$$

Note, however, that the true stress-rate is *not* derivable from a potential when (1.1) holds, unless the material is incompressible.

The convected derivative of the load vector on an infinitesimal plane element, deforming with the material and currently having covariant area dS_α, has components $\dot{\tau}^{\alpha\beta}dS_\alpha$. On the other hand, the rate of change of the load components referred always to the present coordinate configuration (and not the future deformed coordinates) is $\dot{s}^{\alpha\beta}dS_\alpha$, where

$$\dot{s}^{\alpha\beta} = \dot{\tau}^{\alpha\beta} + \tau^{\alpha\gamma}v^{\beta}_{,\gamma}. \tag{1.3}$$

Here v^β is the velocity and the comma denotes covariant differentiation. $\dot{s}^{\alpha\beta}$ is called the 'nominal' stress-rate; it is an unsymmetric tensor and is derivable from a homogeneous potential function of velocity gradient (and not simply strain-rate):

$$\dot{s}^{\alpha\beta} = \frac{\partial U}{\partial v_{\beta,\alpha}}, \tag{1.4}$$

where
$$U(v_{\beta,\alpha}) = W(\varepsilon_{\alpha\beta}) + \tfrac{1}{2}\tau^{\alpha\beta}v_{\gamma,\alpha}v^{\gamma}_{,\beta}.$$

The function U is fundamental to the subsequent analysis.

If equations (1.4) have a unique inverse this is

$$v_{\beta,\alpha} = \frac{\partial U}{\partial \dot{s}^{\alpha\beta}}, \tag{1.5}$$

where $U = \tfrac{1}{2}\dot{s}^{\alpha\beta}v_{\beta,\alpha}$ is expressed as a function of the nominal stress-rate. Similarly, the inverse of (1.1) would be $\varepsilon_{\alpha\beta} = \partial W/\partial\dot{\tau}^{\alpha\beta}$ when W is expressed as a function of the Kirchhoff stress-rate.

The following terminology will be used. Consider a material element in a given mechanical state under a given stress, and imagine it to be subjected to any cycle of deformation that fully restores its shape. If there are certain arbitrarily small cycles which do not restore the stress to its initial value the solid will be called *inelastic* in this state. In particular, if the stress is not recovered after any arbitrarily small cycle of strain the state will be called *plastic*. Again, if the stress is not recovered after any arbitrarily small cycle lying to one side of a surface in Lagrangian-strain space (with the current configuration as origin), while in a neighbourhood on the other side the stress and finite strain are related as in an elastic solid, the state will be called *elastic/plastic*. The following is an example of an elastic/plastic solid where the potential W is analytically composite, with a discontinuity in its second derivatives across a hyperplane surface in strain-rate space:

$$2W = \begin{cases} \boldsymbol{\varepsilon}K\boldsymbol{\varepsilon}, & \mathbf{m}\boldsymbol{\varepsilon} \leq 0 \\ \boldsymbol{\varepsilon}K\boldsymbol{\varepsilon} - (\mathbf{m}\boldsymbol{\varepsilon})^2, & \mathbf{m}\boldsymbol{\varepsilon} \geq 0 \end{cases} \tag{1.6}$$

where, for brevity, the strain-rate is written as a 9-component vector, $\mathbf{m}$ is a vector in the same space, and K is a symmetric matrix; K and $\mathbf{m}$ are constant for the given state of the considered element, but vary with continued deformation. Potentials for a considerably wider class of elastic/plastic solids can be formulated without difficulty [6].

It will be fundamentally important to distinguish (i) *linear* solids for which W is a single quadratic form with a matrix K which is constant for the given state, so that $\dot{\boldsymbol{\tau}} = K\boldsymbol{\varepsilon}$ in the whole of strain-rate space (for example, an elastic solid after any finite strain), and (ii) *non-linear* solids for which equations (1.1) are non-linear in the components of strain-rate, W being a general second-degree homogeneous function, which may in particular be a quadratic whose coefficients take different constant values in different regions of strain-rate space [as in (1.6)]. Note that many linear solids are inelastic, as well as all non-linear ones.

§ 2. The boundary-value problem

The rate equations are associated with the following boundary-value problem. At a generic stage in a process of quasi-static distortion the shape of the body, the internal distribution of stress, the current values of material parameters, and the state of anisotropy and inhomogeneity are supposed to be given or to have been determined already. The velocity vector v_β is now prescribed on a part S of the surface, together with the nominal traction-rate $\dot{t}^\beta$ on the remainder Σ and the nominal body force rate $\dot{g}^\beta$ per unit volume through the volume V. We enquire under what conditions the entire velocity field (or mode of deformation) is thereby uniquely determined. Some preliminary general observations follow.

Any mode, unique or not, is characterized by a variation principle, namely that

$$\delta\left[\int (U - \dot{g}^\beta v_\beta)dV - \int \dot{t}^\beta v_\beta d\Sigma\right] = 0, \tag{2.1}$$

where δ denotes a weak variation in the class of continuous differentiable velocity fields taking the prescribed values on S. In particular, if a bifurcation occurs under dead loading on Σ and in V, with rigid constraints on S, (one mode then being, of course, identically zero) each non-zero mode (eigenfield) is such that

$$\delta \int U dV = 0, \tag{2.2}$$

the stationary value of the functional being itself zero. Conversely, if there is a field for which (2.2) is satisfied, it must be an eigenfield.

Similarly, if (1.5) holds, then

$$\delta\left[\int \dot{t}^{\beta} v_{\beta} dS - \int U dV\right] = 0 \tag{2.3}$$

in the class of piecewise-continuous differentiable nominal stress-rate fields in equilibrium with the prescribed body-force rates in V and the prescribed nominal traction-rates on Σ. For an actual mode the functionals in (2.1) and (2.3) have the same value:

$$\tfrac{1}{2}\left[\int \dot{t}^{\beta} v_{\beta} dS - \int \dot{t}^{\beta} v_{\beta} d\Sigma - \int \dot{g}^{\beta} v_{\beta} dV\right].$$

§ 3. A sufficient condition for uniqueness

Suppose, now, that there could be more than one mode (a bifurcation) and let the prefix Δ denote the difference of corresponding quantities in any two distinct modes. Then, after using the equations of equilibrium

$$\dot{s}^{\alpha\beta}_{,\alpha} + \dot{g}^{\beta} = 0,$$

the boundary conditions, and the divergence theorem, we have

$$\int \Delta \dot{s}^{\alpha\beta} \Delta v_{\beta,\alpha} dV = 0.$$

A sufficient condition for uniqueness is therefore that

$$\int \Delta\left(\frac{\partial U}{\partial v_{\beta,\alpha}}\right) \Delta v_{\beta,\alpha}\, dV > 0 \tag{3.1}$$

for all pairs of continuous differentiable fields taking the given values on S. (The condition would not be strengthened by requiring that the associated traction-rate on Σ should take the prescribed values, since the range of the functional would be the same). This criterion was originally formulated in [3] in the context of finite elasticity, and in complete generality in [6].

The condition (3.1) is, of course, by no means necessary for uniqueness. In particular, in a given state the inequality may be satisfied for certain boundary-values on S and not for others, since the integrand is a multi-valued function of Δv_{β} (regarded as a single field). When the values prescribed for v_{β} on S are zero (rigid constraint) the identically zero field is admissible in (3.1) and so, if (3.1) holds for all fields vanishing on S, then necessarily

$$\int \frac{\partial U}{\partial v_{\beta,\alpha}} v_{\beta,\alpha}\, dV \equiv 2\int U(v_{\beta})\, dV > 0 \tag{3.2}$$

for all fields v_β vanishing on S and not identically zero. This excludes the possibility of an eigenfield (for which the functional in (3.2) would vanish).

The converse need not be true when the solid is non-linear; that is, (3.2) can hold when (3.1) does not. Among other things, this indicates the possibility that, during a monotonic process of distortion with rigid constraints on S, bifurcation under *varying* loads may precede the attainment of an eigenstate. That this in fact occurs for a built-in column under axial compression (as originally conjectured by Shanley) has recently been rigorously proved by Hill and Sewell [7] for certain elastic/plastic solids of type (1.6).

When the solid is linear, however, (3.1) and (3.2) coincide completely and the particular boundary values (for given S and Σ) no longer affect the question of uniqueness. If two modes satisfy the same inhomogeneous boundary conditions, their difference is an eigenfield; conversely, if an eigenfield exists, the solution to any inhomogeneous problem is non-unique to the extent of an arbitrary additive multiple of the eigenfield. Consequently, when the solid is linear, the problem of bifurcation is equivalent to the problem of the existence of eigenstates (an equivalence which is, of course, very familiar when the solid is elastic).

It is to be expected that the practically important eigenstates will be those in which the stationary zero value of $\int U\, dV$ is an absolute minimum in the class of admissible fields vanishing on S. However, conditions guaranteeing the existence of eigenstates with this extremum property (or, what is the same, the proof of a generalized Rayleigh Principle) have not yet been rigorously investigated for either linear or non-linear solids. Again, for a non-linear solid passing through a continuous series of states as a certain parameter is varied, there is the question whether a bifurcation necessarily first becomes possible (under some boundary conditions or other) in the critical state where the functional (3.1) ceases to be one-signed. Circumstances will be mentioned later where a definite answer can be given to this question.

With the help of (1.4) we can rewrite (3.2) as

$$\int [W(\varepsilon_{\alpha\beta}) + \tfrac{1}{2}\tau^{\alpha\beta} v_{\gamma,\alpha} v^{\gamma}_{,\beta}]\, dV > 0. \tag{3.3}$$

The explicit contributions of rotations of material elements now appear separately in the second term. If the body dimensions are such that rotation effects are negligible and if, further, the existing stress is small compared with 'moduli' appearing in W, this term can be disregarded.

In particular, when the solid is elastic and isotropic after infinitesimal prestrain, with Lamé constants λ, μ ($\mu > 0$), (3.3) reduces to

$$\frac{\lambda}{\mu} > -\tfrac{2}{3}(1+c), \text{ where } c = \text{Min}\, \frac{\int_\Sigma (\varepsilon_1 - \varepsilon_2)^2\, dV}{\int (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)^2\, dV} \tag{3.4}$$

in the class of all velocity fields vanishing on S, where ε_1, ε_2, ε_3 are the corresponding principal components of strain rate. Kirchhoff's classical proof of uniqueness required W to be positive definite, or $\lambda/\mu > -\frac{2}{3}$. The present result shows that bifurcation is generally impossible for an even wider range of λ/μ values (admittedly thermodynamically unacceptable). In particular, when $\Sigma = 0$, so that the velocity is prescribed over the entire surface, $c = 2$; this was shown in effect by Ericksen and Toupin [1] from a different standpoint. At the other extreme, when $S = 0$, so that the velocity is prescribed nowhere, $c = 0$ (since a field of uniform dilatation is admissible).

§ 4. Extremum principles

Suppose, now, that there is given a state in which (3.1) holds, so that the mode is certainly unique for the given boundary values. Then it can be asserted that, not only is the functional in (2.1) stationary, but it attains an absolute minimum for the actual mode ([5], p. 244); that is

$$\Delta\left[\int (U - \dot{g}^\beta v_\beta)\, dV - \int \dot{t}^\beta v_\beta\, d\Sigma\right] > 0, \tag{4.1}$$

where (in this formula only) v_β denotes the actual mode, v_β^* any other admissible field, and the Δ difference is to be taken in the order corresponding to $\Delta v_\beta = v_\beta^* - v_\beta$. An improved version of the original proof is now presented. It depends on showing that (3.1) is completely equivalent to

$$\int \left[\Delta U - \frac{\partial U}{\partial v_{\beta,\alpha}}\, \Delta v_{\beta,\alpha}\right] dV > 0, \tag{4.2}$$

where v_β, v_β^* are now any distinct pair of fields. Evidently, when v_β is the actual mode, (4.2) transforms into (4.1).

Suppose, first, that (4.2) is given; then, by interchanging v_β and v_β^*,

$$-\int \left[\Delta U - \frac{\partial U^*}{\partial v^*_{\beta,\alpha}}\, \Delta v_{\beta,\alpha}\right] dV > 0.$$

Addition of the two preceding inequalities gives (3.1) immediately. Conversely, suppose (3.1) is given; with $\Delta v_\beta = \eta w_\beta$, where η is an infinitesimal constant, it implies

$$\int \frac{\partial^2 U}{\partial v_{\beta,\alpha}\partial v_{\delta,\gamma}} w_{\beta,\alpha} w_{\delta,\gamma}\, dV > 0, \tag{4.3}$$

when v_β is any admissible field (taking the given values on S) and w_β is any continuous differentiable field vanishing on S. Denote the functional in (4.3) by $F(v, w)$. Now put $\Delta v_\beta = \lambda w_\beta$ in the functional in (4.2), where λ is a constant parameter of arbitrary magnitude, and regard it as a function of λ:

$$f(\lambda) = \int \left[U(v_{\beta,\alpha} + \lambda w_{\beta,\alpha}) - U(v_{\beta,\alpha}) - \lambda \frac{\partial U}{\partial v_{\beta,\alpha}} w_{\beta,\alpha} \right] dV.$$

Then $f(0) = 0$ and also, in view of the assumed continuity of the first derivatives of U, $f'(0) = 0$. Moreover, since the second derivatives are assumed at least piecewise continuous,

$$f''(\lambda) = F(v + \lambda w, w) > 0$$

by (4.3), since $v_\beta + \lambda w_\beta$ is admissible for any λ. It follows that $f(\lambda)$ is a positive convex function for all λ, zero only when $\lambda = 0$, and hence (4.2) is proved on identifying λw_β with Δv_β.

When (1.5) is valid, an alternative (though possibly less convenient) sufficient condition for uniqueness would be

$$\int \Delta \left(\frac{\partial U}{\partial \dot{s}_{\alpha\beta}} \right) \Delta \dot{s}^{\alpha\beta}\, dV > 0 \tag{4.4}$$

for any distinct pair of piecewise continuous fields of nominal stress-rate in equilibrium with the prescribed body force rates in V and traction rates on Σ. When the criteria (3.1) and (4.4) differ, as happens in general, it is not known which is the stronger. However, when (4.4) is satisfied, we can show (as in the preceding proof) that the functional in (2.3) attains an absolute maximum for the actual mode.

§ 5. Comparison theorems

Since the integrand in (3.1) is multi-valued when the solid is non-linear, this criterion is inconvenient in practice. A more useful criterion can be obtained in terms of a certain linear solid, with potential W_L say, which is such that

$$U - U_L = W - W_L = \Phi(\varepsilon_{\alpha\beta}),$$

say, is a *convex* function of the strain-rate tensor (and therefore also of the velocity gradient) at every point of the body. By this we mean that, everywhere in the body and for any pair of strain-rates, the following equivalent inequalities hold:

$$\text{(i)} \qquad \Delta\left(\frac{\partial \Phi}{\partial \varepsilon_{\alpha\beta}}\right)\Delta\varepsilon_{\alpha\beta} \geq 0,$$

$$\text{(ii)} \qquad \Delta\Phi - \frac{\partial \Phi}{\partial \varepsilon_{\alpha\beta}}\Delta\varepsilon_{\alpha\beta} \geq 0, \tag{5.1}$$

$$\text{(iii)} \qquad \frac{\partial^2 \Phi}{\partial \varepsilon_{\alpha\beta}\partial \varepsilon_{\gamma\delta}}\Delta\varepsilon_{\alpha\beta}\Delta\varepsilon_{\gamma\delta} \geq 0.$$

We shall speak of *strict convexity* when the inequality (and never the equality) always holds for distinct pairs. Conditions (5.1) incidentally imply that $\Phi \geq 0$, since it is here a homogeneous function of second degree. In practice, the convexity of Φ can most conveniently be tested through (iii), which states the positive character of the quadratic form associated with the Hessian matrix of Φ for any strain-rate.

Suppose, now, that

$$\int U_L(\Delta v_\beta)\, dV > 0, \tag{5.2}$$

so that bifurcation is impossible for the linear solid W_L under any boundary conditions on the given S and Σ. But for a linear solid we have identically

$$U_L(\Delta v_\beta) \equiv \Delta U_L - \frac{\partial U_L}{\partial v_{\beta,\alpha}}\Delta v_{\beta,\alpha} \equiv \tfrac{1}{2}\Delta\left(\frac{\partial U_L}{\partial v_{\beta,\alpha}}\right)\Delta v_{\beta,\alpha} \equiv$$
$$\equiv \frac{1}{2}\frac{\partial^2 U}{\partial v_{\beta,\alpha}\partial v_{\delta,\gamma}}\Delta v_{\beta,\alpha}\Delta v_{\delta,\gamma}. \tag{5.3}$$

Hence, from (i) of (5.1) with (5.2) and (5.3), it may be seen that (3.1) is satisfied, and so bifurcation is impossible also for the non-linear solid W under any boundary conditions on the given S and Σ. From this it incidentally follows, in particular, that (3.2) holds for U for fields vanishing on S (or this follows directly from (5.2) since $\Phi = U - U_L \geq 0$); the inequality (3.2) is therefore a pre-requisite for the existence of a function U_L satisfying (5.1) and (5.2). Again, from (ii) of (5.1) with (5.2) and (5.3), we recover (4.2) and then the extremum principle (4.1) immediately without needing the previously proved full equivalence of (3.1) and (4.2).

This comparison method was worked out in detail for elastic/plastic solids of type (1.6) ([5], especially equations (16) and (18) there). An analogous method was also used in connexion with rigid/plastic solids [4]; these do not strictly admit a potential of type (1.6), but may in most respects be regarded as a limiting case.

Of course, (3.1) may hold in certain states even where (5.2) does not, and to this extent (5.2) is a weaker sufficient condition. Evidently, the best choice for the comparison linear solid is that which makes the greatest lower bound of the functional in (5.2) as large as possible, while still maintaining the required convexity of $W - W_L$. It can indeed happen that, in the fundamental eigenstate of the best comparison solid W_L, a bifurcation is actually possible for the non-linear solid W under certain boundary conditions. This can occur when the potential W coincides with the quadrate W_L in a part of strain-rate space (which may vary with position in the body), and when moreover, for certain boundary conditions, a sub-family of the corresponding modes for W_L involve only strain-rates in this part. These modes are then clearly solutions also of the boundary-value problem for the solid W, and therefore constitute a bifurcation.

This situation is found in the column problem when W is the potential (1.6) [8]. Under increasing axial load the column can deflect, in a range of possible ways, at the eigenstate load of a geometrically similar column of the linear solid defined by

$$2W_L = \boldsymbol{\varepsilon} K \boldsymbol{\varepsilon} - (\mathbf{m}\boldsymbol{\varepsilon})^2, \qquad \text{for all } \boldsymbol{\varepsilon},$$

provided suitable restrictions are placed on the matrix K. In this example the convexity of Φ is obvious from (iii) of (5.1), since here Φ is the positive quadratic form $\frac{1}{2}(\mathbf{m}\boldsymbol{\varepsilon})^2$ when $\mathbf{m}\boldsymbol{\varepsilon} \leq 0$ and is identically zero when $\mathbf{m}\boldsymbol{\varepsilon} \geq 0$. The eigenstate for the column of non-linear material is reached at a higher load; between these critical loads, and also in the eigenstate itself, a more restricted type of bifurcation appears possible.

§ 6. Non-linear solids with strictly convex potentials

In conclusion, we mention briefly a special class of non-linear solids such that U is itself a strictly convex function of velocity gradient (and therefore also of nominal stress-rate) at every point of the body. Since the integrands in (3.1), (4.2), and (4.4) are then positive everywhere, bifurcation is impossible for all boundary conditions and the complementary extremum principles are always valid. This happens, in particular, when W is a strictly convex function of strain-rate, provided that

$U \doteq W$ to a sufficient approximation. Under these last conditions the uniqueness theorem and extremum principles were originally obtained in [2], not only for solids characterised by rate equations of type (1.1) (page 69 of the reference), but for any material (solid or fluid) for which the stress itself is derivable from a potential function (not necessarily homogeneous) of strain rate or of infinitesimal strain (the boundary value problem being appropriately set in the corresponding variables in each case). Among such materials were mentioned viscous solids or fluids, viscoplastic solids of Bingham type, and viscoelastic solids of Maxwell and Voigt types [equations (9), (10), (12), respectively, of the reference]. As in the present paper, no restrictions were placed on the state of anisotropy or inhomogeneity in the main theorems.

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A COMPARISON OF YIELD CONDITIONS IN THE THEORY OF PLASTIC SHELLS[1]

PHILIP G. HODGE, Jr.
Chicago, USA

Exact yield conditions for shells whose material satisfies either the Tresca or Mises-Huber-Hencky yield criteria are compared with two approximate linear conditions. Factors are established by which the solution according to any theory can provide both upper and lower bounds on the solution according to any other theory. The results are illustrated with reference to an annular plate under combined tension and bending.

§ 1. Basic concepts

We are concerned with a rigid/perfectly-plastic material which remains rigid under stresses which leave a certain function of stress less than a given constant, flows in accord with the plastic potential flow law [1] when this function equals the given constant, and cannot tolerate stresses such that this function exceeds the given constant. The theory can be stated in terms of the physical components of stress and strain rate. For our purposes, however, it will be more convenient to deal with dimensionless generalized stresses and strain rates which are appropriate to thin shell problems.

§ 1.1. *Generalized shell variables.* Let N_α and M_α ($\alpha = 1, 2$) denote the principal direct stresses and bending moments in the shell and let e_α and κ_α denote the corresponding extension and curvature rates of the middle surface. In a four dimensional stress-resultant or strain-rate-resultant space, let the dimensionless stress and strain-rate vectors be

$$\mathbf{S} = (n_1, n_2, m_1, m_2), \qquad \mathbf{E} = (e_1, e_2, k_1, k_2) \tag{1.1}$$

respectively, where

$$n_\alpha = N_\alpha/N_0, \qquad m_\alpha = M_\alpha/M_0, \qquad k_\alpha = M_0\kappa_\alpha/N_0, \qquad N_0 = 2\sigma_0 H, \qquad M_0 = \sigma_0 H^2. \tag{1.2}$$

Here σ_0 is the tensile yield stress and $2H$ the shell thickness.

[1] The results presented in this paper were obtained in the course of research sponsored by the National Science Foundation.

The internal rate of dissipation of energy per unit area of middle surface is

$$d_i = N_1 e_1 + N_2 e_2 + M_1 \kappa_1 + M_2 \kappa_2 = N_0 \mathbf{S} \cdot \mathbf{E}. \tag{1.3}$$

Therefore, $\mathbf{S}$ and $\mathbf{E}$ satisfy the criterion for proper generalized variables [2].

§ 1.2. *Statement of boundary value problem.* Given a shell with a load distribution prescribed except for its magnitude p, the boundary value problem for a rigid/perfectly-plastic material involves determining a stress field $\mathbf{S}$, velocity field $\mathbf{E}$, and load magnitude p_0 such that:

(1) $\mathbf{S}$ is in internal and external equilibrium with the loads p_0;
(2) the dimensionless yield function $f(\mathbf{S})$ nowhere exceeds one;
(3) $\mathbf{E}$ is derivable from a velocity field which satisfies any velocity constraints;
(4) $\mathbf{E}$ is zero wherever $f(\mathbf{S}) < 1$; $\mathbf{E}$ and $\mathbf{S}$ are related by the plastic potential flow law wherever $f(\mathbf{S}) = 1$.

The load p_0 defined above will be called the yield-point load. Alternative terms are the collapse load or limit load. It has been shown [3, 4] that this load has the same value as the load at which an elastic/perfectly-plastic material can undergo indefinitely large deformations assuming that geometry changes can be neglected. It has also been demonstrated in several examples [5, 6, 7] that the yield-point load is a close approximation to the load at which an elastic/strain-hardening material begins to deform at a rate much greater than it did elastically.

§ 1.3. *Theorems of limit analysis.* Close and sometimes coincident bounds can be obtained for the yield-point load without necessarily solving the preceding boundary value problem. Two theorems formally stating this fact were first proved by Hill [8, 9] and Drucker, Greenberg, and Prager [3, 4, 10].

We first define a statically admissible stress field $\mathbf{S}^\circ$ as any field which satisfies requirements (1) and (2) preceding in equilibrium with a load magnitude p^-. Then the lower bound theorem states

$$p^- \leq p_0. \tag{1.4}$$

The upper bound theorem starts with a kinematically admissible field which consists of any strain-rate field $\mathbf{E}^*$ which satisfies requirement (3) together with any stress field $\mathbf{S}^*$ obtained from it by (4). Let $\mathbf{v}^*$ be the velocity associated with $\mathbf{E}^*$ and define a load magnitude p^+ such that the external rate of dissipation of energy done by the loads p^+ on $\mathbf{v}^*$ is

equal to the internal rate of dissipation of energy obtained by integrating (1.3) for **E***, **S***. The theorem then states that

$$p_0 \leq p^+. \tag{1.5}$$

In the following we shall use two easily proved lemmas of the lower bound theorem. The bounding yield surface lemma states that, if two yield surfaces A and B are such that A is everywhere on or interior to B, then

$$p_{0A} \leq p_{0B}. \tag{1.6}$$

The convexity lemma is concerned with a shell subjected to two or more independently prescribed loads, $p_1, p_2, \cdots, p_n$. The role of the yield-point load is now taken over by the yield-point loading surface which is the collection of all load points $\mathbf{P} = (p_1, p_2, \cdots, p_n)$ in load space for which the boundary value problem of Sec. 1.2 has a solution. The lemma states simply that, if the yield surface $f(\mathbf{S}) = 1$ is convex in stress space, then the yield-point loading surface is convex in load space.

§ 2. Yield surfaces

Thus far we have not discussed the nature of the yield surface. Various exact and approximate yield surfaces for shells have been discussed in the literature. Here we summarize some of the most important ones.

§ 2.1. *Yield condition according to Tresca.* Tresca's yield condition states that the maximum shearing stress must not exceed half the tensile yield stress. Thus, under the essentially plane stress conditions assumed in shell theory

$$\max [|\sigma_1|, |\sigma_2|, |\sigma_1 - \sigma_2|] = \sigma_0 \tag{2.1}$$

at a plastic point. Onat and Prager [11] have derived the shell yield condition and flow law based on (2.1) in terms of three parameters p, q, and r defined by

$$p = -\frac{1}{4}\frac{e_1}{k_1}, \qquad r = -\frac{1}{4}\frac{e_2}{k_2}, \qquad q = -\frac{1}{4}\frac{e_1 + e_2}{k_1 + k_2}. \tag{2.2}$$

Evidently the strain-rate vector (1.1) is given parametrically to within a scalar factor v by

$$\mathbf{E} = v[-4p(q - r), -4r(p - q), (q - r), (p - q)]. \tag{2.3}$$

The resulting yield surface is summarized in Table 1 which is subject to the following interpretation:

TABLE 1. Tresca yield surface for uniform shell

Intermediate Parameter	Stress Resultants			
	n_1	n_2	m_1	m_2
p	$\mp (p+q)$	$\mp (q-r)$	$\pm 1 \mp 2(p^2+q^2)$	$\pm 2(r^2-q^2)$
q	$\mp (p+q)$	$\mp (q+r)$	$\pm 1 \mp 2(p^2+q^2)$	$\pm 1 \mp 2(q^2+r^2)$
r	$\mp (q-p)$	$\mp (q+r)$	$\pm 2(p^2-q^2)$	$\pm 1 \mp 2(q^2+r^2)$

In all cases the upper or lower signs must be used consistently in a given line, the correct choice being that which makes d_i positive.

If p, q, and r are all distinct and all within the interval $(-\frac{1}{2}, \frac{1}{2})$, then the "intermediate parameter" in Table 1 is that one of p, q, r which lies between the other two.

If one or more of p, q, r is greater than $\frac{1}{2}$, then that parameter must be replaced by $\frac{1}{2}$ in Table 1; similarly, if one or more parameter is less than $-\frac{1}{2}$.

If any of the parameters, p, q, or r become indeterminate as computed from Eqs. (2.2), the other two parameters must necessarily be equal. For example, if $e_2 = k_2 = 0$, r is indeterminate and $p = q$. The appropriate yield surface is obtained when p is set equal to q in either of the first two lines in Table 1 and eliminated from those stresses which are independent of r. Following this procedure for each of the three possibilities $p = q$, $q = r$, and $r = p$, we obtain six hyper-cylinders:

$$m_1 = \pm (1 - n_1^2), \tag{2.4}$$

$$m_2 = \pm (1 - n_2^2), \tag{2.5}$$

$$m_1 - m_2 = \pm [1 - (n_1 - n_2)^2]. \tag{2.6}$$

Finally, if $p = q = r$, Table 1 can be used with the convention that the "intermediate parameter" corresponds to the absolutely largest numerator of Eqs. (2.2).

§ 2.2. *Yield condition according to Mises-Huber-Hencky.* If the material of the shell satisfies the Mises-Huber-Hencky criterion of octahedral shearing stress

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma_0^2, \tag{2.7}$$

the yield surface is again given by three parameters p, q, and r. However, in this case no simple physical interpretation such as (2.2) can be

given the parameters. The resulting surface and strain-rate vector are defined parametrically by [12]

$$(n_1, n_2)[\sqrt{3}\sin(p-q)] =$$

$$= \cos p \cos q \cos(r \mp \pi/6) \log \frac{(1+\sin q)(1-\sin p)}{(1-\sin q)(1+\sin p)} +$$

$$+ 2\sin(r \mp \pi/6)(\cos p - \cos q),$$

$$(2.8)\quad (m_1, m_2)[\sqrt{3}\sin^2(p-q)/(2\cos p\cos q)] =$$

$$= [\sin(p+q)\cos(r \mp \pi/6) + \cos p \cos q \sin(r \mp \pi/6)] \cdot$$

$$\cdot \log[(1+\sin q)(1-\sin p)(1-\sin q)^{-1}(1+\sin p)^{-1}] -$$

$$- 4\cos(r \mp \pi/6)(\cos p - \cos q) - 2\sin(r \mp \pi/6)(\sin q - \sin p),$$

$$(e_1, e_2) = \pm 2\nu[\cos q \sin(r - p \pm \pi/6) + \cos p \sin(r - q \pm \pi/6)],$$

$$(k_1, k_2) = \pm \nu \sin(p-q)\cos(r \pm \pi/6).$$

In (2.8) the parameters p and q must lie in one or the other of the domains

$$(2.9a)\qquad -\pi/2 < p \leq q < \pi/2,$$

$$(2.9b)\qquad \pi/2 < p \leq q < 3\pi/2,$$

The boundary between the two parts of (2.8) is given by

$$(n_1, n_2)/t = (m_1, m_2)/(1-t^2) = -(2/\sqrt{3})\sin(r \mp \pi/6), \quad |t| < 1,$$

$$(2.10)\quad (n_1, n_2) = -(2/\sqrt{3})\operatorname{sgn} t \sin(r \mp \pi/6) \quad m_1 = m_2 = 0, \quad |t| \geq 1,$$

$$(e_1, e_2) = \pm 2\nu t \cos(r \mp \pi/6) \quad (k_1, k_2) = \pm \nu \cos(r \pm \pi/6).$$

§ *2.3. Piecewise linear yield conditions.* It would be cumbersome if not impossible to solve any but the simplest of problems according to either of the preceding yield conditions. For this reason, it has seemed worthwhile to develop approximate yield surfaces which are somewhat simpler. The first such surface to be proposed [13, 14] can be given a physical interpretation as the yield surface according to Tresca for an idealized sandwich shell; its equations are

$$(2.11)\quad \max[|n_1 \pm m_1|, |n_2 \pm m_2|, |(n_1 - n_2) \pm (m_1 - m_2)|] = 1.$$

A second and somewhat simpler approximation is the limited interaction surface first proposed in [15]

$$(2.12)\quad \max[|n_1|, |n_2|, |n_1 - n_2|, |m_1|, |m_2|, |m_1 - m_2|] = 1.$$

§ 3. Comparison of yield surfaces

In order to interpret the approximate yield surfaces suggested in § 2.3 in terms of either of the more realistic yield surfaces, we invoke the bounding yield surface lemma. Taking any one yield surface as fixed in stress-resultant space, we can expand or shrink any other surface so that it will lie wholly outside or wholly inside the first surface. Therefore we can apply the lemma to determine both upper and lower bounds on the yield-point load for one yield surface based on the solution for any other surface. It remains to determine the various expansion and contraction coefficients.

To avoid excessive words in the analysis, we shall denote the yield surfaces by f and use subscripts to distinguish them: thus f_T is the yield surface according to Tresca, f_M the surface according to Mises-Huber-Hencky, f_S the sandwich yield surface (2.11), and f_L the limited interaction surface (2.12). Similarly, we shall use p_T, p_M, p_S, and p_L to denote the yield-point loads according to the various theories.

§ 3.1. *Bounds on the Tresca yield surface.* The relation between f_T and the linear surfaces f_S and f_L has been treated previously in [15] where the results

(3.1a, b) $$p_S \leq p_T \leq 1.25 p_S,$$

(3.1c, d) $$0.618 p_L \leq p_T \leq p_L$$

were obtained.

To find the relation between f_T and f_M we note that both are based on actual stress distributions throughout the shell thickness. Since the curve (2.1) is interior to the curve (2.7), any stress distribution satisfying (2.1) is statically admissible according to (2.7). In particular, the collection of stress resultants based on (2.1) is statically admissible according to (2.7); hence f_T is interior to f_M. A similar argument shows that $(\sqrt{3}/2)f_M$ is interior to f_T. Therefore

(3.1e, f) $$0.866 p_M \leq p_T \leq p_M.$$

§ 3.2. *Bounds on the Mises-Huber-Hencky yield surface.* Solving (3.1e, f) for p_M we obtain

(3.2a, b) $$p_T \leq p_M \leq 1.155 p_T.$$

If we now construct continuing inequalities from (3.1) and (3.2a, b) and eliminate p_T we obtain

(3.2c, d) $$0.618 p_L \leq p_M \leq 1.155 p_L,$$

(3.2e, f) $$p_S \leq p_M \leq 1.444 p_S.$$

It remains to be seen if (3.2) are the sharpest possible inequalities.

To show that an inequality cannot be improved upon it is necessary and sufficient to find some stress state for which it is an equality. We consider first simple tension with $n_2 = m_1 = m_2 = 0$. For this case, all four yield surfaces reduce to

$$f_T = f_M = f_S = f_L = n_1 = 1.$$

Therefore, the inequalities (3.2a) and (3.2e) are sharp.

Next, we consider the unequal biaxial tension state $n_2 = 2n_1$, $m_1 = m_2 = 0$. In this case

$$f_T = f_S = f_L = 2n_1 = 1, \qquad f_M = \sqrt{3}n_1 = 1.$$

Therefore $(2/\sqrt{3})f_M$ will predict the same value for n_1 as f_T, f_S, or f_L will; hence (3.2b) and (3.2d) cannot be improved upon.

Now consider the stress field $n_1 = n_2 = m_1 = m_2$. This is part of the boundary curve (2.10) for f_M and corresponds to parameter values

$$r = \pi/2, \qquad t^2 + t - 1 = 0;$$

hence

$$n_1 = n_2 = m_1 = m_2 = (\sqrt{5} - 1)/2 = 0.618.$$

On the other hand, f_L for this stress field yields the values

$$n_1 = n_2 = m_1 = m_2 = 1.$$

Thus f_L must be multiplied by 0.618 to equal f_M which verifies the preciseness of (3.2c).

Finally, we consider the plane $n_1 + m_1 = 1$ of f_S. The corresponding strain-rate vector is in the direction (1, 0, 1, 0). According to f_M, a strain-rate vector in this direction must be on the boundary surface (2.10) with

$$r = -\pi/2, \qquad t = \tfrac{1}{2}, \qquad n_1 = 1/\sqrt{3}, \qquad m_1 = \sqrt{3}/2.$$

Evidently

$$n_1 + m_1 = 5/(2\sqrt{3}) = 1.444$$

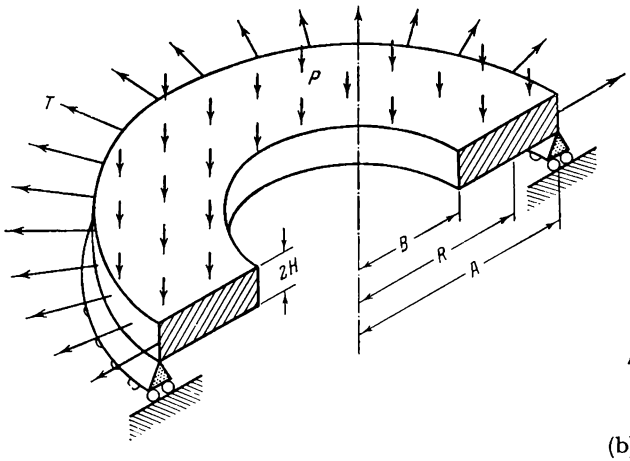
for f_M as compared with $n_1 + m_1 = 1$ for f_S; thus (3.2f) is in its sharpest form.

§ 4. Example

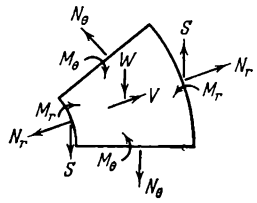
We consider as an example the annular plate shown in Fig. 1a. The outer edge is simply supported against vertical motion and subjected to a radial tensile load T per unit length; the inner edge is free and unloaded; a normal load P per unit area is applied to the top face of the plate.

The principal directions in the plate are fixed by symmetry in the circumferential and radial directions, θ and r, respectively. Fig. 1b shows the sign conventions for forces and velocities. We define dimensionless quantities by (1.2) and

$$r = R/A, \qquad b = B/A, \qquad h = H/A,$$
$$v = V/A, \qquad w = W/H, \qquad s = S/N_0, \tag{4.1}$$
$$t = T/N_0, \qquad p = PA^2/6M_0.$$



(a) Loaded plate



(b) Sign conventions for stresses and velocities.

Fig. 1. Annular plate

Then the equilibrium equations and the strain-rate-velocity relations are given by

$$(rn_r)' - n_\theta = 0, \tag{4.2a}$$

$$(rs)' - 3phr = 0, \tag{4.2b}$$

$$(rm_r)' - m_\theta = -(2/h)rs, \tag{4.2c}$$

$$\mathbf{E} = (v/r, v', -h^2w'/2r, -h^2w''/2), \tag{4.3}$$

where primes denote differentiation with respect to r. The boundary conditions are

$$m_r = w = 0, \qquad n_r = t, \qquad \text{at} \quad r = 1, \tag{4.4a}$$

$$m_r = n_r = s = 0, \qquad \text{at} \quad r = b. \tag{4.4b}$$

We consider first a plate which satisfies the limited interaction yield surface. Given a ratio between p and t, the problem to be solved is to determine six functions n_θ, n_r, m_θ, m_r, w, and v and a load magnitude which satisfy the four requirements of § 1.2 based on (2.12), (4.2), (4.3) and (4.4). For the particular cases where $p = 0$ or $t = 0$ the resulting problems are almost trivially solved, the results being

$$\begin{gathered} n_\theta = 1, \qquad n_r = 1 - (1 - t)/r, \qquad m_\theta = m_r = 0, \\ p = 0, \qquad t = t_0 \equiv 1 - b, \\ v = C > 0, \qquad w = 0 \end{gathered} \tag{4.5}$$

and

$$\begin{gathered} n_\theta = n_r = 0, \qquad m_\theta = 1, \\ m_r = 1 + 3pb^2 - pr^2 - (1 + 3pb^2 - p)/r, \\ p = p_0 \equiv (1 + b - 2b^2)^{-1}, \qquad t = 0, \\ v = 0, \qquad w = D(1 - r), \qquad D > 0. \end{gathered} \tag{4.6}$$

If $0 < p < p_0$ the solution for moments is not uniquely determined. One possible solution is to take an appropriate multiple of (4.6) for the moments and (4.5) for the direct stresses and velocities. Thus

$$\begin{gathered} n_\theta = 1, \qquad n_r = 1 - (1 - t)/r, \qquad m_\theta = \lambda, \\ m_r = \lambda[1 + 3p_0b^2 - p_0r^2 - (1 + 3p_0b^2 - p_0)/r], \\ p = \lambda p_0, \qquad 0 \leq \lambda \leq 1, \qquad t = t_0, \\ v = C > 0, \qquad w = 0. \end{gathered} \tag{4.7}$$

A similar solution is easily constructed for $p = p_0$, $0 \leq t \leq t_0$. Therefore, the yield-point loading curve based on the limited interaction loading surface is the rectangle bounded by the lines

$$t = t_0, \qquad p = p_0. \tag{4.8}$$

Equation (4.8) and inequalities (3.2c, d) can be used to obtain bounds on the yield-point loading curve according to the Mises-Huber-Hencky yield surface. Thus, the desired curve is bounded by the rectangles ABC and DEF in Fig. 2.

We can obtain better bounds by observing that the problems of pure tension ($p = 0$) or pure bending ($t = 0$) are easily solved for the Tresca yield surface. Indeed, for these particular cases f_T and f_L are identical so that (4.5) and (4.6) represent solutions according to the Tresca

yield surface. Therefore, it follows from inequalities (3.2a, b) that the yield-point loading curve hits the t axis somewhere between the points G ($t = 0.8$) and A, whereas it crosses the p axis between H ($p = 0.893$) and C.

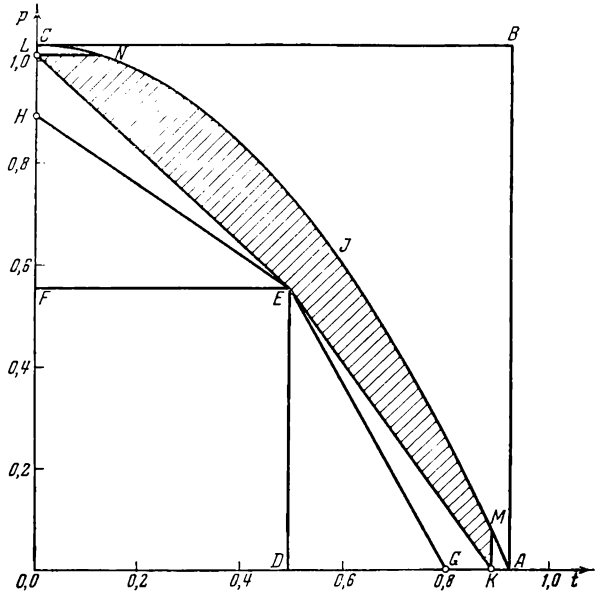


Fig. 2. Bounds on the interaction curve according to Mises-Huber-Hencky ($b = 0.2$).

We now use the convexity lemma to improve the inner bound on the yield-point loading curve. Indeed, since the points E, G, and H are all on or within the yield-point loading curve, the broken line GEH is an inner bound on the curve.

To improve the outer bound, we take as a kinematically admissible velocity field a linear combination of the velocity fields in (4.5) and (4.6) and apply the upper bound theorem. According to (4.3) the strain-rate vector is

$$\mathbf{E} = (Dh^2/r)(\beta, 0, \tfrac{1}{2}, 0), \tag{4.9}$$

where

$$\beta = C/Dh^2.$$

Evidently this vector is normal to the boundary (2.10) of the yield surface f_M. Further, a comparison of (4.9) and the last equation (2.10) shows that

$$r = -\pi/3, \qquad t = \beta, \qquad v = Dh^2/(\sqrt{3}r).$$

The first line of (2.10) then leads to

$$(4.10)\qquad n_\theta = 2n_r = 2\beta/\sqrt{3}, \quad m_\theta = 2m_r = (2/\sqrt{3})(1-\beta^2), \quad 0<\beta<1.$$

It follows next from (1.3), (4.9), and (4.10) that the total internal rate of dissipation of energy of the annulus is

$$(4.11a)\qquad D_i = 2\pi A^2 \int_b^r N_0 \mathbf{S}\cdot\mathbf{E} r dr = 2\pi A^2 N_0 D h^2 (1-b)(1+\beta^2)/\sqrt{3}.$$

Similarly, the external rate of dissipation of energy is

$$(4.11b)\qquad D_e = 2\pi A^2 N_0 \left[3ph^2 \int_b^1 wr dr + tv(1)\right] =$$

$$= \pi A^2 N_0 D h^2 [(1-3b^2+2b^3)p + 2\beta t].$$

Equating the two equations (4.11) in accord with the upper bound theorem we obtain

$$(4.12)\qquad p = [(2/\sqrt{3})(1-b)(1+\beta^2) - 2\beta t](1-3b^2+2b^3)^{-1}.$$

Equation (4.12) is an upper bound on p for a given t for any value of $0<\beta<1$. Obviously the best upper bound will be obtained if p is minimized with respect to β for a given t. Thus

$$(4.13)\qquad \beta = (\sqrt{3}/2)t(1-b)^{-1}$$

and the upper bound on the yield-point loading curve becomes

$$(4.14)\qquad (\sqrt{3}/2)(1-b)(1+2b)p + (\tfrac{3}{4})[t(1-b)^{-1}]^2 = 1,$$

as shown by the curve AJC in Fig. 2. Therefore, the yield-point loading curve must lie somewhere within the region $AJCHEGA$ in Fig. 2.

A further improved lower bound can be obtained fairly easily by finding exact numerical solutions to the problems of pure stretching and pure bending according to the Mises-Huber-Hencky yield surface. In the former case this involves only the straight-forward numerical integration of the equation

$$(4.15)\qquad (rn_r)' - (\tfrac{1}{2})(n_r + \sqrt{4-3n_r^2}) = 0$$

with the initial condition $n_r(b) = 0$. The analysis for pure bending is slightly more complicated but does not offer any essential difficulty [16].

In terms of Fig. 2, the above analysis gives two points K and L which

must lie on the exact yield-point loading curve. Therefore, the convexity lemma shows that KEL is an inner bound. Further, since the curve will obviously be symmetric into either of the adjoining quadrants, and since it must be convex, it can never extend beyond a vertical line through K nor above a horizontal line through L. Therefore, by considering only very much simpler problems, we have shown that the yield-point loading curve according to the Mises-Huber-Hencky yield surface must lie within the shaded region $KMJNLEK$ in Fig. 2.

A similar and somewhat simpler analysis may be carried out for the Tresca yield surface [17]. A typical result is shown in Fig. 3. The points

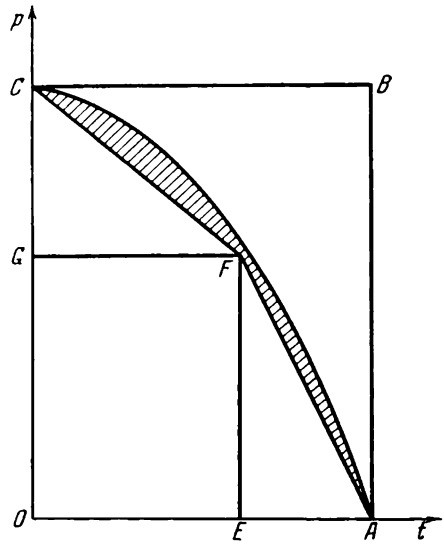


Fig. 3. Bounds on the interaction curve according to Tresca ($b = 0.3$).

A, F, and C are $(t_0, 0)$, $0.618(t_0, p_0)$, and $(0, p_0)$, respectively where t_0 and p_0 are the same as in (4.5) and (4.6). The equation of the upper bound curve is

$$(1 + b - 2b^2)\dot{p} + t^2/(1 - b)^2 = 1. \tag{4.16}$$

As shown in [17], only a slight modification is necessary to find similar bounds for other support conditions.

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THE EFFECT OF MERIDIAN CURVATURE ON THE INFLUENCE COEFFICIENTS OF THIN SPHERICAL SHELLS

(A STUDY OF THE ACCURACY OF THE GECKELER APPROXIMATION)

NICHOLAS J. HOFF
Stanford, U.S.A.

A new approximate solution of the differential equation governing the axisymmetric deformations of thin spherical shells is derived by expanding the trigonometric term of the equation into a Taylor series and retaining the first two terms of the series. Thus the curvature of the meridian is taken into account in an approximate manner. In the Geckeler solution this is not done because the differential equation is reduced to one with constant coefficients. Nevertheless the influence coefficients derived from the new solution differ from the Geckeler solution only by correction factors which are small in the range of the parameters in which the new solution is valid. This range is investigated in some detail. The investigation leads to the conclusion that the influence coefficients of the Geckeler approximation should be accurate enough for most engineering purposes when the loaded edge of the shell is at an angle $30° < \phi < 150°$ and the shell geometry is characterized by $r/h > 100$.

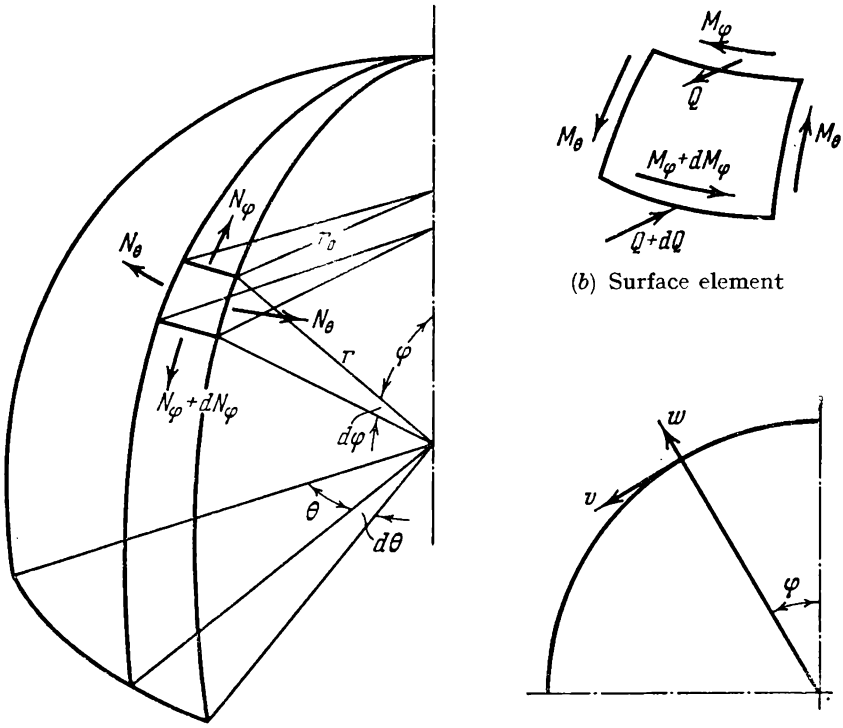
§ 1. Introduction

One of the most important problems in the stress analysis of pressure vessels, fuel tanks and other shell-type structures is the calculation of the stresses at the juncture of two dissimilar shells, such as the cylindrical body and the spherical vessel head. This can be accomplished with ease if the influence coefficients of the shells are known. The influence coefficients are the deformations of the edge of the shell caused by distributed edge loads of unit magnitude.

Since the equations governing the axisymmetric deformations of thin shells have variable coefficients except in the case of the circular cylinder, the determination of the influence coefficients is not an easy task as a rule. This is one of the reasons why influence coefficients derived from the Geckeler approximation have been in general use in engineering. This approximation will be described in more detail later. In it the differential

equation of the problem is reduced to one having constant coefficients which can be solved without difficulty.

It is almost obvious that the Geckeler approximation should be accurate enough for engineering purposes in the case of the sphere when the free edge is located near $\phi = 90°$ (see Fig. 1). It is equally clear that the approximation is inadmissible in the neighborhood of $\phi = 0°$. But it is questionable, unless a more thorough investigation yields satisfactory results, whether the use of influence coefficients derived from the Geckeler solution is permissible in engineering analysis for instance at $\phi = 45°$ for the usual r/h ratios.



(*a*) Section of sphere

(*b*) Surface element

(*c*) Displacements

Fig. 1. Notation
(Moment vectors follow left-hand screw rule)

In the present paper the rigorous equation of the spherical shell is replaced by an approximate one in which the first two terms of the Taylor expansion of the cotangent function are substituted for the cotangent function itself when it appears as a variable coefficient of the equation. Thus the curvature of the meridian is taken into account in an approximate manner. The accuracy of this equation and of its solution is investi-

gated in detail and closed-form expressions are derived for the influence coefficients. The results obtained are compared to the influence coefficients following from the Geckeler approximation.

§ 2. Differential equations

The differential equation governing the deformations of thin spherical shells subjected to axisymmetric edge loading can be given in the form (see, for instance, p. 151 of [1])

$$L^2(Q) + \mu^4 Q = 0, \tag{2.1}$$

or as

$$L^2(\chi) + \mu^4 \chi = 0, \tag{2.2}$$

where Q is the shear stress resultant per unit length, χ the rotation of the meridian tangent, μ a constant defined as

$$\mu^4 = 12(1 - \nu^2)/h^2, \tag{2.3}$$

ν is Poisson's ratio of the material, h the shell thickness, and L the Reissner operator

$$L(y) = (1/r)(y'' + y' \cot \phi - y \cot^2 \phi) \tag{2.4}$$

with r the radius of the meridian surface of the undistorted shell and ϕ the angle measured from the axis (see Fig. 1). The superscript 2 applied to L indicates two consecutive applications of the differential operator. The prime denotes differentiation with respect to ϕ:

$$' = d/d\phi. \tag{2.5}$$

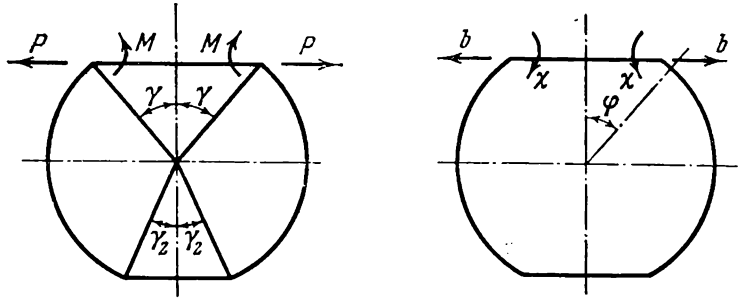
The definition given in (2.3) is correct only if

$$1 \gg \frac{1}{12} \frac{\nu}{1 - \nu^2} \frac{h^2}{r^2}. \tag{2.6}$$

If ν can be taken as 0.3, the right-hand member of the inequality is smaller than 10^{-2} when $r/h > 2$, and it is smaller than 10^{-4} when $r/h > 30$. The approximation involved in (2.3) is therefore always acceptable in the analysis of practical shells.

The determination of the influence coefficients consists of prescribing the stress resultant P perpendicular to the axis of the shell and the moment resultant M_ϕ at an edge $\phi = \gamma$ of the shell (see Figs. 1 and 2) and of calculating the displacement b perpendicular to the axis and the rotation χ of the meridian at the edge $\phi = \gamma$. Since the deformations and stresses are attenuated rapidly with distance from γ, it is usually un-

necessary to stipulate P and M_ϕ at a second angle $\phi = \pi - \gamma_2$. In other words, the interaction between the two edges is negligible.



(*a*) Forces and moments (*b*) Displacements and rotations

Fig. 2. Sign convention for influence coefficients

The differential equations can be solved with the aid of hypergeometric series (see, for instance, p. 460 of [2]). Unfortunately, the series converge very slowly in the range of the parameter r/h most often encountered in practical applications ($r/h > 30$). For this reason approximate solutions have been developed for the two limiting cases when ϕ is small and when ϕ is near $\pi/2$. One solution near $\pi/2$ is known as the Geckeler solution (see p. 469 of [2]).

§ 3. The Geckeler solution

In the derivation of the Geckeler solution, one starts from the observation that in the related problem of the edge loading of the circular cylindrical shell all the displacements and stresses oscillate rapidly and are attenuated a short distance from the edge. For this reason y'' is large compared to $y' \cot \phi$, and this again is large compared to $y \cot^2 \phi$, if ϕ is in the neighborhood of $\pi/2$. If the largest term alone is retained, equations (2.1) and (2.2) become

$$Q^{IV} + 4k^4 Q = 0, \tag{3.1}$$

$$\chi^{IV} + 4k^4 \chi = 0, \tag{3.2}$$

where

$$4k^4 = 12(1 - \nu^2)(r/h)^2. \tag{3.3}$$

Since the coefficients of these differential equations are constants, the effect of the change in the meridian slope with ϕ has been eliminated from the problem. The solution of this simplified problem is easy to obtain. If the results presented on p. 471 of [2] are modified to apply to

the shell of Fig. 2, the notation and sign convention of the present paper are used, and if the influence coefficients are defined by the equations

$$b = k_{bP}P + k_{bM}M, \tag{3.4}$$

$$\chi = k_{\chi P}P + k_{\chi M}M, \tag{3.5}$$

the results of the analysis can be given as

$$k_{\chi M} = -4[3(1-\nu^2)]^{\frac{1}{4}}(1/E)(r^{\frac{1}{2}}/h^{\frac{3}{2}}), \tag{3.6}$$

$$k_{bM} = -k_{\chi P} = 2[3(1-\nu^2)]^{\frac{1}{4}}(1/E)(r/h^2)\sin\gamma, \tag{3.7}$$

$$k_{bP} = 2[3(1-\nu^2)]^{\frac{1}{4}}(1/E)(r/h)^{\frac{3}{2}}\sin^2\gamma. \tag{3.8}$$

Comparison with published results shows that these are the influence coefficients of a circular cylindrical shell of radius r except for the trigonometric terms. But the component of the radial displacement w in the direction b is $w\sin\gamma$ and the component of the shear force Q in the direction of P is $Q\sin\gamma$. This explains the need for the sine terms.

Although the influence coefficients given by (3.4)–(3.8) have been widely used, the range of the parameters in which they are sufficiently accurate is not known with any precision. Obviously they cannot be correct for very small values of ϕ because then $\cot\phi$ becomes very large and the order of magnitude considerations used in simplifying equations (2.1) and (2.2) are invalid.

§ 4. Effect of the curvature of the meridian

The range of validity of the influence coefficients of the Geckeler approximation will be investigated in the present paper by means of a more accurate solution of the problem obtained through replacement of the cotangent function in (2.4) by the linear expression

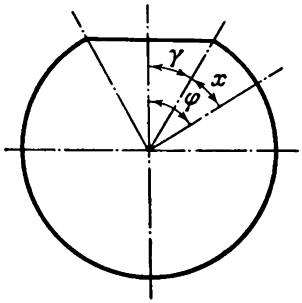


Fig. 3. New angular coordinate x

$$\cot\phi = \alpha - \beta x. \tag{4.1}$$

Since the Taylor series of the cotangent function is

$$\cot(\gamma + x) = \cot\gamma - (x/\sin^2\gamma) + (x^2\cot\gamma/\sin^2\gamma) + \cdots, \tag{4.2}$$

one obtains

$$\alpha = \cot\gamma, \quad \beta = 1/\sin^2\gamma, \tag{4.3}$$

if $\phi = \gamma$ is the angle defining the free edge of the shell and x is the angle (in radians) measured from the free edge (see Fig. 3). The maximum error committed when this approximation is used is given in Table 1 for the case when $x = \pm 0.1$; that is, when $\gamma - 0.1 < \phi < \gamma + 0.1$.

TABLE 1

Error involved in approximation for cotangent function when $x = \pm 0.1$

γ degrees	20	30	45	60	90
Error % when $x = +0.1$	9	4	2	2	0
Error % when $x = -0.1$	8	4	2	1	0

This means that the approximation is certainly acceptable if $\gamma \geq 30°$ and if the shell dimensions are such that the stresses and deflections are attenuated sufficiently within 0.1 radian $\cong 6$ degrees of the free edge $\phi = \gamma$. The expression $\gamma - 0.1$ is relevant when the cap of the sphere is under consideration, and $\gamma + 0.1$ when the rest of the sphere is studied.

When the attenuation of the deflections and stresses is slower, a longer portion of the shell is affected. To cover this possibility, maximum error values have been collected in Table 2 for the case when $x = \pm 0.2$; that is, when $\gamma - 0.2 < \phi < \gamma + 0.2$.

TABLE 2

Error involved in approximation for cotangent function when $x = \pm 0.2$

γ degrees	20	30	45	60	90
Error % when $x = +0.2$	37	18	10	6	1
Error % when $x = -0.2$	33	15	7	4	1

With this slower attenuation the error involved in the approximation at 30° may be too large for engineering purposes. It should be acceptable at $\gamma = 45°$.

§ 5. Solution of the equations

It is easy to prove (see, for instance, p. 151 of Ref. [1]) that the solutions of the two equations

$$L(Q) + i\mu^2 Q = 0, \tag{5.1'}$$

$$L(Q) - i\mu^2 Q = 0 \tag{5.1''}$$

are also solutions of equation (2.1). Since the complex function that is the

solution of equation (5.1′) is the conjugate of the complex function that is the solution of equation (5.1″), in the analysis that follows it will suffice to restrict our attention to the solution of equation (5.1′).

Substitution of the assumption contained in equation (4.1) into the Reissner operator yields

$$L(y) = (1/r)[y'' + (\alpha - \beta x)y' - (\alpha - \beta x)^2 y]. \tag{5.2}$$

If a new independent variable ξ is introduced as

$$\xi = (\alpha/\beta) - x = (\alpha/\beta) + \gamma - \phi, \tag{5.3}$$

equation (5.2) becomes

$$L(y) = (1/r)[y'' - \beta\xi y' - \beta^2\xi^2 y], \tag{5.4}$$

where the prime denotes differentiation with respect to ξ:

$$' = d/d\xi. \tag{5.5}$$

Because of (5.4) one can write equation (5.1′) in the form

$$(1/r)[Q'' - \beta\xi Q' - \beta^2\xi^2 Q] + i\mu^2 Q = 0, \tag{5.6}$$

that is

$$Q'' - \beta\xi Q' - (\beta^2\xi^2 - ir\mu^2)Q = 0. \tag{5.7}$$

For the solution assume

$$Q = u \exp(\beta\xi^2/4). \tag{5.8}$$

Substitution yields the requirement

$$u'' - [5(\beta\xi/2)^2 - (\beta/2) - ir\mu^2]u = 0. \tag{5.9}$$

This can be reduced through further substitutions to

$$u'' - 2[3(1 - \nu^2)]^{\frac{1}{4}}(r/h)[\Omega(h/r) - i]u = 0, \tag{5.10}$$

where

$$\Omega = \tfrac{1}{4}[3(1 - \nu^2)]^{-\frac{1}{4}}(\sin\gamma)^{-2}\left\{\tfrac{5}{2}\left[\cos\gamma - \frac{x}{\sin\gamma}\right]^2 - 1\right\}. \tag{5.11}$$

In (5.10) the absolute value of $\Omega(h/r)$ is small compared to that of i (which is unity). It is permissible therefore to disregard the term $\Omega(h/r)$ and thus to reduce the equation to one having constant coefficients. In Table 3 values of Ω are given for various angles γ and various angular distances x from the free edge. Again, negative x values refer to a spherical cap and positive values to the remainder of the sphere. Since these values

of Ω have to be divided by r/h, which is generally greater than 30, the absolute value of the multiplier of u in equation (5.10) does not change much from $x = -0.2$ to $x = 0.2$.

TABLE 3

Values of Ω

x \ γ	20°	30°	45°	60°	90°
− 0.2	6.18	1.81	0.44	0.07	− 0.14
− 0.1	3.58	1.11	0.24	− 0.01	− 0.15
0	1.55	0.53	0.08	− 0.08	− 0.15
0.1	0.06	0.07	− 0.06	− 0.13	− 0.15
0.2	− 0.88	− 0.28	− 0.17	− 0.17	− 0.14

It is possible now to establish a criterion whether $\Omega(h/r)$ can be disregarded in equation (5.10). One may require, for instance, that the absolute value of $\Omega(h/r)$ be 0.05 or less. On this basis the minimum values of r/h can be computed from Table 3. This has been done and the results have been collected in Table 4.

TABLE 4

Minimum required values of r/h

x \ γ	20°	30°	45°	60°	90°
− 0.2	124	36	9	2	3
− 0.1	72	22	5	1	3
0.0	31	11	2	2	3
0.1	2	2	2	3	3
0.2	18	6	4	4	3

In the case of $\gamma = 20°$, Table 4 shows that the simplification suggested for equation (5.10) is permissible for spherical caps having r/h values equal to or greater than 124 if the affected boundary region extends to $x = -0.2$, and for spherical caps with $r/h \geq 72$ if the affected region extends only to $x = -0.1$. The restriction is even lighter for the remaining portion of the spherical shell. The minimum required value of r/h is 31 if the affected region extends either to $x = \pm 0.1$ or $x = \pm 0.2$. At $\gamma = 30°$ an r/h ratio of 36, and at $\gamma = 45°$ a ratio of 9 suffices when the boundary region does not extend beyond ± 0.2.

The simplified form of equation (5.10) is

$$u'' + i2k^2u = 0 \tag{5.12}$$

with

$$k^2 = \tfrac{1}{2}r\mu^2 = [3(1 - \nu^2)]^{\frac{1}{4}}(r/h). \tag{5.13}$$

The solution of (5.12) is assumed as

$$u = \exp p\xi. \tag{5.14}$$

Substitution yields the condition

$$p^2 = -i2k^2, \tag{5.15}$$

from which follows

$$p = \pm (1 - i)k. \tag{5.16}$$

The solution of equation (5.7) can therefore be written in the form

$$Q = e^q \tag{5.17}$$

with

$$q = (\beta\xi^2/4) + p\xi. \tag{5.18}$$

In a similar way it can be shown that the solution of equation (5.1″) is again given by (5.17) and (5.18), but with p representing the quantity

$$p = \pm (1 + i)k. \tag{5.19}$$

§ 6. Deformations and stress resultants at edge

The derivatives of q and Q with respect to ξ are

$$q' = (\beta\xi/2) + p, \tag{6.1}$$

$$q'' = \beta/2, \tag{6.2}$$

$$Q' = q'e^q, \tag{6.3}$$

$$Q'' = [q'^2 + (\beta/2)]e^q, \tag{6.4}$$

$$Q''' = q'[q'^2 + (3\beta/2)]e^q. \tag{6.5}$$

From (4.1) and (5.3) one obtains

$$\cot \phi = \beta\xi. \tag{6.6}$$

With this expression it follows from established relations (see, for

instance, [1]) that

$$N_\phi = -Q\cot\phi = -\beta\xi e^q. \tag{6.7}$$

Moreover,

$$N_\theta = -dQ/d\phi. \tag{6.8}$$

Since from (5.3)

$$d\xi/d\phi = -1, \tag{6.9}$$

equation (6.8) becomes

$$N_\theta = -(dQ/d\xi)(d\xi/d\phi) = q'e^q = [(\beta\xi/2) + p]e^q. \tag{6.10}$$

The membrane strains in the shell are (see, for instance, p. 50 of [1])

$$\varepsilon_\phi = (1/Eh)(N_\phi - \nu N_\theta) = (1/Eh)(-Q\cot\phi + \nu Q'), \tag{6.11}$$

$$\varepsilon_\theta = (1/Eh)(N_\theta - \nu N_\phi) = (1/Eh)(-Q' + \nu Q\cot\phi), \tag{6.12}$$

where the prime again indicates differentiation with respect to ϕ.

The relation between the meridional displacement v and the membrane strains is (see, for instance, p. 58 of [1])

$$(dv/d\phi) - v\cot\phi = r(\varepsilon_\phi - \varepsilon_\theta) = f(\phi). \tag{6.13}$$

Substitution from (6.11) and (6.12) yields

$$f(\phi) = (1+\nu)(r/Eh)(Q' - Q\cot\phi), \tag{6.14}$$

where the prime denotes differentiation with respect to ϕ.

The solution of (6.13) can be written as

$$v = \sin\phi\left[\int (1/\sin\phi)f(\phi)\,d\phi + C'\right], \tag{6.15}$$

where C' is a constant of integration. Observe that

$$\frac{d}{d\phi}\left(\frac{Q}{\sin\phi}\right) = \frac{Q'}{\sin\phi} - \frac{Q\cos\phi}{\sin^2\phi}. \tag{6.16}$$

Hence the expression under the integral sign of (6.15) is an exact differential and the expression for v becomes

$$v = (1+\nu)(r/Eh)Q + C'\sin\phi. \tag{6.17}$$

From this expression w can be calculated with the aid of the equation (see p. 58 of [1])

$$w = r\varepsilon_\theta - v\cot\phi. \tag{6.18}$$

Substitutions from equations (6.12) and (6.17) give

$$w = -(r/Eh)(Q' + Q \cot \phi) - C' \cos \phi. \tag{6.19}$$

The displacement b perpendicular to the axis follows then as

$$b = v \cos \phi + w \sin \phi = (r/Eh)(\nu Q \cos \phi - Q' \sin \phi). \tag{6.20}$$

If the prime is again to denote differentiation with respect to ξ, equation (6.20) must be changed to read

$$b = (r/Eh)(\nu Q \cos \phi + Q' \sin \phi). \tag{6.21}$$

It follows from (6.6) that

$$\sin \phi = (1 + \beta^2\xi^2)^{-\frac{1}{2}}, \quad \cos \phi = \beta\xi(1 + \beta^2\xi^2)^{-\frac{1}{2}}. \tag{6.22}$$

Substitution from equations (5.17), (6.3) and (6.22) into (6.21) yields

$$b = (r/Eh) \frac{[\frac{1}{2} + \nu]\beta\xi + p}{(1 + \beta^2\xi^2)^{\frac{1}{2}}} e^q. \tag{6.23}$$

Moreover

$$\begin{aligned} \chi &= (1/r)[(dw/d\phi) - v] = \\ &= -(1/Eh)[Q'' + Q' \cot \phi + (\nu - \cot^2 \phi)Q] = \\ &= -(1/Eh)(\nu - i\mu^2 r)Q = -(1/Eh)(\nu + p^2)Q. \end{aligned} \tag{6.24}$$

It follows then that

$$\chi = -(1/Eh)(\nu + p^2)e^q. \tag{6.25}$$

At the edge of the shell

$$b_{\phi=\gamma} = (r/Eh) \frac{[\frac{1}{2} + \nu]\alpha + p}{(1 + \alpha^2)^{\frac{1}{2}}} e^{q^*}, \tag{6.26}$$

$$\chi_{\phi=\gamma} = -(1/Eh)(\nu + p^2)e^{q^*}, \tag{6.27}$$

where q^* is the value of q at $\phi = \gamma$.

The value of the force resultant P perpendicular to the axis of rotation is

$$P = Q/\sin \phi = (1 + \beta^2\xi^2)^{\frac{1}{2}} e^q \tag{6.28}$$

which becomes at the edge

$$P_{\phi=\gamma} = (1 + \alpha^2)^{\frac{1}{2}} e^{q^*}. \tag{6.29}$$

The bending moment can be calculated from * (see p. 148 of [1])

$$(6.30)\quad M_\phi = -[12(1-\nu^2)]^{-1}(h^2/r)\Big\{Q''' + (1+\nu)\cot\phi Q'' + \\ + \Big(\nu - \frac{1}{\sin^2\phi} + \nu\cot^2\phi - \cot^2\phi\Big)Q' + \\ + \Big(\frac{2\cos\phi}{\sin^3\phi} + \nu^2\cot\phi - \nu\cot^3\phi\Big)Q\Big\}.$$

Because

$$(6.31)\qquad rL(Q) = -ir\mu^2 Q = p^2 Q,$$

equation (6.30) can be simplified to read

$$(6.32)\quad M_\phi = -[12(1-\nu^2)]^{-1}(h^2/r)\Big\{-\Big[p^2+\nu-\frac{1}{\sin^2\phi}\Big]Q' + \\ + \Big[p^2+\nu+\frac{2}{\nu\sin^2\phi}\Big]Q\nu\cot\phi\Big\},$$

provided the prime indicates differentiation with respect to ξ.

Substitutions yield

$$(6.33)\quad M_\phi = -[12(1-\nu^2)]^{-1}(h^2/r)\Big\{\Big[(p^2+\nu)(\nu-\tfrac{1}{2}) + \frac{5}{2\sin^2\phi}\Big]\cot\phi - \\ - \Big[p^2+\nu-\frac{1}{\sin^2\phi}\Big]p\Big\}e^q.$$

At the edge this simplifies to

$$(6.34)\quad M_{\phi=\gamma} = -[12(1-\nu^2)]^{-1}(h^2/r)\{-p^3 + \alpha(\nu-\tfrac{1}{2})p^2 - (\nu-1-\alpha^2)p + \\ + [\nu(\nu-\tfrac{1}{2}) + \tfrac{5}{2}(1+\alpha^2)]\alpha\}e^{q^*}.$$

§ 7. Change to real quantities

Equations (5.1′) and (5.1″) contain terms with both real and imaginary coefficients. Consequently their solutions are complex. But the linear differential equation listed as equation (2.1), which was replaced by (5.1′) and (5.1″), has only real coefficients. If it has a complex solution, both the real and the imaginary parts of the solution must individually satisfy the differential equation. It can be concluded therefore that the first of the solutions given in (5.17), namely

$$(7.1)\qquad Q = e^q = e^{\beta\xi^2/4}e^{(1-i)k\xi} = e^{\beta\xi^2/4}e^{k\xi}(\cos k\xi - i\sin k\xi)$$

represents one complex solution of (5.1'), but two real solutions of (2.1). Multiplied by arbitrary constants of integration the two solutions can be given as

$$(7.2)\quad Q = \exp(\beta\xi^2/4)\{A\ \mathrm{Re}[\exp(1-i)k\xi] + B\ \mathrm{Im}[\exp(1-i)k\xi]\} = \\ = \exp[(\beta\xi^2/4) + k\xi][A\cos k\xi - B\sin k\xi].$$

Using the negative sign in equation (5.16) we obtain in a similar manner

$$(7.3)\qquad Q = e^{q} = e^{\beta\xi^2/4}e^{-(1-i)k\xi} = e^{\beta\xi^2/4}e^{-k\xi}(\cos k\xi + i\sin k\xi),$$

from which follow the two remaining solutions of (2.1) as

$$(7.4)\quad Q = \exp(\beta\xi^2/4)\{C\ \mathrm{Re}[\exp(-1+i)k\xi] + D\ \mathrm{Im}[\exp(-1+i)k\xi]\} = \\ = \exp[(\beta\xi^2/4) - k\xi][C\cos k\xi + D\sin k\xi].$$

The solutions of (5.1") lead to the same results, since p in (5.19) is the conjugate of the complex number p in (5.16).

At the edge of the shell

$$(7.5)\qquad q = q^* = (\alpha^2/4\beta) + (1-i)(k\alpha/\beta),$$

and thus equation (7.2) becomes

$$(7.6)\qquad Q = \exp[(\alpha^2/4\beta) + (k\alpha/\beta)][A\cos(k\alpha/\beta) - B\sin(k\alpha/\beta)].$$

In the calculation of the real expression for the moment corresponding to the solution given in (7.2) [and not in (7.3)], we have to make use of the expressions

$$(7.7)\qquad p = (1-i)k,\quad p^2 = -i2k^2,\quad p^3 = -2(1+i)k^3.$$

If the symbol B^* is introduced to represent the expression in braces in (6.34) and the coefficients of the real and imaginary parts are denoted B_1 and B_2, one obtains

$$(7.8)\qquad B^* = B_1 + iB_2,$$

where

$$(7.9)\qquad B_1 = 2k^3 - (\nu - 1 - \alpha^2)k + [\nu(\nu - \tfrac{1}{2}) + (\tfrac{5}{2})(1+\alpha^2)]\alpha,$$

$$(7.10)\qquad B_2 = 2k^3 - k^2\alpha(2\nu - 1) + k(\nu - 1 - \alpha^2).$$

With the new symbol

$$(7.11)\qquad E^* = e^{(\alpha/\beta)(\alpha+4k)/4},$$

one may write

$$(7.12)\quad M_{\phi=\gamma} = -[12(1-\nu^2)]^{-1}(h^2/r)(B_1 + iB_2)E^*[\cos(\alpha k/\beta) - i\sin(\alpha k/\beta)].$$

If the real part of the solution is multiplied by the integration constant A and the imaginary part by B, the moment becomes

$$(7.13)\quad M_\gamma = -[12(1-\nu^2)]^{-1}(h^2/r)E^*\{A[B_1\cos(\alpha k/\beta) + B_2\sin(\alpha k/\beta)] + \\ + B[B_2\cos(\alpha k/\beta) - B_1\sin(\alpha k/\beta)].$$

With the same notation the expression for the force resultant becomes along the edge

$$(7.14)\qquad P_\gamma = (1+\alpha^2)^{\frac{1}{2}}E^*\{A\cos(\alpha k/\beta) - B\sin(\alpha k/\beta)\}.$$

§ 8. Influence coefficients

It will now be conjectured that the present solution will reveal the same kind of a boundary effect as has been observed with thin circular cylindrical shells subjected to axisymmetric shear and moment resultants. In the study of the spherical shell extending from $\phi = \gamma$ in the direction of increasing ϕ we wish to retain only the two solutions whose integration constants are A and B and wish to disregard the solutions containing C and D. Since increasing ϕ implies decreasing ξ, this choice corresponds completely to the approach in general use in the calculation of the influence coefficients of circular cylindrical shells. Naturally, the rejection of C and D will have to be justified later.

Under the assumptions made, specification of the edge loading Q and M_ϕ at $\phi = \gamma$ completely determines the solution. We can solve therefore equations (7.13) and (7.14) for the integration constants A and B and obtain

$$(8.1)\quad A = (M^*/B_2)\sin(\alpha k/\beta) + P^*[\cos(\alpha k/\beta) - (B_1/B_2)\sin(\alpha k/\beta)],$$

$$(8.2)\quad B = -P^*[(B_1/B_2)\cos(\alpha k/\beta) + \sin(\alpha k/\beta)] + (M^*/B_2)\cos(\alpha k/\beta),$$

where

$$(8.3)\qquad M^* = -12(1-\nu^2)(r/h^2E^*)M_\gamma,$$

$$(8.4)\qquad P^* = (1/E^*)(1+\alpha^2)^{-\frac{1}{2}}P_\gamma.$$

Next, the edge displacement and edge rotation will be calculated. The edge rotation can be written as

$$(8.5)\qquad \chi = -(1/Eh)\{A\ \mathrm{Re}[(\nu + p^2)e^{q^*}] + B\ \mathrm{Im}[(\nu + p^2)e^{q^*}]\}.$$

Since $p^2 = -i2k^2$, one has

$$\text{(8.6)} \qquad \mathrm{Re}[(\nu + p^2)e^{q^*}] = E^*[\nu \cos(\alpha k/\beta) - 2k^2 \sin(\alpha k/\beta)],$$

$$\text{(8.7)} \qquad \mathrm{Im}[(\nu + p^2)e^{q^*}] = -E^*[2k^2 \cos(\alpha k/\beta) + \nu \sin(\alpha k/\beta)].$$

Substitutions yield

$$\text{(8.8)} \qquad \chi = (E^*/Eh)\{2k^2(M^*/B_2) - P^*[\nu + (B_1/B_2)2k^2]\}.$$

The radial displacement b of the edge is given by

$$\text{(8.9)} \quad b = \frac{(E^*/E)(r/h)}{2(1+\alpha^2)^{\frac{1}{2}}} \cdot \{-2k(M^*/B_2) + P^*\{(1+2\nu)\alpha + 2k[1+(B_1/B_2)]\}\}.$$

Introduction of the influence coefficient notation yields the defining equations

$$\text{(8.10)} \qquad \chi = k_{\chi M} M_\gamma + k_{\chi P} P_\gamma,$$

$$\text{(8.11)} \qquad b = k_{bM} M_\gamma + k_{bP} P_\gamma.$$

The influence coefficients are

$$\text{(8.12)} \qquad k_{\chi M} = (E^*/Eh)(2k^2/B_2)(M^*/M_\gamma) = 12(1-\nu^2)\frac{2rk^2}{Eh^3 B_2}.$$

The expression for B_2 can be rewritten from (7.10) in the form

$$\text{(8.13)} \qquad B_2 = 2k^3\left\{1 + \frac{(1-2\nu)\alpha}{2k} - \frac{1+\alpha^2-\nu}{2k^2}\right\} = 2k^3 B^*.$$

Substitution in (8.12) yields

$$\text{(8.14)} \qquad k_{\chi M} = -4[1-\nu^2)]^{\frac{3}{4}}(1/E)(r^{\frac{1}{2}}/h^{\frac{1}{2}})(1/B^*)$$

with

$$\text{(8.15)} \qquad B^* = 1 + \frac{(1-2\nu)\alpha}{2k} - \frac{1+\alpha^2-\nu}{2k^2}.$$

If ν is disregarded when it is added to $2k^2(B_1/B_2)$, equation (8.8) yields for the second influence coefficient

$$\text{(8.16)} \qquad k_{\chi P} = -2[3(1-\nu^2)]^{\frac{3}{4}}(1/E)(r/h^2)(1+\alpha^2)^{-\frac{1}{2}}B^{**},$$

where

$$\text{(8.17)} \quad \frac{B_1}{B_2} = B^{**} =$$

$$= \frac{1 + [(1+\alpha^2-\nu)/2k^2] + [\nu(2\nu-1) + 5(1+\alpha^2)](\alpha/4k^3)}{1 + (1-2\nu)(\alpha/2k) - [(\alpha^2+1-\nu)/2k^2]}.$$

From equation (8.10) it follows that

$$k_{bP} = [3(1 - \nu^2)]^{\frac{1}{4}}(1/E)(r/h)^{\frac{1}{2}} \frac{1 + B^{**} + (1 + 2\nu)(\alpha/2k)}{(1 + \alpha^2)}. \tag{8.18}$$

From the same equation one can calculate

$$k_{bM} = 2[3(1 - \nu^2)]^{\frac{1}{4}}(1/E)(r/h^2)(1 + \alpha^2)^{-\frac{1}{4}}(1/B^*). \tag{8.19}$$

The numerical values of k_{bM} and $k_{\chi P}$ differ, because $1/B^*$ is not equal to B^{**}. However, the difference becomes negligible, if

$$\alpha^2/2k^2 \ll 1. \tag{8.20}$$

This quantity is only about 0.023 at $\gamma = 20°$, when $r/h = 100$, and, of course, it decreases with increasing r/h. At $\gamma = 30°$, the same error is made if r/h is only about 40. Hence the error committed in setting $B^* = 1/B^{**}$ is acceptable for engineering purposes, if

$$\begin{aligned} &\gamma = 20° \text{ and } r/h > 100, \\ &\gamma = 30° \text{ and } r/h > \ 40. \end{aligned} \tag{8.21}$$

For this accuracy one may use the simplified expression

$$B^* = 1/B^{**} = 1 + \tfrac{1}{2}(1 - 2\nu)(\alpha/k). \tag{8.22}$$

With the same degree of accuracy

$$1/B^* = B^{**} = 1 - \tfrac{1}{2}(1 - 2\nu)(\alpha/k). \tag{8.23}$$

If one remembers that

$$(1 + \alpha^2)^{-1} = \sin^2\gamma, \quad (1 + \alpha^2)^{-\frac{1}{2}} = \sin\gamma, \tag{8.24}$$

equations (8.14), (8.16), (8.18) and (8.19) can also be written in the form

$$k_{\chi M} = -\,4[3(1 - \nu^2)]^{\frac{1}{4}}(1/E)(r^{\frac{1}{2}}/h^{\frac{1}{2}})[1 - (1 - 2\nu)(\alpha/2k)], \tag{8.25}$$

$$k_{bM} = -\,k_{\chi P} = 2[3(1 - \nu^2)]^{\frac{1}{4}}(1/E)(r/h^2)\sin\gamma[1 - (1-2\nu)(\alpha/2k)], \tag{8.26}$$

$$k_{bP} = 2[3(1 - \nu^2)]^{\frac{1}{4}}(1/E)(r/h)^{\frac{1}{2}}\sin^2\gamma[1 + \nu(\alpha/k)]. \tag{8.27}$$

The difference in the signs of k_{bM} and $k_{\chi P}$ is caused by the sign convention.

Comparison of equations (8.25) – (8.27), (3.6)–(3.8) shows that the influence coefficients derived from the present theory are the influence coefficients of the Geckeler approximation multiplied by modifying factors. These factors are in square brackets in (8.25)–(8.27); their numerical values are small as can be seen from Table 5. The computation of these values was carried out on the assumption that $\nu = 0.3$.

TABLE 5

Modifying factors of $k_{\chi M}$, k_{bM} and $k_{\chi P}$

r/h \ γ	20°	30°	45°	60°	90°
50	0.940	0.962	0.978	0.987	1
100	0.957	0.973	0.984	0.991	1
400	0.979	0.987	0.992	0.996	1

The correction is 50 percent higher for k_{bP}. For instance, for $\gamma = 20°$ and $r/h = 50$ the modifying factor is 0.91; and for $\gamma = 30°$ and $r/h = 100$ it is 0.96.

§ 9. Attenuation of stresses

It should be remembered now that the solution of the problem was found and the influence coefficients were calculated on the assumption that the stresses and the deformations were attenuated rapidly with distance from the edge $\phi = \gamma$. Before the results can be accepted, it must be checked whether this assumption was justified. To do so one can observe the variation with x, that is, with the angular distance from the edge of the shear resultant Q as expressed in (7.2). For a given shell this variation is governed by the parameters β and k. According to (4.3), the former is simply $1/\sin^2\gamma$, while the second depends only on the ratio r/h [see (3.3)], if ν is taken as 0.3 for all the metallic materials of construction. Finally, the independent variable ξ is defined as

$$\xi = \cos\gamma \sin\gamma - x. \tag{9.1}$$

This follows from equations (4.3) and (5.3).

The shear stress resultant can be given therefore as

$$Q = T \exp[(\beta\xi^2/4) + k\xi], \tag{9.2}$$

where T represents trigonometric functions. The attenuation depends on the exponential function multiplying T. Equation (9.2) can be written after substitutions as

$$Q = T \exp\left\{\frac{(\cos\gamma \sin\gamma - x)^2}{4 \sin^2\gamma} + k(\cos\gamma \sin\gamma - x)\right\}. \tag{9.3}$$

The ratio R_1 of the value of Q (except for the trigonometric terms) at a

distance x from the edge to the value at the edge is

$$R_1 = \exp\left\{\frac{-2x\cos\gamma\sin\gamma + x^2}{4\sin^2\gamma} - kx\right\}. \tag{9.4}$$

Values of R_1 have been computed for two values of the edge angle γ (30° and 45°), three values of the r/h ratio (50, 100, and 400) and three values of the distance x from the edge (0.1, 0.2 and 0.3). The results are presented in Table 6. Also included are values for the far edge (or for the spherical cap). In their calculation k must be replaced by $-k$, and x by $-x$ in equation (9.4).

It can be seen from Table 6 that γ has little effect on the attenuation of Q. On the other hand, the attenuation is very much accelerated when r/h is increased. The attenuation is a little slower for the spherical cap than for the remainder of the shell. At $\gamma = 30°$ the major stresses appear in a boundary layer of width x not much greater than 0.1 when $r/h > 100$, and at $\gamma = 45°$ the same can be said when $r/h > 50$.

TABLE 6

Attenuation of shear stress resultant Q with distance x from edge
(Values of R_1)

γ	ϕ	$\lvert x\rvert$				r/h
		0	0.1	0.2	0.3	
30°	Increasing	1	0.37	0.14	0.056	50
30°	Increasing	1	0.25	0.066	0.018	100
30°	Increasing	1	0.06	0.005	0.0004	400
30°	Decreasing	1	0.44	0.20	0.092	50
30°	Decreasing	1	0.30	0.093	0.030	100
30°	Decreasing	1	0.085	0.0073	0.00006	400
45°	Increasing	1	0.38	0.15	0.058	50
45°	Increasing	1	0.26	0.07	0.02	100
45°	Increasing	1	0.073	0.005	0.0004	400
45°	Decreasing	1	0.43	0.18	0.08	50
45°	Decreasing	1	0.29	0.086	0.025	100
45°	Decreasing	1	0.081	0.007	0.0005	400

Since all the deformation and stress quantities can be expressed in terms of Q and Q', the width of the boundary layer can be completely

established, if the attenuation of Q' is also investigated. As

$$Q' = dQ/d\phi = (dq/d\phi)Q = -[(\beta\xi/2) + (1 - i)k]Q, \tag{9.5}$$

the magnitude of the ratio of Q' (except for the trigonometric terms) at a distance x from the edge to the value at the edge is R_1 as given in (9.4) multiplied by the ratio R_2 defined as

$$R_2 = \frac{\cos\gamma \sin\gamma - x + 2k \sin^2\gamma}{\cos\gamma \sin\gamma + 2k \sin^2\gamma}. \tag{9.6}$$

This formula is valid for increasing x, that is, for the remainder of the shell rather than for the spherical cap. If x is replaced by $-x$, and k by $-k$, the new formula refers to decreasing x, that is, to the spherical cap. Numerical values were computed for various values of the parameters involved. All the values of R_2 are less than unity (except at $x = 0$ where the value is unity) which indicates that Q' is attenuated more rapidly than Q. The difference in the rapidity of the attenuation is small because all the values computed are close to unity. Sample values are given in Table 7.

TABLE 7

Values of numerical factor R_2 for $r/h = 50$ and $\gamma = 30°$

$\lvert x\rvert$	0	0.1	0.2	0.3
ϕ increasing	0	0.98	0.96	0.94
ϕ decreasing	1	0.98	0.95	0.93

§ 10. Conclusions

Expressions have been derived for the influence coefficients of edge-loaded thin spherical shells. In the derivation the effect of the change in the direction of the meridian tangent was taken into account approximately in a manner that is satisfactory everywhere except in the neighborhood of $\phi = 0$ and $\phi = \pi$. It was shown that the formulas for the influence coefficients should yield values accurate enough for engineering purposes for an edge characterized by $\gamma = 30°$ if $r/h > 100$, and for an edge characterized by $\gamma = 45°$ if $r/h > 50$.

The influence coefficients are given in a form consisting of the product of two factors. The first is identical with the influence coefficient derivable from the Geckeler approximation, and the second is a correction factor. In

the range of applicability just defined the correction factors are so small that the Geckeler approximation also appears accurate enough for engineering purposes.

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ON THE THEORY OF TRANSIENT CREEP

D. D. Ivlev
Voronezh, U.S.S.R.

V. V. Novoshilov and Iu. I. Kadashevich [1] have shown recently the expediency of the use of dynamic moduli for the explanation of qualitative features of the behavior of a plastic material. Apparently, they first used a two-dimensional dynamic model which permitted them to explain the peculiarity of the associative law of flow in the case of an anisotropically consolidating material.

The idea of the introduction of "microstresses" into a phenomological theory, in spite of the apparent paradoxical nature, has been found to be natural, fruitful and undoubtedly forward looking.

In the present note the particular considerations following from the study of the corresponding moduli are placed at the base of a theory of creep. It is shown that the concept of micro-stresses, apparently, must play an important role in the design of a theory of creep, where their introduction affects in an essential manner the form of the relationship between the states of stress and strain.

We will assume that the mechanism of creep is composed of combined action of three simple elements: elastic, viscous and plastic which are schematically represented in Fig. 1. The symbols v and p denote the viscous and the plastic (mechanism of dry friction) elements respectively.

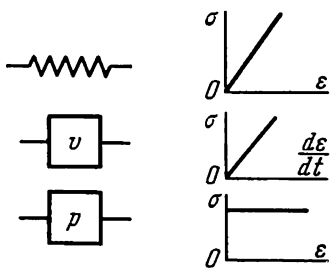

Fig. 1

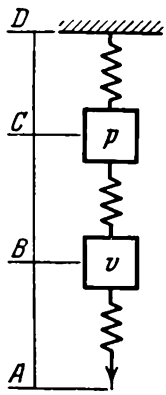

Fig. 2

Since the analysis of the properties of the simplest combinations occur in the literature [2], we will, right from the start, consider the model of Fig. 2. We will call the elastic spring on the segment AB: the element AB,

the viscous mechanism: the element B, the spring on the segment BC: the element BC, the mechanism of dry friction: the element C and, finally, the spring on the segment CD: the element CD.

We will imagine that a tensile force is applied to the body, the qualitative character of the properties of which is shown in Fig. 2. In this case there occurs immediately an instantaneous elastic deformation (on the segment AB); further, a creep deformation begins to develop, conditioned by the viscous mechanism (on the segment BC) and, finally, when the dry friction of the element has been overcome, there arises an irreversible residual creep deformation (on the segment CD). The element BC causes reversible creep, the element CD is the effect of the acquired anisotropy (Bauschinger effect).

Obviously, an entire range of properties, observed in experiments, is qualitatively described by the model under consideration.

In the paper [1], a model was studied which consisted of the elements BC, C and CD (Fig. 2). The stresses acting on the segment CD were called micro-stresses. The element BC is found in an analogous situation in the model under consideration. The stresses acting on this element will likewise be included with the number of micro-stresses. Thus, in fact, the microstresses cause reversible effects.

We will consider the relations between the stresses and the strains in a body the properties of which qualitatively correspond to the model of Fig. 2. For simplicity, we will assume the body to be incompressible. In order to arrive at the general case, we will use the two-dimensional generalization of this model which is shown in Fig. 3.

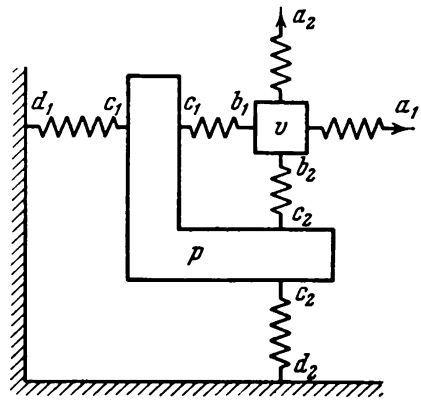

Fig. 3

Following the ideas of the Prandtl-Reiss theory, we will assume that the total increase of the strain consists of three components

$$d\varepsilon_{ij} = d\varepsilon^e_{ij} + d\varepsilon^v_{ij} + d\varepsilon^p_{ij}, \tag{1}$$

where $d\varepsilon^e_{ij}$ is the increase of the elastic strain, $d\varepsilon^v_{ij}$ is that of the viscous component of the creep strain and $d\varepsilon^p_{ij}$ is that of the plastic component of the creep strain.

We will denote by σ_{ij} the stress tensor which satisfies the equilibrium equations, by s_{ij} the micro-stress tensor causing reversible creep and by

t_{ij} the micro-stress tensor causing the Bauschinger effect. The stresses s_{ij} are due to the elastic elements b_1c_1, b_2c_2, the stresses t_{ij} due to the elastic elements c_1d_1, c_2d_2 (Fig. 3).

In accordance with the assumption introduced we will have

$$d\varepsilon_{ij}^e = \frac{1}{2G}\, d\sigma_{ij}', \tag{2}$$

where G is the shear modulus.

Here and in what follows we will agree to mark the deviator tensor by dashes.

The viscous components of the strain increase are caused by the sum of the effects of the tensors σ_{ij} and s_{ij}. Consequently, one may write

$$d\varepsilon_{ij}^v = \mu\, dt(\sigma_{ij}' - s_{ij}'), \tag{3}$$

where μ is a function of the invariants of the tensor ε_{ij}^v.

In accordance with the introduced assumptions regarding the nature of the micro-stress tensor s_{ij}, we will write

$$\varepsilon_{ij}^v - \varepsilon_{ij}^p = g_1 s_{ij}', \tag{4}$$

where g_1 is a function of the invariants of the tensor $\varepsilon_{ij}^v - \varepsilon_{ij}^p$.

Finally, the increase of the plastic strains is caused by the sum of the effects of the tensors s_{ij} and t_{ij}. Then, if we take the Mises plasticity condition

$$(s_{ij}' - t_{ij}')(s_{ij}' - t_{ij}') = 2k^2,\ k = \text{const.}, \tag{5}$$

we obtain

$$d\varepsilon_{ij}^p = d\lambda(s_{ij}' - t_{ij}'). \tag{6}$$

In agreement with the assumptions regarding the character of the micro-stress tensor t_{ij}, we write

$$\varepsilon_{ij}^p = g_2 t_{ij}', \tag{7}$$

where g_2 is a function of the invariants of the tensor ε_{ij}^p.

Finally, we will have

$$d\varepsilon_{ij} = \frac{1}{2G}\, d\sigma_{ij}' + \mu\, dt(\sigma_{ij}' - s_{ij}') + d\lambda(s_{ij}' - t_{ij}'), \tag{8}$$

where

$$d\varepsilon_{ij}^v = \mu\, dt(\sigma_{ij}' - s_{ij}'), \quad \varepsilon_{ij}^v - \varepsilon_{ij}^p = g_1 s_{ij}',$$
$$d\varepsilon_{ij}^p = d\lambda(s_{ij}' - t_{ij}'), \quad \varepsilon_{ij}^p = g_2 t_{ij}',$$
$$(s_{ij}' - t_{ij}')(s_{ij}' - t_{ij}') = 2k^2.$$

These relations have a very complex form, but we have produced them here only to demonstrate the possibility which exists in creep theory for the utilization of the concepts of micro-stresses.

We note that for $\mu = 0$ the results of [1] apply.

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RUPTURE TIME UNDER CREEP CONDITIONS

L. M. KACHANOV
Leningrad, U.S.S.R.

§ 1. On rupture time under creep conditions

Under conditions of creep the failure stress depends essentially on the duration of the effort. One may assume that rupture will take place for every stress; for high stresses it occurs after a short time, for low stresses the time before the instant of failure turns out to be substantial. Under these conditions the traditional concept of a limit strength loses its meaning and engineers use another characteristic – the *endurance strength*, i.e., the stress leading to failure after a definite time period (the "life" time of a part, for example, for turbine discs is 100,000 hours) at a given temperature.

The estimate of the time to failure has great practical importance in the design of machine parts, exposed to creep conditions. Creep laboratories give much attention to the experimental study of the characteristics of endurance strength. However, an experimental study of the endurance strength of a machine part exposed to a stress field which is not uni-axial and homogeneous raises difficulties and often requires much time. Therefore importance is attached to attempts at the theoretical determination of rupture time, based on the utilization of the methods of solid mechanics.

The first attempt in this direction was made in the work of N. J. Hoff [1]. He determined the failure time by analyzing the unbounded quasi-viscous flow of a machine part under loading. For constant applied stresses and "viscous" (i.e., attended by large deformations) failure the concept due to N. J. Hoff apparently led to results which agreed satisfactorily with experiments.

The concept of N. J. Hoff has a limited character and, besides, it can not explain "brittle" (small strain) failures. The last are connected with the gradual development of fields of cracks. An attempt to consider crack formation and to describe under one scheme "viscous" as well as "brittle" failure was presented in my paper [2]. The present work contains the development of the earlier proposed method and its application to a number of concrete problems.

§ 2. The scheme of N. J. Hoff

The work of N. J. Hoff considers the stretching of a cylindrical rod subject to a constant force ("load") P. We will denote by l, F the current values of the length and cross-sectional area of the rod; at the initial instant $t = 0$ we have $l = l_0$, $F = F_0$.

From the condition of incompressibility one has

$$lF = l_0 F_0. \tag{2.1}$$

We will assume that the strain velocity $\xi = 1/l\ (dl/dt)$ is related to the stress $\sigma = P/F$ by the power law

$$\xi = B_1 \sigma^m, \tag{2.2}$$

where B_1, m are constants (for given temperature; B_1 is the creep coefficient, m the creep index). Introducing into (2.2) the values ξ, σ, eliminating with the help of (2.1) the area F, executing the integration and determining the constant from the initial condition, we obtain

$$\xi_0 mt = 1 - \left(\frac{l_0}{l}\right)^m, \tag{2.2'}$$

where $\xi_0 = B_1 \sigma_0^m$ is the strain velocity for the initial stress. We determine the time to failure t_1 as the time for which the area $F = 0$ (or $l = \infty$). It is obvious that

$$t_1 = \frac{1}{m\xi_0}. \tag{2.3}$$

Thus, in logarithmic coordinates, we have the straight line of endurance strength which is a mirror reflection of the straight line corresponding to the creep law (2.2). This deduction satisfactorily corroborates experimental evidence.

According to the solution obtained

$$\frac{F}{F_0} = \left(1 - \frac{t}{t_1}\right)^{1/m}, \quad t \leq t_1. \tag{2.4}$$

The method of N. J. Hoff has been used for the determination of the failure time of tubes subject to internal pressure (Sh. M. Kats [3], Rimrott, F. [4]) and of rotating discs (V. I. Rozenblium [5]). However, it is easy to state problems for the solution of which Hoff's method is, in general, not suitable. Thus, by Hoff's scheme, the creep process for torsion cannot lead to failure which contradicts experimental evidence.

It is impossible to discuss on the basis of Hoff's scheme failure under relaxation conditions, failure for constant rate of elongation and many other problems.

It is obvious that for a constant rate of elongation an infinite time is necessary for the unbounded elongation of the rod, i.e., according to the concept of N. J. Hoff the time to failure is infinite, which does not quite correspond to experimental results. One might attempt to supply in the capacity of a failure criterion some finite ("limit") elongation; such a point of view has often been proposed in the technical literature and it has a certain foundation. However, criteria of limit strains do not agree with certain experimental results. Thus, by the experiment of A. V. Staniukovich [6], the strain to the instant of failure ε' is equal to

$$\varepsilon' = D\xi^k, \tag{2.5}$$

where ξ is the strain rate, D and k are constants and

$$0 \leq k \leq 0.37. \tag{2.5'}$$

Hence, the "limit strain" depends on the rate of deformation which excludes the above stated possibility of broadening Hoff's scheme. The stated generalization also does not agree with the change of the character of failure (from "viscous" to "brittle") for large and small stresses.

§ 3. The continuity ψ

"Brittle" failure is explained by the gradual development of cracks, and it is realized mainly along grain boundaries. "Viscous" failure is a consequence of the flow in the material of the grains themselves.

We will interpret the process of failure as a process of crack formation spreading against a background of growing creep deformations. The development of cracks under conditions of a complex state of stress has hardly been studied. According to experimental evidence brittle failure is determined by the maximum tensile stress $\sigma_{\max}$.

The state of partial cracking (or "partial damage") of a material is characterized by the scalar quantity $0 \leq \psi \leq 1$; we will call it the continuity.

At the initial instant of time, in the absence of initial partial cracking, we have $\psi = 1$. In the course of time the continuity ψ decreases. If the material is homogeneous with respect to the development of damage, one may assume that the value $\psi = 0$ corresponds to its failure. In the

case of a non-homogeneous material one must assume that for failure $\psi = \psi_0(x, y, z) \geq 0$.

The function ψ characterizes the distributed failure of the material. For a high degree of damage the distributed character becomes unstable, in weak parts dominant cracks arise. In the present paper, similar effects concerning the final stages of rupture are not considered (just as in the theory of viscous failure the appearance of "necks" is ignored) which to a certain extent may be justified by the relative shortness of the duration of these stages.

We will assume that [2]

$$\frac{d\psi}{dt} = -A\left(\frac{\sigma_{\max}}{\psi}\right)^n, \tag{3.1}$$

where $A > 0$, $n \geq 0$ are constants (A is the coefficient, n the index of crack formation). We note that the power law has been selected for the sake of simplicity. This law must be interpreted not as a physical relationship, but only as one that is convenient for the purpose of approximation. It is completely possible to use other analytic laws (for example, the exponential).

We will understand by $\sigma_{\max}$ the true maximum tensile stress at a given point. If a compressive stress is present, we will assume that the magnitude of the tensile stress is not small compared with the magnitude of the compressive stresses. The ratio $\sigma_{\max}/\psi$ may be interpreted as a certain effective stress.

The influence of variable temperature and of certain structural changes may be taken into consideration approximately by assuming the parameters A, n to be functions of the temperature and the parameter A also a function of the time.

Finally, we assume that the process of crack-formation, on the whole, does not influence the creep deformations [2].

In the absence of initial damage (i.e., $\psi = 1$ for $t = 0$) we obtain from the differential relation (3.1)

$$1 - \psi^{n+1} = A(n+1)\int_0^t \sigma^n{}_{\max}(\tau)\, d\tau. \tag{3.2}$$

§ 4. Extension of a rod for constant load

In this problem the "load" $P = \text{const}$. We will assume that failure takes place for small deformations, so that one may neglect changes of

the cross-section of the rod. Then

$$\sigma_{\max} = \frac{P}{F_0} \equiv \sigma_0 = \text{const.},$$

and it follows from (3.2) that

$$1 - \psi^{n+1} = A(n+1)\sigma_0{}^n t. \tag{4.1}$$

At the moment of failure t' the continuity $\psi = 0$. Thus

$$t' = \frac{1}{(n+1)A\sigma_0{}^n}. \tag{4.2}$$

We will agree to call t' the time to brittle failure. Generally speaking, failures which develop in the presence of small strains, in the future, will be called *brittle*.

We will now turn to the analysis of failure under conditions involving considerable creep deformations. Introducing F from (2.4) into (3.2) and completing the integration, we find the failure time t_* corresponding to $\psi = 0$:

$$\frac{t_*}{t_1} = 1 - \left(1 - \frac{m-n}{m}\frac{t'}{t_1}\right)^{m/(m-n)}. \tag{4.3}$$

This equation has sense for $t_* \leq t_1$. Consequently

$$\sigma_0 \leq \left(\frac{1}{\kappa(m-n)}\right)^{1/(m-n)} \equiv \bar{\sigma}_0, \quad \kappa \equiv \frac{B_1}{A(n+1)}. \tag{4.4}$$

Failure which occurs as a consequence of the vanishing of the continuity ψ, but which involves more or less significant creep strains, we will call *non-brittle* (for this t_* is the time to non-brittle failure, and the strain may be found from (2.4) for $t = t_*$). For large stresses (i.e., for $\sigma_0 > \bar{\sigma}_0$) failure occurs in agreement with the scheme of N. J. Hoff, i.e., the cross-sectional area of the rod vanishes ($F = 0$), before the continuity ψ is exhausted. Failure of this type will be assumed to be *viscous* (for this t_1 is the time to viscous failure).

These cases are shown in Fig. 1 on a logarithmic scale. The straight line *abc* corresponds to Hoff's solution, the straight line *de* is the law (4.2) for brittle fracture. Non-brittle failure is described by the curve *be*.

The entire curve of the endurance strength, corresponding to the solution produced, is shown in Fig. 1 by the solid line. For stresses $\sigma_0 > \bar{\sigma}_0$, the curve of the endurance strength runs into the straight segment *ab* and the smaller are the stresses the more significance is acquired by

brittleness and the curve approaches the straight line de. The constants A, n may be determined from the usual experimental evidence on endurance strength (for small strains) by the position of the line de. From experimental results we have

(4.5) $$m \geq n.$$

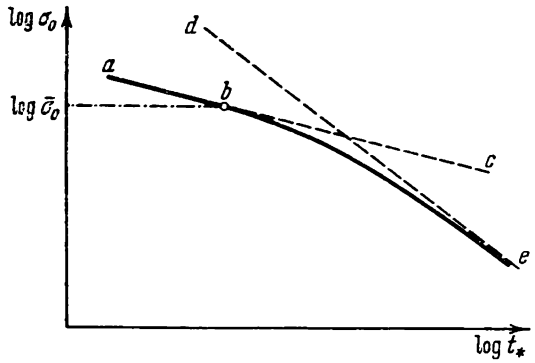

Fig. 1

In the case $m = n$, the solution has the form

(4.6) $$t^* = q_1 t_1, \quad q_1 = 1 - e^{-m\kappa} \leq 1.$$

Here only brittle or non-brittle failure occurs (viscous rupture is impossible) which is realized for one and the same (for all stresses σ_0 for a given temperature) relative elongation. We note that

(4.7) $$q_1 = 1 - \left(\frac{l_0}{l_*}\right)^m,$$

where l_* is the length of the rod at failure. The case $m = n$ must refer to material with a sufficiently stable structure.

If the experimental points lie on a straight line (in a logarithmic scale), a suitable choice of the constant A may achieve a good approximation which is the advantage of the present method over that of N. J. Hoff, since in the latter the constant B_1 is fixed.

§ 5. Failure of a rod for constant rate of elongation

In this case $dl/dt = \text{const.} = v$, i.e.,

(5.1) $$l = l_0 + vt.$$

In the creep equations (2.2) the rate of deformation ξ is known, and consequently the stress σ is determined. Introducing the value found

into (3.2) and assuming that at the instant of failure $t = t^*$ one has $\psi = 0$, we obtain

$$t' = \int_0^{t^*} \frac{dt}{\left(1 + \frac{v}{l_0} t\right)^{n/m}}, \tag{5.2}$$

where t' is the time to brittle fracture in accordance with (4.2) and σ_0 is determined by (2.2).

The case $m = n$. In this case the time of failure is

$$t^* = \frac{l_0}{v} (e^{\kappa} - 1). \tag{5.3}$$

It is readily seen that the deformation at the instant of failure is equal to $e^{\kappa} - 1$ and that it does not depend on the duration of the experiment, i.e., on v.

The case $n < m$. In this case we have

$$t^* = \frac{l_0}{v} \left[\left(1 + \kappa \frac{m - n}{m} \sigma_0^{m-n} \right)^{m/(m-n)} - 1 \right]. \tag{5.4}$$

The strain at the instant of failure is

$$\varepsilon^* = \frac{v}{l_0} t^*$$

and it attains 30–40%, although in most cases it is considerably less. The second term in the bracket is positive and small compared with unity. In the expansion of the bracket one may then limit consideration to the first approximation of two terms, and hence obtain

$$\varepsilon^* \approx \kappa B_1^{(n-m)/m} \xi_0^{(m-n)/m}.$$

This formula agrees well with the experimental formula (2.5) of A. V. Staniukovich; since [6] for a number of alloys $n \approx 0.7m$, one has in this case

$$\frac{m - n}{n} \approx 0.3.$$

§ 6. Remarks on other cases of failure

Obviously the equation (3.2) permits study of failure under different loading conditions. In particular, it is easy to consider failure for linearly changing stresses as well as for stepped loading.

In the case of simple relaxation (when there is no additional loading) failure does not always take place.

In the case of relaxation in the presence of additional loading, supporting higher levels of stresses, failure of the ends finally occurs [9].

§ 7. Front of failure

If the state of stress of the body is non-homogeneous, one has to differentiate two stages in the analysis of the process of failure of a body. In the first phase (stage of latent failure) $0 \leq t < t_1$, one has at each point of the body $\psi > 0$. At the instant $t = t_I$, at some point (or region) of the body local failure occurs. The spreading of the local failures leads to failure of the entire body at time $t = t_{II}$. We will agree to call the interval $t_I < t \leq t_{II}$ *the stage of the propagation of the failure*.

Let at time $t > t_I$ the failure occupy the region V_2 (Fig. 2); the region V_2 is separated from the region V_1 by the surface Σ, the front of failure. At the surface Σ one has $\psi = 0$, and hence $d\psi/dt + \partial\psi/\partial\nu \; d\nu/dt = 0$, where ν is the normal to the front. Using (3.2), we find the equation of motion of the failure front

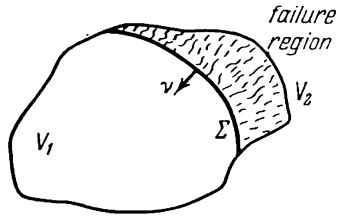

Fig. 2

$$\frac{d\nu}{dt} = -\frac{(\sigma^n{}_{\max})_\Sigma}{\left[\dfrac{\partial}{\partial\nu}\displaystyle\int_0^t \sigma^n{}_{\max}\, dt\right]_\Sigma}. \tag{7.1}$$

In the case of a homogeneous state of stress the denominator vanishes and the failure front propagates instantly. For a non-homogeneous state of stress the front of failure propagates at a finite velocity.

The equation of the front of failure may be presented in a somewhat different form, which is at times more convenient for applications (V. I. Rozenblium has drawn attention to this circumstance). At the instant τ the stress $\sigma_{\max}$, at some fixed point, is a function of the coordinates and τ. At time t, the failure front passes through the point under consideration when $\psi = 0$ and we obtain from (3.2)

$$A(n+1)\int_0^t \sigma^n{}_{\max}(\tau)\, d\tau = 1. \tag{7.2}$$

Equation (7.1) is readily deduced from the relation (7.2).

§ 8. Brittle failure in tubes under the influence of internal pressure

We shall consider the problem of brittle fracture of thick-walled tubes (for small strains). The initial dimensions of the tube are characterized by the radii a_0, b_0 (Fig. 3). If the maximum normal stress is attained on the outside surface, the tube fails on the outside. Let the radius of the failure front at time t be $b(t)$.

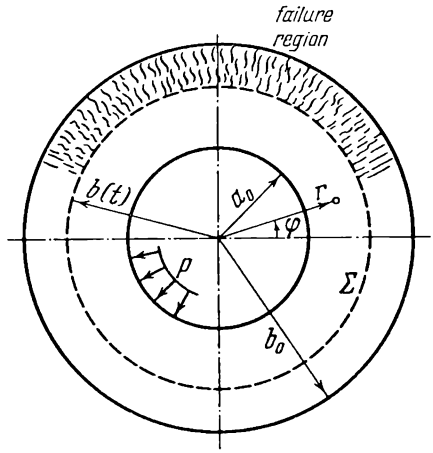

Fig. 3

The distributions of the stresses σ_r, σ_φ in the tube with dimensions a_0, b in a state of steady creep are described by the well-known formulae [7]:

$$(8.1) \qquad \sigma_r = s\left[1 - \left(\frac{b}{r}\right)^{2\mu}\right], \quad \sigma_\varphi = s\left[1 + (2\mu - 1)\left(\frac{b}{r}\right)^{2\mu}\right],$$

where $\mu = 1/m$, m is the creep index, $s = s(\beta) = p(\beta^{2\mu} - 1)^{-1}$, $\beta = b/a_0$, $\beta_0 = b_0/a_0$, p is the internal pressure, $s_0 = s(\beta_0)$.

For $m = 2$, the stress σ_φ is constant through the thickness of the tube and failure ensues immediately in the entire region. For $m < 2$, failure spreads from the inside surface of the tube. For $m > 2$ and $\beta \leq \leq (1 - 2\mu)^{-m/2}$, the stress σ_φ is tensile everywhere and it attains its maximum on the outside surface of the tube. In this case, failure develops following the scheme shown in Fig. 3; since in most cases $m > 2$, only this case has been analyzed below.

Introducing the value $\sigma_{\max} = \sigma_\varphi$ into (7.2), setting $s = s[\beta(\tau)]$, $b = b(\tau)$, $r/a_0 = \beta(t)$ and differentiating them with respect to t, we

arrive at the integro-differential equation for $\beta = \beta(t)$

$$\frac{d\beta}{dt} = \tag{8.2}$$

$$= -\frac{(2\mu)^{n-1}}{n(1-2\mu)} \cdot \frac{s^n[\beta(t)]}{\int_0^t s^n[\beta(\tau)]\left[1 + (2\mu - 1)\left(\frac{\beta(\tau)}{\beta(t)}\right)^{2\mu}\right]^{n-1}\left(\frac{\beta(\tau)}{\beta(t)}\right)^{2\mu}\frac{d\tau}{\beta(\tau)}}.$$

Obviously the initial conditions are

$$\beta = \beta_0 \quad \text{for} \quad t = t_I, \tag{8.3}$$

where

$$t_I = \frac{1}{A(n+1)(2\mu s_0)^n}. \tag{8.4}$$

The solution of the equation (8.2) may be obtained numerically, for example, by Euler's method. For this purpose it is convenient to break up the integral into the sum of two integrals for the intervals $0 \leq \tau \leq t_I$ and $t_I < \tau \leq t$. Since in the first interval $\beta(\tau) = \text{const.} = \beta_0$, the corresponding integral can be evaluated directly. Figure 4 shows the function $\beta = \beta(t/t_1)$, found by numerical integration for the cases $\beta = 2$, $m = 4$ and $m = 6$.

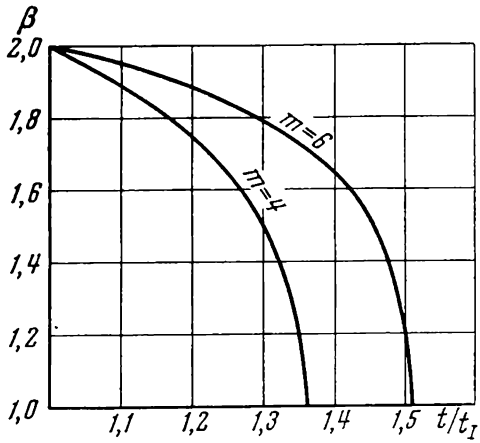

Fig. 4

It is of interest to note that at first the failure front moves comparatively little and that in the following process of failure it proceeds with great velocity. The duration of the propagation stage of failure

may form a considerable part of the stage of latent failure. In other words, the tube with a partially failed cross-section may still resist the external load for a rather long time.

§ 9. Brittle failure in bending

We shall consider the problem of brittle fracture of a rod of rectangular cross-section in pure bending. We will assume that the fracture develops under conditions of small strains and that the stress distribution is close to steady state.

During the stage of partial failure ($0 \leq t < t_I$) we have $\psi > 0$, and the entire section of the beam resists the bending. The stress distribution is described by the formula [7]

$$\sigma = \frac{M}{I_{m0}} y_0^{\mu}, \quad (y_0 > 0), \tag{9.1}$$

where M is the bending moment, x_0, y_0 are the coordinates and the axis Ox_0 is directed along the axis of the rod. The generalized moment of inertia is equal to

$$I_{m0} = \frac{2b}{1+\mu/2} h_0^{2+\mu}, \tag{9.2}$$

where $2b$ is the width of the section and $2h_0$ its height.

In the extended zone $y_0 > 0$, the continuity is to be determined in accordance with (3.2); the compressed zone $y_0 < 0$ does not fail, i.e., we have in it $\psi = 1$.

The stress σ has its maximum for $y_0 = h_0$ and we find readily from (3.2) for $\psi = 0$

$$t_I = \left[A(n+1) \left(\frac{M}{I_{m0}} \right)^n h_0^{n/m} \right]^{-1}. \tag{9.3}$$

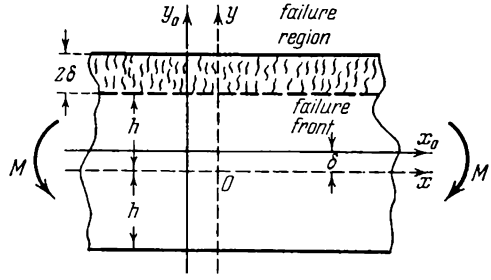

Fig. 5

At the instant $t = t_I$, the surface layer $y_0 = h_0$ fails. In the subsequent period, the failure front Σ moves down into the rod (Fig. 5). Let at the

time t the thickness of the failed layer be 2δ; it is obvious that $h_0 = h + \delta$.

At some instant $\tau > t_I$ the stress will be

$$\sigma = \frac{M}{I_m} y^{\mu}, \tag{9.4}$$

where the generalized moment of inertia

$$I_m = \frac{2b}{1 + \frac{\mu}{2}} h^{2+\mu}$$

is a function of τ and y is measured from the new position of the neutral axis, so that $y = y_0 + h_0 - h$.

Let at time t the failure front pass the fixed point with coordinate y_0. By (7.2), we have

$$M^n A(n+1) \int_0^t [y_0 + h_0 - h(\tau)]^{n/m} \frac{d\tau}{I_m{}^n(\tau)} = 1.$$

However, at the stated instant t, one has $y = h(t)$; hence $y_0 + h_0 = 2h(t)$, and the last relation takes the form

$$A(n+1)M^n \int_0^t [2h(t) - h(\tau)]^{n/m} \frac{d\tau}{I_m{}^n(\tau)} = 1. \tag{9.5}$$

For the sake of simplicity we will assume that $m = n$ and differentiate (9.5) with respect to t; then

$$2\frac{dh}{dt} \int_0^t h^{-1-2m}(\tau) d\tau + h^{-2m} = 0. \tag{9.6}$$

The initial condition is such that

$$h = h_0 \quad \text{for} \quad t = t_I. \tag{9.7}$$

We find likewise from (9.6) that

$$\frac{dh}{dt} = -\frac{h_0}{2t_I} \quad \text{for} \quad t = t_I. \tag{9.8}$$

Differentiating (9.6) with respect to the time t and eliminating the integral, we obtain the differential equation for $h(t)$

$$\frac{d^2h}{dt^2} + 2(m-1)\frac{1}{h}\left(\frac{dh}{dt}\right)^2 = 0.$$

Integrating this equation and determining the arbitrary constants from the conditions (9.7) and (9.8) we find:

$$\frac{t}{t_I} = 1 + \frac{2}{2m-1}\left[1 - \left(\frac{h}{h_0}\right)^{2m-1}\right]. \tag{9.9}$$

For $h = 0$, complete failure of the beam takes place. The failure time t' is given by

$$\frac{t'}{t_I} = 1 + \frac{2}{2m-1}. \tag{9.10}$$

Hence it follows that the stage of propagation of the failure is characterized by a considerable duration (thus, for $m = 3$, $t' = 1.4t_I$).

Following the traditional scheme of the technical bending theory, one may apply the results obtained to the analysis of the fracture of beams under arbitrary loading. Then the stress σ is determined, as before, by the formula (9.4), but the bending moment $M = M(x)$ is a function of x.

Let the largest bending moment occur for $x = x^*$. Failure arises in the surface layer of the extended zone of the beam at the point $x = x^*$ at time t_I^*, determined by the formula (9.3) for $M = M(x^*)$.

In expanding the zone of failure will embrace the entire section $x = x^*$ at the moment

$$t_*' = t_I^*\left(1 + \frac{2}{2m-1}\right). \tag{9.11}$$

At time $t = t_*'$, failure covers some region of wedge-shaped form adjoining the extended region of the beam in the neighborhood of the section $x = x^*$. The outline of this zone $h = h(x)$ may be found from (9.9), if one puts there

$$t = t_*', \quad t_I = t_I(x),$$

where $t_I(x)$ is the time of termination of the stage of partial failure of the surface layer of the beam at the section determined by the abscissa x. With the help of (9.3) and (9.11) we obtain

$$\left[\frac{M(x)}{M(x^*)}\right]^m = \frac{2m-1}{2m+1}\left\{1 + \frac{2}{2m-1}\left[1 - \left(\frac{h(x)}{h_0}\right)^{2m-1}\right]\right\}. \tag{9.12}$$

Given the value of $h(x)/h_0$, we calculate from the equation (9.12) the corresponding coordinate x.

For example, consider the failure of a beam (of length $2l$), lying on two supports and bent by a force P applied in the middle; then

$M(x)/M(x^*) = 1 - x/l$. Failure envelops a narrow region, ending in a point (Fig. 6).

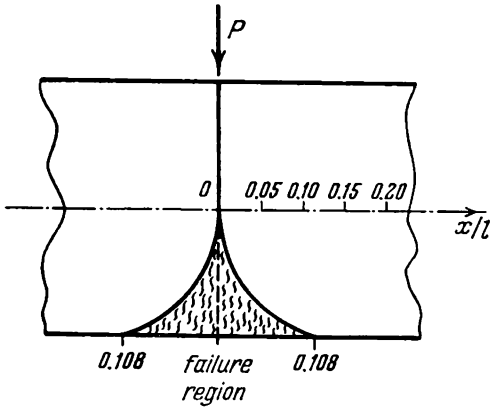

Fig. 6

§ 10. Brittle failure of a stretched plate with a hole

We will consider the problem of brittle failure of a circular plate of constant thickness h. We will denote by a_0, b_0 the initial radii of the plate (Fig. 7). The internal contour of the plate $r = a_0$ is free, but the outer contour is subject to uniform tension q. We denote by $a(t)$ the radius of the failure front Σ at the instant t. As always, we will assume that $\sigma_z = 0$. Let $v(r)$ be the radial velocity of a point of the plate; the rates of deformation will be

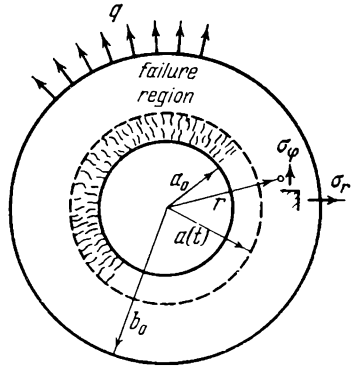

Fig. 7

$$\xi_r = \frac{\partial v}{\partial r}, \quad \xi_\varphi = \frac{v}{r}. \tag{10.1}$$

In order to simplify the mathematical analysis of the problem, we will go over to the scheme of maximal shear stress for the associated flow law of W. Prager [8]. Figure 8 shows the intersection of the Tresca prism with the plane $\sigma_z = 0$. The point depicting $(\sigma_r, \sigma_\varphi)$ lies inside one of the sides of the hexagon or at one or the other of its vertices. According to the law of associated flow, the latter develops along the normal to the surface of the prism at the point under consideration.

If $\sigma_\varphi > \sigma_r > 0$, then $\tau_{\max} = \frac{1}{2}\sigma_\varphi$ and σ_r is the intermediate stress.

The illustrating point lies inside the segment $A_{11}A_{12}$, the normal to which is perpendicular to the axis σ_r. Therefore, for the power creep law, we have

$$\xi_r = 0, \quad \xi_\varphi = B_1\sigma_\varphi^m, \quad \xi_z = -\xi_\varphi. \tag{10.2}$$

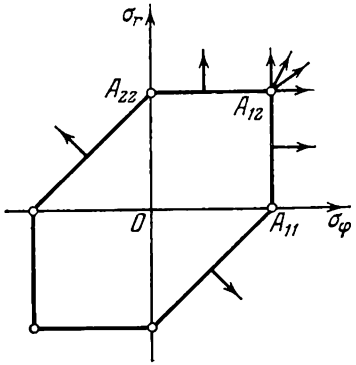

Fig. 8

However, if the illustrating point lies at the vertex A_{12}, then $\sigma_r = \sigma_\varphi$ and flow may develop in any direction within the limits of the angle, formed by the normals to the neighboring sides.

Generally speaking, there may exist in the plate some zones in each of which one or the other flow is realized. We will limit consideration to the case when the flow (10.2) applies in the entire region. This condition to a certain extent limits the dimensions of the plate, as will be seen from what follows.

The condition of compatibility of the strain velocities and the equilibrium equation respectively have the forms

$$\frac{\partial \xi_\varphi}{\partial r} + \frac{\xi_\varphi - \xi_r}{r} = 0, \tag{10.3}$$

$$\frac{\partial}{\partial r}(r\sigma_r) - \sigma_\varphi = 0. \tag{10.4}$$

The boundary conditions are

$$\sigma_r = 0 \quad \text{for} \quad r = a, \tag{10.5}$$

$$\sigma_r = q \quad \text{for} \quad r = b_0. \tag{10.6}$$

If the flow (10.2) is realized in the entire plate, $\partial v/\partial r = 0$; hence

$$v = v_0(t), \quad \xi_\varphi = \frac{v_0(t)}{r}, \tag{10.7}$$

where $v_0(t)$ is an arbitrary function of the time. Evaluating the stress σ_φ from the relationship (10.2), introducing it into the equilibrium equation and executing the integration for the stated boundary conditions, we obtain

$$r\sigma_r = b_0 q \frac{r^{1-\mu} - a^{1-\mu}}{b_0^{1-\mu} - a^{1-\mu}},$$

$$\sigma_\varphi = b_0 q \frac{r^{-\mu}}{b_0^{1-\mu} - a^{1-\mu}}(1-\mu).$$

This solution holds true if $0 < \sigma_r/\sigma_\varphi < 1$. It is readily seen that then the following condition must be fulfilled:

$$\frac{b_0}{a_0} < m^{m/(m-1)}.$$

Introducing the non-dimensional variables

$$\frac{r}{b_0} = \rho, \quad \frac{a}{b_0} = \alpha, \quad \frac{a_0}{b_0} = \alpha_0,$$

we find that the stress σ_φ is given by

$$\sigma_\varphi = s(\alpha)\rho^{-\mu}, \tag{10.8}$$

where

$$s(\alpha) = \frac{(1-\mu)q}{1-\alpha^{1-\mu}}. \tag{10.9}$$

In the stage of partial failure $t < t_I$ we have $\alpha = a_0$; hence the stress σ_φ does not depend on the time. We determine from (3.2) for $\psi = 0$ the instant of time

$$t_I = \frac{1}{A(n+1)} \cdot \frac{\alpha_0{}^{n/m}}{s^n(\alpha_0)}. \tag{10.10}$$

For $t > t_I$, failure of the material takes place in the region adjoining the hole. Introducing into the equation of motion of the failure front (7.2) the value $\sigma_{\max} = \sigma_\varphi$, setting $s(\alpha) = s[\alpha(\tau)]$, $\rho = \alpha(t)$ and differentiating them with respect to t, we arrive at the differential equation for $\alpha(t)$

$$\frac{d\alpha}{dt} = \frac{m}{n} \frac{\alpha_0}{t_I} \frac{s^n(\alpha)}{s^n(\alpha_0)} \left(\frac{\alpha}{\alpha_0}\right)^{1-(n/m)}. \tag{10.11}$$

The initial condition has the form

$$\alpha = \alpha_0 \quad \text{for} \quad t = t_I.$$

Integrating (10.11), we obtain

$$\frac{t}{t_I} - 1 = \frac{n}{m} \frac{1}{\alpha_0} \int_{\alpha_0}^{\alpha} \left(\frac{1-\alpha^{1-\mu}}{1-\alpha_0{}^{1-\mu}}\right)^n \left(\frac{\alpha_0}{\alpha}\right)^{(m-n)/m} d\alpha. \tag{10.12}$$

Setting $\alpha = 1$, we find the time of failure t'

$$\frac{t'}{t_I} = 1 + \frac{n}{m} \frac{1}{\alpha_0} \int_{\alpha_0}^{1} \left(\frac{1-\alpha^{1-\mu}}{1-\alpha_0{}^{1-\mu}}\right)^n \left(\frac{\alpha_0}{\alpha}\right)^{(m-n)/m} d\alpha.$$

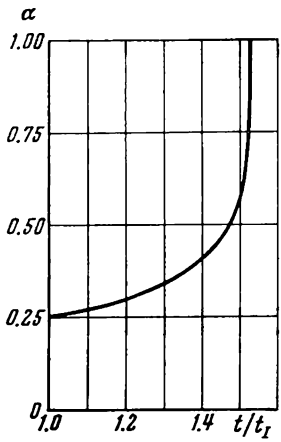

Fig. 9

For integral n, the integral in (10.12) is readily evaluated.

Figure 9 shows the graph of the position of the failure front as a function of the time for the case $m = n = 4$, $\alpha_0 = \frac{1}{4}$, evaluated from the solution (10.12). As in the other examples, the failure initially propagates comparatively slowly; then the rate of formation of the failed region increases and in the last stage (approximately from $\alpha = 0.5$ to $\alpha = 1.0$) failure bears an "avalanche" character. The plate fails completely at time $t' = 1.53t_I$.

I wish to seize this opportunity to express my gratitude to V. I. Rozenblium and Ia. B. Fridman for a number of important remarks.

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ON A METHOD OF SOLUTION OF PROBLEMS OF THE THEORY OF ELASTICITY FOR THE SEMI-CIRCLE

A. I. KALANDIIA
Tbilisi, U.S.S.R.

This note presents a method which makes it possible to find effective (approximate) solutions to certain problems of bending of thin plates, and likewise to plane problems of the theory of elasticity for the case when the elastic medium occupies the region of a semi-circle. The problem is solved by reduction to a singular integral equation with a Cauchy type kernel and subsequent application to this equations of known numerical methods. The method is expounded with reference to the problem of bending of an elastic plate having the shape of a semi-circle which is clamped along the circumference and free along the diameter.

§ 1. We consider a thin isotropic elastic plate having the shape of a semi-circle, bent by uniform normal loading q, distributed over the middle surface. As the physical plane we take the plane of the variable $z = x + iy$ and we place our semi-circle of radius unity in the upper half plane. We will denote by γ the part of the contour which represents the circumference of the region.

Then, as is known, the problem under consideration reduces to finding a solution of the equation

$$\Delta\Delta w = \frac{q}{D} \tag{1.1}$$

which is regular in the region $|z| < 1$, $\operatorname{Im} z > 0$ for the boundary conditions

$$w = \frac{dw}{dn} = 0 \quad \text{on } \gamma, \tag{1.2}$$

$$\begin{aligned} G(w) &\equiv \nu\Delta w + (1-\nu)[w_{xx}\cos^2\theta + w_{xy}\sin 2\theta + w_{yy}\sin^2\theta] = 0, \\ S(w) &\equiv \frac{d\Delta w}{dn} + \\ &+ (1-\nu)\frac{d}{ds}[w_{xy}\cos 2\theta + (w_{yy} - w_{xx})\sin\theta\cos\theta] = 0, \\ &\text{for } -1 < x < 1, \end{aligned} \tag{1.3}$$

where ν is Poisson's ratio, θ is the angle between the external normal $\mathbf{n}$ and the axis x.

We will now given an outline of the method of solution.

The given semi-circle will be completed into a full circle, clamped along its entire circumference and bent by normal loading of the same intensity q. In this circle a certain generalized loading $p(x)$ is introduced distributed along its diameter $-1 < x < 1$ and represented by a complex function [1] of x which is selected in such a manner that the condition (1.3) is fulfilled. In other words, the solution $w(x, y)$ is represented in the form of the sum of two functions

$$w(x, y) = \tilde{w}(x, y) + w_0(x, y), \tag{1.4}$$

where $\tilde{w}(x, y)$ is the deflection in the circular plate clamped along its contour and subject to the given loading q, and $w_0(x, y)$ is the deflection of the same plate, bent by "forces" $p(x)$ subject to determination which are distributed along its diameter $-1 < x < 1$.

The function $\tilde{w}(x, y)$ is known to have the form

$$\tilde{w}(x, y) = \frac{q}{64D}[(x^2 + y^2)^2 - 2(x^2 + y^2) + 1]. \tag{1.5}$$

We will now search for the function w_0 and will use for it its complex representation

$$w_0(x, y) = \bar{z}\varphi_0(z) + z\overline{\varphi_0(z)} + \chi_0(z) + \overline{\chi_0(z)}. \tag{1.6}$$

§ 2. We will determine the functions $\varphi(z)$, $\psi(z)$, analytic in the unit circle and having singularities of the form

$$\varphi(z) = p \log (z - z_0) + \varphi_*(z), \quad \psi(z) = \bar{p} \log (z - z_0) - \frac{p\bar{z}_0}{z - z_0} + \psi_*(z),$$

where p is a constant, z_0 is a fixed point inside the region, φ_*, ψ_* are holomorphic functions, for the boundary condition

$$\varphi(\sigma) + \sigma\overline{\varphi'(\sigma)} + \overline{\psi(\sigma)} = 0, \quad \sigma = e^{i\alpha}. \tag{2.1}$$

The functions φ, ψ are readily constructed by use of the well-known method of N. I. Muskhelishvili. We find for them, under the usual conditions of normalization,

$$\varphi(z) = p \log (z - z_0) - p \log (1 - z\bar{z}_0) - \bar{p}\frac{z(z - z_0)}{1 - z\bar{z}_0} - \tfrac{1}{2}(p\bar{z}_0 + \bar{p}z_0)z,$$

$$\psi(z) = \bar{p} \log (z - z_0) - \frac{p\bar{z}_0}{z - z_0} - \bar{p} \log (1 - z\bar{z}_0) + \tag{2.2}$$

$$+ \frac{\bar{p}\bar{z}_0(z - z_0)}{(1 - z\bar{z}_0)^2} - \frac{p\bar{z}_0^2 - 2p + \bar{p}z_0\bar{z}_0}{1 - z\bar{z}_0}.$$

[1] In certain cases it is apparently sufficient to introduce for the auxiliary loading

We will now, in accordance with (2.2), seek the functions $\varphi_0(z)$, $\psi_0(z) = \chi_0'(z)$, holomorphic in the semi-circle, in the form

$$\varphi_0(z) = \frac{1}{16\pi D}\int_{-1}^{1}\Big\{p(t)[\log(z-t) - \log(1-zt)] - \overline{p(t)}\,\frac{z(z-t)}{1-zt} - \\ - \tfrac{1}{2}[p(t) + \overline{p(t)}]tz\Big\}\,dt,$$

(2.3)

$$\psi_0(z) = \frac{1}{16\pi D}\int_{-1}^{1}\Big\{\overline{p(t)}[\log(z-t) - \log(1-zt)] - \frac{p(t)t}{z-t} + \\ + H(p, z; t)\Big\}\,dt,$$

$$H(p, z; t) = \frac{\overline{p(t)}t(z-t)}{(1-zt)^2} - \frac{[p(t) + \overline{p(t)}]t^2 - 2\overline{p(t)}}{1-zt},$$

where $p(x)$ is a new unknown complex function of x, defined on the interval $(-1, 1)$.

It is easy to see that the introduced functions (2.3) are holomorphic in any finite region of the z plane which does not contain points of the real axis. More exactly, the functions $\varphi'_0(z)$, $\psi_0(z)$ consist of Cauchy type integrals, taken along a finite and an infinite segment of the Ox axis which complement each other into a complete straight line. In what follows these functions have to be considered by us exclusively in the closed upper semi-circle, where we will understand by $\log(z-t)$ for fixed t the branch for which $0 \leq \operatorname{Im}\log(z-t) \leq \pi$. In addition, by (2.1), the functions (2.3) will satisfy, exactly apart from a constant, the boundary condition (1.2) for any $p(x)$.

We obtain from the first of the formulae (2.3) by differentiation

$$\varphi_0'(z) = \frac{1}{16\pi D}\int_{-1}^{1}\Big\{\frac{p(t)}{z-t} - zH(p, z; t) + \tfrac{1}{2}[p(t+ \overline{p(t)}]t\Big\}\,dt. \tag{2.4}$$

§ 3. Now, let $\varphi(z)$, $\chi(z)$ be functions, holomorphic in some finite simply connected region T, w the regular biharmonic function which corresponds to them by (1.6), and G, S the left hand sides of (1.3), constructed for the function w. Then, the formula of S. G. Lekhnitskii [1] and I. N.

$p(x)$ a real (for example, a normal) load. In those cases $p(x)$ is, of course, a real function.

Vekua [2]

$$(3.1)\quad G(w) + i\int_s^s S(w)ds = -2(1-\nu)\frac{d}{dt}[-\kappa\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)}]$$

applies on any smooth part L of the boundary of the region T. Here t is a point on L, corresponding to the arc coordinate s, $\varphi(t)$, $\varphi'(t)$, $\psi(t)$ are the limiting values on L of the corresponding functions of z and $\kappa = (3+\nu)/(1-\nu)$.

On the basis of (3.1), denoting by G_0 and S_0 the values of G and S for the unknown function w_0, we will have on the interval $(-1, 1)$

$$(3.2)\quad G_0(x) + i\int_x^x S_0(t)dt = -2(1-\nu)\frac{d}{dx}[-\kappa\varphi_0(x) + x\overline{\varphi_0'(x)} + \overline{\psi_0(x)}].$$

With the help of (2.3) and (2.4) we find

$$(3.3)\quad \Omega[p; x] \equiv -\kappa\varphi_0(x) + x\overline{\varphi_0'(x)} + \overline{\psi_0(x)} =$$

$$= \frac{1}{16\pi D}\int_{-1}^{1}\left\{\left[-\kappa\log(x-t) + \overline{\log(x-t)} + (\kappa-1)\log(1-xt) + \right.\right.$$

$$+ \frac{(t^2-1)(x^2-1)}{1-xt} + \frac{(t^2-1)(x^2-1)}{(1-xt)^2}\Big]p(t) +$$

$$\left. + \left[\kappa + (\kappa+t^2)\frac{x^2-1}{1-xt}\right]\overline{p(t)} + \frac{\kappa+1}{2}[p(t) + \overline{p(t)}]tx\right\}dt.$$

On the other hand, for the operators $\tilde{G}$ and $\tilde{S}$, corresponding to the function $\tilde{w}$, we have on the same interval

$$(3.4)\qquad \tilde{G}(x) = \frac{q}{16D}[(3\nu+1)x^2 - 1 - \nu],\ \tilde{S}(x) = 0.$$

Forming now the sums $\mathbf{G} = \tilde{\mathbf{G}} + \mathbf{G}_0$ and $\mathbf{S} = \tilde{\mathbf{S}} + \mathbf{S}_0$ for the unknown solution (1.4) and expressing the conditions

$$G(x) = S(x) = 0 \text{ on } (-1, 1),$$

we obtain for the determination of $p(x)$ the equality

$$(3.5)\qquad 2(1-\nu)\frac{d}{dx}\Omega[p; x] = \tilde{G}(x),$$

where Ω and $\tilde{G}$ are given by (3.3) and (3.4) respectively.

We will assume that the function $p(x)$ satisfies on the interval $(-1, 1)$

a Hölder condition. Then it is known that the equalities

$$\frac{d}{dx}\int_{-1}^{1}\log(x-t)p(t)dt = -\pi i p(x) - \int_{-1}^{1}\frac{p(t)dt}{t-x},$$

$$\frac{d}{dx}\int_{-1}^{1}\overline{\log(x-t)}p(t)dt = \pi i p(x) - \int_{-1}^{1}\frac{p(t)dt}{t-x}, \tag{3.6}$$

are valid for any x of this interval.

Taking into consideration the preceding formulae, we now execute the differentiation in (3.5). Then, after certain simplifications, we obtain the equation for the determination of $p(x)$

$$\lambda p(x) + \frac{1}{\pi i}\int_{-1}^{1}\frac{p(t)dt}{t-x} + \frac{1}{\pi i}\int_{-1}^{1}[n_0(x,t)p(t) + k_0(x,t)\overline{p(t)}]dt = f_0(x), \tag{3.7}$$

$$n_0(x,t) = -\frac{t}{1-xt} + \frac{\mu}{2}(t^2-1)\left[\frac{x}{1-xt} + \frac{x-t}{(1-xt)^2} + \frac{2(x-t)}{(1-xt)^3}\right] + \frac{\lambda}{2}t,$$

$$k_0(x,t) = \frac{\mu}{2}(\kappa + t^2)\left[\frac{x}{1-xt} + \frac{x-t}{(1-xt)^2}\right] + \frac{\lambda}{2}t, \tag{3.8}$$

$$f_0(x) = \frac{q}{4(1+\nu)i}[(3\nu+1)x^2 - 1 - \nu],\ \lambda = \frac{2}{1-\nu},\ \mu = \frac{1-\nu}{1+\nu}.$$

Equation (3.7) is a singular integral equation with a Cauchy type kernel.

In accordance with the general theory of singular integral equations [3], the solution of (3.7), must be sought in the class of functions, satisfying a Hölder condition on any closed part of the interval $(-1, 1)$ and assuming for $x = \pm 1$. singularities of order less than unity. One has to still have here in mind that the functions $n_0(x, t)$ and $k_0(x, t)$, in contrast to the usual situation in the theory of singular equations, are continuous functions of their arguments only on the interval $(-1, 1)$; for $x = t = \pm 1$, these functions cease to be defined and, consequently, the kernel of the equation (3.7) admits besides the moving singularity inherent in the Cauchy type kernel also immovable singularities at the ends of the segment.

Similar integral equations find application in boundary problems of the

theory of equations of the mixed type (cf., for example, [4]). It will be of interest to us to study the general properties of such equations and, in particular, to investigate the questions of their solubility within one or the other class of functions. A detailed study of the equation (3.7) would, for example, permit one to judge the possibility of a strict solution of our problem in terms of potentials (2.3).

Below, we will occupy ourselves with the approximate solution of the equation (3.7) and we will assume, starting from physical considerations, that this equation admits a solution $p(x)$, satisfying a Hölder condition on the entire segment $-1 \leq x \leq 1$. An additional condition which will be introduced here with respect to the unknown function $p(x)$ consists essentially of the requirement that it is to be bounded near the ends of the segment.

The last requirement is equivalent to a completely determined hypothesis regarding the behavior of characteristic physical quantities inside the closed semi-circle. Under the above assumptions regarding $p(x)$ the first derivatives of w, as follows from the formulae (2.3), (2.4), will be continuous in the closed region, and hence conditions will be fulfilled which guarantee the uniqueness of the solution of the boundary problem (cf., for example, [5]).

As regards the above mentioned physical quantities – the bending and twisting moments M_x, M_y, H_{xy}, they will be continuous in the closed semi-circle except at the angular points near which one has the estimates (cf., for example, [6]),

$$|M_x|, |M_y|, |H_{xy}| < A \log \frac{1}{|z \pm 1|}, \quad (A = \text{const.}). \tag{3.9}$$

Under these conditions it may be asserted that the equation (3.7) admits one unique solution.

We will set in equation (3.7) $p(x) = \alpha(x) + i\beta(x)$ and separate real and imaginary parts. Then we obtain for the determination of α and β the system of two equations

$$\begin{aligned} &\frac{\lambda}{2}\alpha(x) + \frac{1}{2\pi}\int_{-1}^{1}\frac{\beta(t)dt}{t-x} + \frac{1}{2\pi}\int_{-1}^{1} n(x,t)\beta(t)dt = 0, \\ &\frac{\lambda}{2}\beta(x) - \frac{1}{2\pi}\int_{-1}^{1}\frac{\alpha(t)dt}{t-x} + \frac{1}{2\pi}\int_{-1}^{1} k(x,t)\alpha(t)dt = f(x), \end{aligned} \tag{3.10}$$

$$n(x,t) = -\frac{1}{1-xt}\left[t + \lambda x + \frac{\lambda(x-t)}{1-xt} - \mu\frac{(x-t)(t^2-1)}{(1-xt)^2}\right],$$

$$k(x,t) = -\frac{1}{1-xt}\left[-t + (1+\mu t^2)x + \frac{(1+\mu t^2)(x-t)}{1-xt} + \right. \tag{3.11}$$

$$\left. + \mu\frac{(x-t)(t^2-1)}{(1-xt)^2}\right] - \lambda t, \quad f(x) = \frac{1}{2i} f_0(x).$$

Using the formulae of N. I. Muskhelishvili [3], characterizing the behavior of Cauchy type kernels near ends of lines of integration, it may be stated that, if $\alpha(x)$, $\beta(x)$ are solutions of the system (3.10) satisfying a Hölder condition on the closed interval $[-1, 1]$, then $\beta(x)$ must vanish at the ends of the interval

$$\beta(1) = \beta(-1) = 0.$$

Further, it is not difficult to prove that the functions $\alpha(x)$ and $\beta(x)$, respectively, are odd and even with respect to x

$$\beta(x) = \beta(-x), \quad \alpha(x) = -\alpha(-x), \quad -1 < x < 1.$$

These properties of α and β will be used below during the construction of the numerical solutions of the equations (3.10).

After the function $p(x)$ has been found, the formulae (2.3) give us the unknown holomorphic functions $\varphi_0(z)$, $\psi_0(z)$. As regards the function $\chi_0(z)$ in the expression for the deflection, it may be represented in the form

$$\chi_0(z) = \chi_*(z) + C,$$

where $\chi_*(z)$ is some function originating from the function $\psi_0(z)$ and C is a real constant. It is sufficient for the determination of C to require that the first condition (1.2) be satisfied at any point of the arc γ, say, at the point $\sigma = i$. Then we will have

$$\mathrm{Re}\{\bar{\sigma}\varphi_0(\sigma) + \chi_*(\sigma)\} = -C \text{ for } \sigma = i. \tag{3.12}$$

All the unknown quantities which characterize the state of stress in the plate may be determined by use of the functions $\varphi_0(z)$, $\psi_0(z)$. For example, the bending moment M ($M = -DG$) along the circular part of the boundary γ will be expressed by the formula

$$M = -\frac{q}{8} + \frac{D(1-\nu)}{8}\,\mathrm{Re}\left\{\frac{d}{d\sigma}\left[-\kappa\varphi_0(\sigma) + \sigma\overline{\varphi_0'(\sigma)} + \overline{\psi_0(\sigma)}\right]\right\},$$

which after some simple calculations may be given the form

$$(3.13)\qquad -M = \frac{q}{8} + \frac{1}{2\pi}\operatorname{Re}\int_{-1}^{1}\left\{\frac{p(t)}{\sigma - t} + \frac{t[p(t) - \overline{p(t)}]}{1 - \sigma t} - \right.$$

$$\left. - \frac{p(t)(\sigma - t)}{1 - \sigma t}\left[1 + \frac{1}{1 - \sigma t}\right] - \tfrac{1}{2}[p(t) + \overline{p(t)}]t\right\} dt,$$

$$\sigma = e^{i\alpha},\ 0 < \alpha < \pi.$$

The bending moment along the diameter of the plate, which must vanish by the conditions of the problem, is equal to

$$(3.14)\quad -M = \frac{q}{16}[(3\nu + 1)x^2 - 1 - \nu] + \tfrac{1}{2}\left[\beta(x) - \frac{1 + \nu}{2\pi}\int_{-1}^{1}\frac{\alpha(t)dt}{t - x} + \right.$$

$$\left. + \frac{1 + \nu}{2\pi}\int_{-1}^{1} k(x, t)\alpha(t)dt\right],\quad -1 < x < 1.$$

For the deflection $w(x, y)$ we will have

$$(3.15)\qquad w(x, y) = \tilde{w}(x, y) + 2\operatorname{Re}\{\bar{z}\varphi_0(z) + \chi_0(z)\}.$$

§ 4. The approximate solution (3.10) is conveniently found with the help of a method given in [7] (p. 717, no. 3). In accordance with this method we will approximate the unknown function $p(x)$ by an interpolated trigonometric polynomial of ϑ ($x = \cos\vartheta$), constructed for the angles

$$(4.1)\qquad x_m = \cos\vartheta_m,\ \vartheta_m = \frac{m\pi}{n + 1},\quad (m = 1, 2, \cdots, n),$$

in the form

$$(4.2)\qquad L_n[p; x] = \frac{2}{n + 1}\sum_{k=1}^{n} p_k \sum_{m=1}^{n}\sin m\vartheta_k \sin m\vartheta,$$

$$p_k = p(x_k) = \alpha(x_k) + i\beta(x_k),$$

where n is an integer. This polynomial has as a function of x the form $\sqrt{1 - x^2}\,P_{n-1}(x)$, where $P_{n-1}(x)$ is a polynomial of x with complex coefficients of order $\leq n - 1$.

We now substitute (4.2) for $p(x)$ in (3.10) and give to the argument ϑ in the approximate equality obtained consecutively the values $\vartheta_1, \cdots, \vartheta_n$.

The equation (3.10) then becomes the following system of linear algebraic equations for the α_k and β_k:

$$\begin{aligned} &\frac{\alpha_m}{\nu+1} - \sum_{k=1}^{n} a_{mk}\beta_k = 0, \\ &\frac{\beta_m}{\nu+1} + \sum_{k=1}^{n} b_{mk}\alpha_k = f_m, \end{aligned} \qquad (m = 1, \cdots, n) \tag{4.3}$$

where

$$a_{mk} = \frac{\sin\vartheta_k}{n+1}\left[\frac{\varepsilon_{mk}}{\cos\vartheta_m - \cos\vartheta_k} - \tfrac{1}{2}n(\cos\vartheta_m, \cos\vartheta_k)\right],$$

$$b_{mk} = \frac{\sin\vartheta_k}{n+1}\left[\frac{\varepsilon_{mk}}{\cos\vartheta_m - \cos\vartheta_k} + \tfrac{1}{2}k(\cos\vartheta_m, \cos\vartheta_k)\right],$$

$$f_m = -\frac{q}{8(1+\nu)}[(3\nu+1)\cos^2\vartheta_m - 1 - \nu],$$

$$\varepsilon_{mk} = \begin{cases} 1, & |m-k| = 1, 3, \cdots \\ 0, & |m-k| = 0, 2, \cdots \end{cases} \tag{4.4}$$

Solving (4.3) and substituting the corresponding values α_k, β_k in (4.2), we obtain an approximate solution of the system (3.10) in the form of a polynomial of ϑ, and subsequently by use of (2.3) an approximate solution of the boundary problem. The physical quantities at the different points are determined for this in the form of simple integrals the evaluation of which in accordance with (4.2) is conveniently achieved by a formula of the Gauss type

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{1-t^2}\,F(t)dt = \frac{1}{n+1}\sum_{k=1}^{n}\sin^2\vartheta_k F(\cos\vartheta_k). \tag{4.5}$$

The system (4.3) has been solved [1] for $n = 8$. On the basis of the above properties of α and β we will have a system of eight equations. We will give here the numerical values of the unknowns (Poisson's Ratio was set equal to $\frac{1}{3}$)

$$\begin{aligned} \alpha_1 &= 0.055262q, & \beta_1 &= 0.001388q, \\ \alpha_2 &= 0.040054q, & \beta_2 &= 0.011940q, \\ \alpha_3 &= 0.024582q, & \beta_3 &= 0.017921q, \\ \alpha_4 &= 0.009685q, & \beta_4 &= 0.025691q, \\ \alpha_k &= -\alpha_{9-k}, & \beta_k &= \beta_{9-k}, \quad k = 5, 6, 7, 8. \end{aligned} \tag{4.6}$$

[1] The computations leading to the following results were performed by S. B. Fain.

In order to assess the accuracy of the approximate solution $p(x)$, determined by the values (4.6), we evaluate the bending moment M at the point $x = y = 0$, lying in the middle of the segment between the two adjacent corners. Applying the formula (4.5) to the integral on the right hand side of (3.14) and substituting (4.6), we find

$$M|_{x=y=0} = 0.005295q.$$

The bending moment along γ, determined by (3.13), will have its maximum at the point $\sigma = i$. Substituting in this formula $\sigma = i$ and evaluating the integral as above we obtain

$$M|_{\sigma=i} = -0.136627q.$$

Finally, for the deflection at the point $x = y = 0$, we will have

$$w|_{x=y=0} = 0.098105 \frac{q}{D}.$$

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THE PROPAGATION OF ELASTO-PLASTIC WAVES IN A HALF-SPACE IN A MAGNETIC FIELD FOR A PERFECT CONDUCTOR

S. KALISKI
Warsaw, Poland

§ 1. Introduction

Some of the present author's papers, [3, 4, 6], stated the equations and obtained solutions for the Cauchy problem and the basic solution for an infinite space for the equations of elasto-dynamics of solid isotropic bodies in a magnetic field. Similar solutions were also obtained for anisotropic bodies. The object of the present paper is to obtain a solution to the elasto-plastic one-dimensional problem of the perfectly conducting half-space bordered by a vacuum, assuming the existence of a primary constant magnetic field and any increasing and then decreasing pressure on the surface of the half-space. Solutions of this type will contain a moderate range of mechanical pressures. The problem of high pressures in magnetic fields may be considered on the basis of the theory of plastic gas as a model of a solid body in the range of very high pressures.

Secondary electro-magnetic effects accompanying the processes of propagation of strong non-linear waves in the range of moderate and high pressures in primary magnetic fields, and also the perturbation of the mechanical processes themselves in magnetic fields, may indicate the way to new methods of recording mechanical phenomena by measuring secondary electro-magnetic effects. A number of new phenomena peculiar to coupled non-linear mechanical-electromagnetic fields such as new kinds of shock waves in solid bodies, unloading waves for coupled fields and generalization of the motion of the so-called Rakhmatulin wave to the case of coupled fields suggest a number of investigations interesting both from the physical and the mathematical points of view, and present possibilities of experimental recording of the mechanical waves by means of electromagnetic measurements. This creates the possibility of investigations, different from those undertaken hitherto, of phenomena connected with high pressures such as of cumulative effects or problems of propagation of strong explosion waves or of underground explosions in the magnetic field of the earth, measurements of which are still made by means of the seismic method.

The literature of the problem, as far as the author knows, does not include work on the propagation of elasto-plastic waves in magnetic

fields. There exist only broad investigations of the problem of shock waves in magnetodynamics.

The present work constitutes one of the first sub-problems of the more general problem. Its object is to establish the equations and give the solution for one of the simplest cases of propagation of non-linear elasto-plastic waves in magnetic fields.

An accurate solution for a loading wave in a magnetic field was obtained for elasto-plastic bodies when the $\sigma-\varepsilon$ characteristic represents an angled line and perfect electric conductivity is assumed. In the case of arbitrary $\sigma-\varepsilon$ characteristic an approximate solution is obtained. For "small" initial magnetic fields it involves an accuracy of the order of a^2/c^2 (where a/c is the ratio of the sound velocity in the body to the light velocity in vacuum). The problem of an unloading wave has only been formulated for a body with rigid unloading characteristic. Solution of this problem will be given in a separate paper.

§ 2. The propagation of an elastic wave in a magnetic field

We assume the system of co-ordinates for the half-space and the neighbouring vacuum shown in Fig. 1.

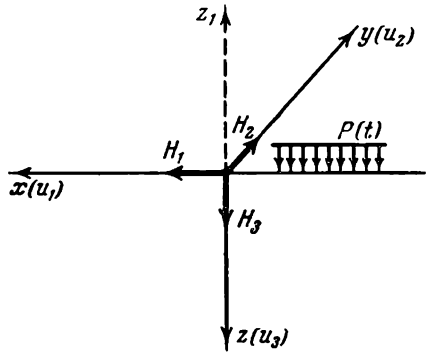

Fig. 1.

We make the following assumptions.

1. The load on the surface of the half-space depends on the time t only and is independent of x and y.
2. The conduction is perfect.
3. $H_2 = H_3 = 0$, $H_1 = H$.
4. $\mu \approx 1$ for the medium in the half-space.
5. Displacement currents are disregarded in the equations of coupled fields for a perfect conductor.
6. Lagrangean co-ordinates areused.

According to [3] the equations of the problem for a real conductor have the form:

a) for the body:

$$\text{rot } \mathbf{h} = \frac{4\pi}{c}\mathbf{j} + \frac{\varepsilon}{c}\frac{\partial \mathbf{E}}{\partial t}, \qquad \text{rot } \mathbf{E} = -\frac{\mu}{c}\frac{\partial \mathbf{h}}{\partial t}, \tag{2.1}$$

$$\mathbf{j} = \lambda_0 \left[\mathbf{E} + \frac{\mu}{c} \left(\frac{\partial \mathbf{u}}{\partial t} \times \mathbf{H} \right) \right],$$

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = G\nabla^2 \mathbf{u} + (\lambda + G)\text{grad div } \mathbf{u} + \frac{\mu}{c} (\mathbf{j} \times \mathbf{H}); \tag{2.1}$$

b) for the vacuum:

$$\text{rot } \bar{\mathbf{h}} = \frac{1}{c} \frac{\partial \bar{\mathbf{E}}}{\partial t},$$

$$\text{rot } \bar{\mathbf{E}} = -\frac{1}{c} \frac{\partial \bar{\mathbf{h}}}{\partial t}. \tag{2.2}$$

For a perfect conductor they are:

a) for the body:

$$\text{rot } \mathbf{h} = \frac{4\pi}{c} \mathbf{j},$$

$$\text{rot } \mathbf{E} = -\frac{\mu}{c} \frac{\partial \mathbf{h}}{\partial t},$$

$$\mathbf{E} + \frac{\mu}{c} \left(\frac{\partial \mathbf{u}}{\partial t} \times \mathbf{H} \right) = 0, \tag{2.3}$$

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = G\nabla^2 \mathbf{u} + (\lambda + G)\text{grad div } \mathbf{u} + \frac{\mu}{c} (\mathbf{j} \times \mathbf{H});$$

b) for the vacuum, the equations are identical with (2.2).

Eliminating **h**, **j**, **E** in terms of **u**, we obtain

$$G\nabla^2 \mathbf{u} + (\lambda + G)\text{grad div } \mathbf{u} - \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} + \frac{\mu}{4\pi} [\text{rot rot}(\mathbf{u} \times \mathbf{H})] \times \mathbf{H} = 0, \tag{2.4}$$

and for the vacuum the equations (2.2).

If in the following we write $u_3 = u$ and assume, in accordance with the assumptions above, $u_1 = u_2 = 0$, the equations (2.4) for the body take the form

$$\rho \frac{\partial^2 u}{\partial t^2} = \kappa \frac{\partial^2 u}{\partial z^2} + \frac{\partial \sigma}{\partial z}, \tag{2.5}$$

where

$$\kappa = \frac{\mu H^2}{4\pi},$$

or

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial z^2}, \tag{2.6}$$

where

$$a = \sqrt{\frac{\lambda + 2G + \kappa}{\rho}}. \tag{2.7}$$

The equations (2.2) for the vacuum take the form

$$\frac{\partial^2 \bar{h}_i}{\partial z_1^2} - \frac{1}{c^2} \frac{\partial^2 \bar{h}_i}{\partial t^2} = 0, \quad \frac{\partial^2 \bar{E}_i}{\partial z_1^2} - \frac{1}{c^2} \frac{\partial^2 \bar{E}_i}{\partial t^2} = 0, \quad i = 1, 2, \tag{2.8}$$

$$\bar{E}_3 = \bar{h}_3 = \text{const.} = 0.$$

The boundary conditions for the surface $z = 0$, $z_1 = 0$, subject to the action of normal pressure $P(t)$ which varies with time, will have the form

$$\begin{aligned} &\sigma_{3k} + T_{3k} - \bar{T}_{3k} = -\delta_{3k} P(t), \quad k = 1, 2, 3, \\ &h_3 = \bar{h}_3 \quad (\mu \approx 1), \\ &E_1 = \bar{E}_1, \\ &E_2 = \bar{E}_2, \end{aligned} \tag{2.9}$$

where

σ_{ik} is the mechanical stress tensor,

T_{ik} is the tensor of the Maxwellian stresses in the body,

$\bar{T}_{ik}$ is the tensor of the Maxwellian stresses in the vacuum.

Using the relation $u_1 = u_2 = 0$ and expressing the tensor difference of the Maxwellian stresses $T_{33} - \bar{T}_{33}$ in terms of the Lorentz forces [1] by means of the vector $\mathbf{u}$ and the surface currents [2], we obtain

$$\begin{aligned} a^2 \frac{\partial u}{\partial z} &= -\frac{B}{4\pi\rho} \bar{h}_1 - \frac{1}{\rho} P(t), \\ -\frac{\partial \bar{h}_1}{\partial z_1} &+ \frac{B}{c^2} \frac{\partial^2 u}{\partial t^2} = 0. \end{aligned} \tag{2.10}$$

Thus, finally, we use for the determination of the solution the boundary conditions (2.10) and the equations for u and $\bar{h}_1$ in the form

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= a^2 \frac{\partial^2 u}{\partial z^2}, \\ \frac{\partial^2 \bar{h}_1}{\partial z_1^2} &- \frac{1}{c^2} \frac{\partial^2 \bar{h}_1}{\partial t^2} = 0. \end{aligned} \tag{2.11}$$

The quantities $\bar{h}_2$, $\bar{h}_3$ do not enter into the solution sought for u in the medium and may be disregarded. The quantities $\bar{E}_i$ will be determined on

the basis of (2.2) by means of $\bar{h}_1$. The quantities h_i, E_i will be determined according to (2.3) in terms of u. We will proceed to the solution of the system of equations (2.11) with the boundary conditions (2.10).

The solution of the system of equations (2.11) will be sought in the form

$$\begin{aligned} \bar{h} &= \psi(z_1 - ct), \\ u &= \Phi(z - at), \end{aligned} \tag{2.12}$$

where, for the sake of brevity, $\bar{h}_1 = \bar{h}$.

Substituting (2.12) in (2.10) we have

$$\begin{aligned} a^2\Phi'(-at) &= -\frac{B}{4\pi\rho}\psi(-ct) - \frac{1}{\rho}P(t), \\ -\psi'(-ct) &+ \frac{Ba^2}{c^2}\Phi''(-at) = 0. \end{aligned} \tag{2.13}$$

Letting

$$-at = \zeta,$$

we find

$$\begin{aligned} a^2\Phi'(\zeta) &= -\frac{B}{4\pi\rho}\psi\left(\frac{c}{a}\zeta\right) - \frac{1}{\rho}P\left(-\frac{\zeta}{a}\right), \\ -\psi'\left(\frac{c}{a}\zeta\right) &+ \frac{Ba^2}{c^2}\Phi''(\zeta) = 0, \end{aligned} \tag{2.14}$$

or differentiating the first of the equations (2.14) and obtaining from the second $\Phi''(\zeta)$ in the form

$$\Phi''(\zeta) = \frac{c^2}{a^2B}\psi'\left(\frac{c}{a}\zeta\right), \tag{2.15}$$

we obtain, on substituting (2.15) into the first of the equations (2.14) differentiated with respect to ζ,

$$\left(\frac{Bc}{4\pi\rho a} + \frac{c^2}{B}\right)\psi'\left(\frac{c}{a}\zeta\right) - \frac{1}{\rho a}P'\left(-\frac{\zeta}{a}\right) = 0. \tag{2.16}$$

Letting

$$\frac{c}{a}\zeta = \eta = -ct,$$

we have, after integrating equation (2.16),

$$\psi(\eta) = -\frac{1}{\dfrac{B}{4\pi} + \dfrac{\rho ac}{B}}P\left(t - \frac{z_1}{c}\right) + A. \tag{2.17}$$

Hence, on the basis of (2.13), we find after some manipulations

$$\Phi(\zeta) = -\frac{1}{\rho a^2 + \frac{B^2}{4\pi}\frac{a}{c}} \int_0^{\zeta} P\left(-\frac{\zeta_1}{a}\right) d\zeta_1 + A\zeta + D. \tag{2.18}$$

From the condition $z = 0$, $t = 0$, i.e., $\zeta = 0$: $u = 0$, we have $D = 0$. From the condition $z = 0$, $t = 0$ with $P(0) = 0$: $\bar{h} = 0$, i.e., $\psi(0) = 0$, we obtain $A = 0$.

Thus, finally, reverting to the initial variables, we have

$$\bar{h} = \psi(z_1 - ct) = -\frac{1}{\frac{B}{4\pi} + \frac{\rho a c}{B}} P\left(t - \frac{z_1}{c}\right),$$

$$u = \Phi(z - at) = -\frac{1}{\rho a^2 + \frac{B^2}{4\pi}\frac{a}{c}} R(z - at), \tag{2.19}$$

where

$$R(z - at) = \int_0^{z-at} P\left(-\frac{\zeta_1}{a}\right) d\zeta_1.$$

We observe that if $B = 0$ then $\bar{h} = 0$, and u becomes the classical solution for the propagation of elastic waves in the half-space. On the basis of (2.3), we also have

$$h = -\frac{\partial u}{\partial z} H, \quad E_2 = -\frac{\mu}{c}\frac{\partial u}{\partial t} H. \tag{2.20}$$

§ 3. Propagation of a non-linear loading wave in a magnetic field for angled σ-ε characteristics

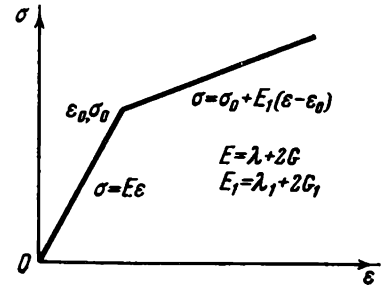

Fig. 2.

Let us proceed to the construction of the solution for loading wave propagation in an elasto-plastic body in a magnetic field. We shall assume first an angled $\sigma-\varepsilon$ characteristic (Fig. 2).

The solution in the phase plane will be described for each of the three regions of Fig. 3.

The solution in zone I, i.e., for $\sigma < \sigma_0$ and $t < t_0 + z/a$, coincides with that for the elastic body and is given by the equations (2.19).

The solution for each particular zone will be denoted in the following by the subcripts I, II, III.

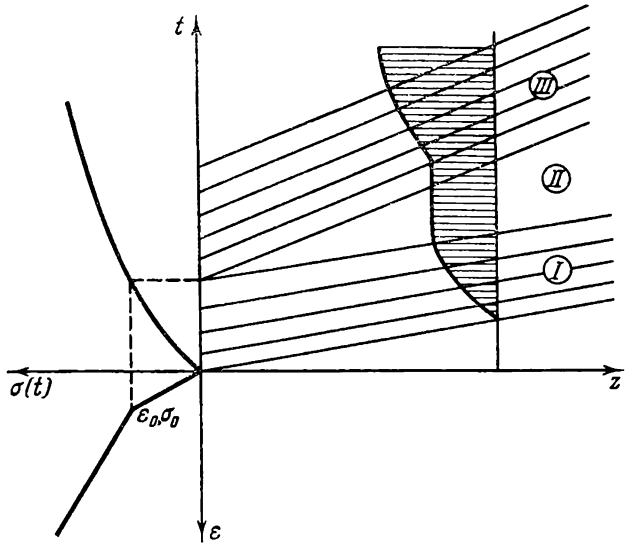

Fig. 3.

In the zone II, i.e., for

$$t_0 + \frac{z}{a} \leq t \leq t_0 + \frac{z}{a_1}, \tag{3.1}$$

where

$$a_1 = \sqrt{\frac{\lambda_1 + 2G_1 + \kappa}{\rho}},$$

the solution is obtained by starting, as before, from the equations (2.11) and from the boundary conditions (2.10). The load $P(t)$ for $t > t_0$ is $\sigma_0 = P(t_0)$. The excess pressure $P(t) > \sigma_0$ appears first in zone III.

Then, on the basis of (2.19), we obtain after integration

$$\bar{h}_{II} = \frac{\sigma_0}{\frac{B}{4\pi} + \frac{\rho a c}{B}},$$

$$u_{II} = \frac{1}{\rho a^2 + \frac{B^2}{4\pi}\frac{a}{c}} \{R_0(t_0) - \sigma_0[a(t - t_0) - z]\}, \tag{3.2}$$

where

$$R_0(t_0) = R(-at_0) - R(0) \tag{3.3}$$

and, according to assumption, $R(0) = 0$, which, of course, is not essential.

Let us pass now to the solution in zone III, i.e., for

$$t > t_0 + \frac{z}{a_1}. \tag{3.4}$$

The solution in zone III is a superposition of the solutions $\bar{h}_{II}$ and u_{II} extended into zone III and the residues

$$\begin{aligned} u_r &= u_{III} - u_{II}, \\ \bar{h}_r &= \bar{h}_{III} - \bar{h}_{II}, \end{aligned} \tag{3.5}$$

described by equations of the type (2.11) and conditions of the type (2.10). For, in the case of angled $\sigma-\varepsilon$ characteristic and a monotonically increasing load, we can use, in a manner analogous to the case of purely mechanical perturbations, the principle of superposition of solutions. The equations of motion in a magnetic field and an electro-magnetic field in the vacuum for u_r and $\bar{h}_r$ in zone III will have the form

$$\begin{aligned} a_1^2 \frac{\partial^2 u_r}{\partial z^2} - \frac{\partial^2 u_r}{\partial t^2} &= 0, \\ c^2 \frac{\partial^2 \bar{h}_r}{\partial z_1^2} - \frac{\partial^2 \bar{h}_r}{\partial t^2} &= 0. \end{aligned} \tag{3.6}$$

The boundary conditions will be as follows: for $t \geq t_0$, $z = z_1 = 0$,

$$\begin{aligned} \rho a_1^2 \frac{\partial u_r}{\partial z} &= -\frac{B}{4\pi} \bar{h}_r - (P - \sigma_0), \\ -\frac{\partial \bar{h}_r}{\partial z_1} + \frac{B^2}{c^2} \frac{\partial^2 u_r}{\partial t^2} &= 0. \end{aligned} \tag{3.7}$$

The solution of the system of equations (3.6) with the boundary conditions (3.7) will be sought in the form

$$\begin{aligned} \bar{h}_r &= \psi[z_1 - c(t - t_0)], \\ u_r &= \Phi[z - a_1(t - t_0)]. \end{aligned} \tag{3.8}$$

Substituting (3.8) into (3.7), we obtain

$$\begin{aligned} \rho a_1^2 \Phi'[-a_1(t - t_0)] &= -\frac{B}{4\pi} \psi[-c(t - t_0)] - P(t) + \sigma_0, \\ -\psi'[-c(t - t_0)] + \frac{B a_1^2}{c^2} \Phi''[-a_1(t - t_0)] &= 0. \end{aligned} \tag{3.9}$$

Letting

$$-a_1(t-t_0)=\zeta \tag{3.10}$$

in these expressions, we find

$$\begin{aligned}\rho a_1^2\Phi'(\zeta) &= -\frac{B}{4\pi}\psi\left(\frac{c}{a_1}\zeta\right)-P_1\left(-\frac{\zeta}{a_1}\right),\\ -\psi'\left(\frac{c}{a_1}\zeta\right)&+\frac{Ba_1^2}{c^2}\Phi''(\zeta)=0,\end{aligned} \tag{3.11}$$

where

$$P(t)-\sigma_0=P_1(t-t_0). \tag{3.12}$$

The transformation (3.12) is, of course, always possible. Hence, bearing in mind that

$$\begin{aligned}\Phi''(\zeta) &= \frac{c^2}{Ba_1^2}\psi'\left(\frac{c}{a_1}\zeta\right),\\ \left[\frac{Bc}{4\pi a_1}+\frac{\rho c^2}{B}\right]&\psi'\left(\frac{c}{a_1}\zeta\right)=\frac{1}{a_1}P_1'\left(-\frac{\zeta}{a_1}\right),\end{aligned} \tag{3.13}$$

we find

$$\psi\left(\frac{c}{a}\zeta\right)=-\frac{4\pi B}{B^2+4\pi c\rho a_1}P_1\left(-\frac{\zeta}{a_1}\right)+A, \tag{3.14}$$

$$\Phi(\zeta)=-\frac{1}{\rho a_1^2+\dfrac{B^2a_1}{4\pi c}}\int_0^{\zeta}P_1\left(-\frac{\zeta_1}{a_1}\right)d\zeta_1+A\zeta+D. \tag{3.15}$$

From the condition $u_r=\Phi(0)=0$ for $z=0$, $t=t_0$, i.e., $\zeta=0$, it follows that $D=0$.

From the condition $\bar{h}_r=\psi(c/a_1 0)=0$ and $P_1(0)=0$ for $z_1=0$, $t=t_0$, i.e., $\zeta=0$, it follows that $A=0$.

Thus, finally, we have

$$\bar{h}_r=\psi[z_1-c(t-t_0)]=-\frac{1}{\dfrac{B}{4\pi}+\dfrac{\rho c a_1}{B}}P_1\left[t-t_0-\frac{z_1}{c}\right], \tag{3.16}$$

$$u_r=\Phi[z-a_1(t-t_0)]=-\frac{1}{\rho a_1^2+\dfrac{B^2}{4\pi}\dfrac{a_1}{c}}R_1[z-a_1(t-t_0)], \tag{3.17}$$

where

$$R_1[z - a_1(t - t_0)] = \int_0^{z-a_1(t-t_0)} P_1\left(-\frac{\zeta_1}{a_1}\right) d\zeta. \tag{3.18}$$

Using the equations (3.5), (3.2) and (3.16), (3.17), we find the solution sought for zone III:

$$\bar{h}_{III} = \bar{h}_{II} + \bar{h}_{II} = \tag{3.19}$$

$$= -\frac{1}{\frac{B}{4\pi} + \frac{\rho c a_1}{B}} P_1\left[t - t_0 - \frac{z_1}{c}\right] - \frac{\sigma_0}{\frac{B}{4\pi} + \frac{\rho c a}{B}},$$

$$u_{III} = u_{II} + u_{II} = -\frac{1}{\rho a_1^2 + \frac{B^2}{4\pi}\frac{a_1}{c}} R_1[z - a_1(t - t_0)] - \tag{3.20}$$

$$-\frac{1}{\rho a^2 + \frac{B^2}{4\pi}\frac{a}{c}} \{R(-at_0) - \sigma_0[a(t - t_0) - z]\}.$$

The values of the electro-magnetic field in the body h_{III}, E_{2III} are computed from equations identical to (2.20). If $B = 0$, the solution (3.20) coincides with that for the loading wave in a body with angled $\sigma-\varepsilon$ characteristic in the pure mechanical case. The quantities v, ε, σ will be found from (3.20) by differentiation.

§ 4. Approximate solution for a non-linear unloading wave in a magnetic field for any σ-ε characteristic

In the case of any (convex) $\sigma-\varepsilon$ characteristic (Fig. 4) the problem becomes considerably more complicated.

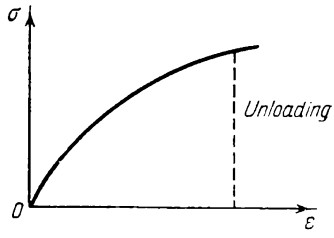

Fig. 4.

Using the equations of the fields in the body and the vacuum in the form (2.11), where

$$a = \sqrt{\frac{E(\varepsilon) + \kappa}{\rho}}, \tag{4.1}$$

and the boundary conditions (2.10), we obtain on the basis of the first of the

boundary conditions (2.10), seeking the solution for $\bar{h}$ in the form

$$\bar{h} = \psi(z_1 - ct), \tag{4.2}$$

the following expression for $u(t, 0)$

$$u(t) = \int_0^{at} \varepsilon(t, z)\, dz = \int_0^{at} \frac{1}{a^2}\, \Phi\left(t - \frac{z}{a}\right) dz, \tag{4.3}$$

where

$$\Phi(t) = -\frac{B}{4\pi\rho}\, \psi(-ct) - \frac{1}{\rho} P(t),$$

$$a = f(\varepsilon) = f\left(\frac{\partial u}{\partial z}\right) = f\left[\frac{1}{a^2}\, \Phi\left(t - \frac{z}{a}\right)\right], \tag{4.4}$$

$$a_0 = f(\varepsilon)|_{\varepsilon=0}.$$

Substituting (4.3) into the second of the boundary conditions (2.10), we obtain the equation

$$\psi'(-ct) = \frac{B^2}{c}\, \frac{\partial^2}{\partial t^2} \int_0^{at} \frac{1}{a^2}\, \Phi\left(t - \frac{z}{a}\right) dz, \tag{4.5}$$

whence the function ψ, and then ε, $a(\varepsilon)$, etc., can be obtained.

The equation (4.5) is a non-linear functional integro-differential equation, the numerical solution of which is very complicated. This difficulty may be avoided by assuming the σ—ε curve in the form of a polygonal line. Then the solution will be reduced to a repeated application of the formulae of the preceding section. However, the problem may be simplified by discussing the terms in the boundary conditions (2.10).

Let us observe that in the second of the boundary conditions (2.10) the coefficient of $\partial^2 u/\partial t^2$, i.e., B/c^2, is a very small quantity. If we assume that $B/c^2 \approx 0$, we get $\partial\bar{h}/\partial z_1 = 0$, and hence $\bar{h} = \bar{h}(t)$. In particular, one may have $\bar{h} = 0$ [if $P(0) = 0$].

Then the solution will coincide with that of the simple boundary-value problem with $\varepsilon(0, t)$ given at the edge. The solution will be obtained, using the characteristics in the same manner as for the classical problem of the loading wave.

To obtain a more accurate solution, we can also use the iteration method for the boundary conditions (2.10). We can construct a solution by disregarding $\bar{h}$ in the first boundary condition, then determining $\bar{h}$ from the second boundary condition and, finally, constructing the second

solution on the basis of the first boundary condition, etc. In general, as a result of the smallness of B/c^2, we obtain, after the first approximation, very good results equivalent to the representation of the solution in the form of a Taylor series as a function of a small parameter and to the retainment of the first term only. This may be illustrated in a very clear manner by the example of a linear $\sigma-\varepsilon$ characteristic. The accurate solution has in this case the form determined by (2.19).

This solution will be constructed by the method of successive approximations for the boundary conditions (2.10). Assuming $\bar{h} = 0$ for the first approximation we obtain for u the following equation and boundary condition:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial z^2}, \tag{4.6}$$

$$a^2 \frac{\partial u}{dz} = -\frac{P(t)}{\rho} \quad \text{for } z = 0. \tag{4.7}$$

Hence,

$$u = -\frac{1}{\rho a^2} \int_0^{z-at} P\left(-\frac{\zeta}{a}\right) d\zeta. \tag{4.8}$$

Substituting (4.8) into the second of the boundary conditions (2.10), we obtain

$$\frac{\partial \bar{h}}{\partial z_1} = -\frac{1}{\rho a^2} P'\left(t - \frac{z}{a}\right) \frac{Ba}{c^2} = -\frac{Ba}{c^2} \frac{1}{\rho a^2} P'(t). \tag{4.9}$$

Hence we find

$$\bar{h} = -\frac{Ba}{c} \frac{1}{\rho a^2} P\left(t - \frac{z_1}{c}\right) + E. \tag{4.10}$$

Assuming for the sake of definiteness $P(0) = 0$, we obtain $E = 0$. Substituting next (4.10) into the first of the boundary conditions (2.10), we obtain

$$a^2 \frac{\partial u}{\partial z} = -\frac{1}{\rho} P(t) + \frac{B^2 a}{4\pi\rho c} \frac{1}{\rho a^2} P(t). \tag{4.11}$$

Hence

$$u = -\frac{1}{\rho a^2} \int_0^{z-at} P\left(-\frac{\zeta}{a}\right) d\zeta \left[1 - \frac{B^2 a}{4\pi c} \frac{1}{\rho a^2}\right]. \tag{4.12}$$

It is seen from the equation (4.12) that this solution possesses an expansion in Taylor series of the solution (2.19) for u with a degree of accuracy of the first term in the small parameter $(B^2/4\pi)(a/c)$. The solution (4.10) for $\bar{h}$ is obtained from (2.19) by rejecting the small term $B/4\pi$ in the denominator. From the second iteration we should obtain further terms of the Taylor expansion, the second for u, the first for h.

In practical cases, since $a/c \ll 1$, it will be sufficient to stop at the first approximation above. The orders of magnitude will be similar in the non-linear case, where, proceeding as before, the first iteration gives highly accurate results. The values of $\bar{h}$ and E are computed from (2.20).

Finally, let us observe that the above assumption $P(0) = 0$ is not essential and that the considerations may be transferred without difficulty to the case of pressure which increases monotonically and appears abruptly.

§ 5. Example

To illustrate the solutions obtained, let us give a simple example, the case of an angled $\sigma-\varepsilon$ characteristic. The notation is explained in Fig. 2. The external load is assumed to vary linearly with the time, i.e.,

$$P(t) = P_0 t. \tag{5.1}$$

Then, by the equations (2.19) and (2.20), we obtain in zone I:

$$\bar{h}_{\mathrm{I}} = -\frac{P_0}{\dfrac{B}{4\pi} + \dfrac{\rho c a}{B}}\left(t - \frac{z_1}{c}\right), \tag{5.2}$$

$$u_{\mathrm{I}} = \frac{P_0}{\rho a^2 + \dfrac{B^2}{4\pi}\dfrac{a}{c}} \cdot \frac{(z - at)^2}{2a}, \tag{5.3}$$

$$h_{\mathrm{I}} = P_0 H \cdot \frac{at - z}{\rho a^2 + \dfrac{B^2}{4\pi}\dfrac{a}{c}} \cdot \frac{1}{a}, \tag{5.4}$$

$$E_{2\mathrm{I}} = -\frac{\mu H P_0}{c\left[\rho a^2 + \dfrac{B^2 a}{4\pi c}\right]}(at - z); \tag{5.5}$$

in zone II:

$$\bar{h}_{II} = -\frac{1}{\frac{B}{4\pi}+\frac{\rho ac}{B}} P_0 t_0 = \frac{\sigma_0}{\frac{B}{4\pi}+\frac{\rho ac}{B}}, \tag{5.6}$$

$$u_{II} = \frac{\sigma_0}{\rho a^2 + \frac{B^2}{4\pi}\frac{c}{a}}\left\{\frac{at_0}{2} + [a(t-t_0)-z]\right\}, \tag{5.7}$$

$$h_{II} = \frac{\sigma_0 H}{\rho a^2 + \frac{B^2}{4\pi}\frac{a}{c}}, \tag{5.8}$$

$$E_{2II} = -\frac{\mu a \sigma_0 H}{c\left(\rho a^2 + \frac{B^2}{4\pi}\frac{a}{c}\right)}; \tag{5.9}$$

in zone III:

$$\bar{h}_{III} = -\frac{P_0}{\frac{B}{4\pi}+\frac{\rho c a_1}{B}}\left[t - t_0 - \frac{z_1}{c}\right] - \frac{\sigma_0}{\frac{B}{4\pi}+\frac{\rho ca}{B}}, \tag{5.10}$$

$$u_{III} = \frac{P_0}{\rho a_1^2 + \frac{B^2}{4\pi}\frac{a_1}{c}}[z - a_1(t-t_0)]^2 \frac{1}{2a_1} + \frac{1}{\rho a^2 + \frac{B^2}{4\pi}\frac{a}{c}}\left\{\frac{at_0^2}{2} + t_0[a(t-t_0)-z]\right\}, \tag{5.11}$$

$$h_{III} = -P_0 H\left[\frac{1}{\rho a_1^2 + \frac{B^2}{4\pi}\frac{a_1}{c}}\frac{z - a_1(t-t_0)}{a_1} - \frac{t_0}{\rho a^2 + \frac{B^2}{4\pi}\frac{a}{c}}\right], \tag{5.12}$$

$$E_{2III} = -\frac{\mu P_0 H}{c\left[\rho a_1^2 + \frac{B^2}{4\pi}\frac{a_1}{c}\right]}[a_1(t-t_0) - z] - \frac{\mu a \sigma_0 H}{c\left[\rho a^2 + \frac{B^2}{4\pi}\frac{a}{c}\right]}. \tag{5.13}$$

In the case $a = a_1$, the solutions tend to those for the elastic case (2.19).

If $B \to 0$, the solution tends to the known solution for a loading wave for an angled $\sigma-\varepsilon$ characteristic. It is also easy to observe that in the front of the wave the momentum conditions are satisfied in their classical form. Similarly, we can construct a solution for the present example for any (convex) curvilinear $\sigma-\varepsilon$ characteristic on the basis of the method of approximations to the boundary conditions of the preceding section. We shall confine ourselves to the above simple example which illustrates the orders of magnitude of the quantities involved.

§ 6. The unloading wave for a body with rigid unloading characteristic

We will consider only the idea of the problem of the unloading wave, taking into account qualitatively the body with rigid unloading characteristic only.

Assuming in the case of a body with rigid unloading characteristic (Fig. 4) the equation of the unloading wave in the form

$$x = \varphi(t), \tag{6.1}$$

we can write down the equation of motion of the body

$$\frac{d}{dt}\left(\rho\varphi(t)v_2 + g\right) = P_2(t) - \bar{T}_0 \tag{6.2}$$

(in this process the body behaves like a rigid body with initial permanent strain),

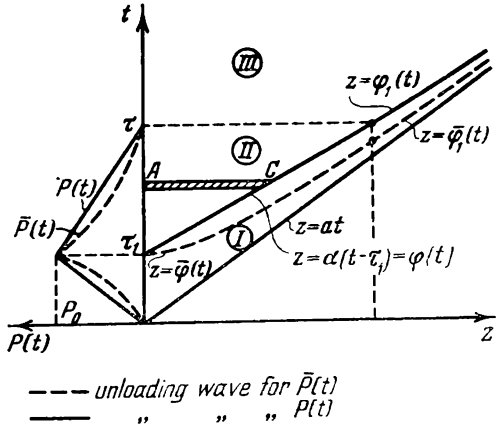

Fig. 5.

where g is the momentum of the magnetic field

$$g = \frac{1}{4\pi c} \int_V \bar{E} H \, dV$$

and v_2 is the velocity of the rigid body in zone II (Fig. 5). For $z = z_1 = 0$, the following condition must be satisfied:

$$\sigma_0 + T_0 - \bar{T}_0 = - P_2(t). \tag{6.3}$$

Neglecting the small quantity g, equation (6.2) may be rewritten in the form

$$\rho\varphi(t) \frac{dv_2}{dt} = P_2(t) + \frac{B\bar{h}_0}{4\pi} + \sigma_c + T_c, \tag{6.4}$$

where σ_c and T_c are the mechanical stress and the Maxwell stress respectively at the wave front.

The initial conditions are

$$v_1(0, \tau_1) = v_2(\tau_1). \tag{6.5}$$

The form of the unloading wave will be calculated from the condition of continuity of the velocity at the unloading wave, i.e.,

$$v_1(\varphi(t), t) = v_2(\varphi(t), t) = v_c(t). \tag{6.6}$$

In the case of rectilinear form of the σ—ε curve in the loading-zone (Fig. 6), the unloading wave for $t < \tau$ will also be rectilinear (the load increasing and decreasing linearly).

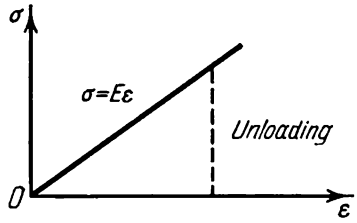

Fig. 6.

If the load (on the surface) appears suddenly and then decreases monotonically [5], the unloading wave, for $t < \tau$, will take the form

$$x = at \tag{6.7}$$

and the equation of motion in zone II will become

$$\rho at \frac{dv_2}{dt} + \rho a v_2 = P^*(t), \tag{6.8}$$

where

$$P^*(t) = P(t) + \frac{B}{4\pi} \bar{h}(t). \tag{6.9}$$

Integrating (6.7) we have

$$v_2(t) = \frac{1}{t\rho a}\int_0^t P^*(t_1)\,dt_1 + \frac{C}{t}.$$

It is easily shown that

$$C \equiv 0.$$

The function $\bar{h}(t)$ included in P^* can be determined from the second boundary condition (2.10), assuming the form of $\bar{h}$ to be

$$\bar{h} = \psi(z_1 - ct). \tag{6.10}$$

In zone III, the equation of the problem will be similar, except that $P(t) \equiv 0$ and the unloading wave does not remain rectilinear [5]. The magnitudes σ, ε in the zones under consideration may be found as in [5].

As has been mentioned above, because of its complexity, the problem of the unloading wave will not be discussed here in detail; it will be discussed in a separate paper.

§ 7. Conclusion

The problem of the propagation of a non-linear loading wave, and, briefly, the problem of the unloading wave for a body with rigid unloading characteristic in a magnetic field has been considered for a perfect conductor in a half-space adjacent to a vacuum. This problem is one of the simplest problems in the propagation of nonlinear waves in magnetic fields.

It seems that the above problem in connection with the possibility of recording strong mechanical perturbations by means of secondary electromagnetic phenomena has some practical aspects.

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AN INFINITE ROW OF PARALLEL CRACKS IN AN INFINITE ELASTIC SHEET

W. T. KOITER
Delft, Holland

§ 1. Introduction

The problem of the increased shear flexibility of an elastic sheet, due to a row of rectangular cutouts, is of primary importance in the investigation of the vibration characteristics of railway carriages. In an earlier paper [1] a limiting case was discussed of an infinite sheet, weakened by a row of collinear cracks, and the solution of this problem was obtained in a simple closed form. The present paper deals with the other limiting case, i.e., *a row of parallel cracks, perpendicular to the direction connecting homologous points on the various cracks*. The two limiting cases are intended as checks on an approximate method for arbitrary dimensions of the rectangular holes, which is now under development in the author's laboratory. [1]

Our present problem does not admit as simple a solution as the problem of a row of collinear cracks. In fact, the general solution would require for its evaluation a numerical solution of an integral equation. However, for our purpose it is sufficient to obtain an approximate solution provided its accuracy is adequate. This object is achieved by considering two special cases, i.e., the case of *widely spaced cracks* ($b/c \to \infty$, where $2b$ is the pitch between cracks and $2c$ is the crack width) and the case of *closely spaced cracks* ($b/c \to 0$). It will appear from our results that the asymptotic behaviour at these two limits specifies the behaviour for intermediate values of b/c with adequate accuracy.

The approximate solution for widely spaced cracks is obtained from Muskhelishvili's well-known solution for a single crack [5, § 120]. The approximate solution for closely spaced cracks is obtained by Fourier-transforms and the Wiener-Hopf technique; the latter solution closely resembles a similar solution for closely spaced cracks normal to the boundary of a half-plane [6]. It is felt that this combination of Mus-

[1] The problem of an infinite row of congruent holes may also be formulated in terms of an integral equation for one of the complex stress functions of Kolosov-Muskhelishvili [2, 3, 4], but the numerical solution of this equation is rather laborious.

khelishvili's powerful methods in two-dimensional elasticity with the equally powerful method of Fourier-transforms may be a particularly appropriate subject for a paper in a volume in honor of the famous scientist Professor N. I. Muskhelishvili.

§ 2. Approximate solution for widely spaced cracks

The average displacements u, v in the problem of generalized plane stress in a sheet, i.e., the mean values of the displacements over the sheet thickness, can always be expressed by means of two analytic functions $\varphi(z)$, $\psi(z)$ of the complex variable $z = x + iy$, the complex stress functions of Kolosov-Muskhelishvili [5, § 32]:

$$2G(u + iv) = \kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}. \tag{2.1}$$

A dash denotes differentiation with respect to z, and a bar denotes that the conjugate value of the barred function is to be taken. The number κ is given by

$$\kappa = \frac{3 - \nu}{1 + \nu}, \tag{2.2}$$

where ν is Poisson's ratio. Finally, G denotes the shear modulus. The average stresses, again defined as mean values over the sheet thickness, are expressed by derivatives of $\varphi(z)$ and $\psi(z)$

$$\sigma_x + \sigma_y = 2[\varphi'(z) + \overline{\varphi'(z)}], \tag{2.3}$$

$$\sigma_y + i\tau_{xy} = \varphi'(z) + \overline{\varphi'(z)} + \bar{z}\varphi''(z) + \psi'(z). \tag{2.4}$$

We first consider the case of a *single crack* along the x-axis, from $z = -c$ to $z = +c$. We restrict our attention to the case of a *uniform shear stress* τ *at infinity*. The stress functions $\varphi(z)$, $\psi(z)$ are now given by (cf. [5, § 120])

$$\varphi(z) = -\tfrac{1}{2}i\tau[\sqrt{z^2 - c^2} - z], \tag{2.5}$$

$$\psi(z) = \tfrac{1}{2}i\tau\left[\frac{c^2}{\sqrt{z^2 - c^2}} + 2\sqrt{z^2 - c^2}\right], \tag{2.6}$$

where the positive branch of the square root is to be taken on the positive x-axis: $x > c$. This solution may be considered as a superposition of a uniform shear stress τ in the entire sheet, described by

$$\varphi_0(z) = 0, \quad \psi_0(z) = i\tau z, \tag{2.7}$$

and a correction, due to the removal of shear stresses τ along the edges of

the crack $-c < x < c$, described by

$$\varphi^*(z) = -\tfrac{1}{2}i\tau[\sqrt{z^2 - c^2} - z], \tag{2.8}$$

$$\psi^*(z) = \tfrac{1}{2}i\tau\left[\frac{c^2}{\sqrt{z^2 - c^2}} + 2\sqrt{z^2 - c^2} - 2z\right]. \tag{2.9}$$

The discontinuity in the displacements across the cut is obtained by substituting (2.8) into (2.1). We obtain for $-c < x < c$

$$2G[u_+ - u_- + i(v_+ - v_-)] = (\kappa + 1)\tau\sqrt{c^2 - x^2}, \tag{2.10}$$

and the increase in elastic strain energy, due to the crack, per unit thickness of the sheet is

$$\Delta A = \int_{-c}^{c} \tfrac{1}{2}\tau(u_+ - u_-)\,dx = \frac{\pi}{8}(\kappa + 1)\frac{\tau^2 c^2}{G}. \tag{2.11}$$

We now consider a sheet with a row of cracks parallel to the x-axis, spaced at z_0, where $|z_0| > 2c$. The cracks are then described by

$$-c < z + kz_0 < +c, \tag{2.12}$$

where k is integral and runs from $-\infty$ to ∞. We now assume stress functions $\varphi^*(z)$, $\psi^*(z)$, to be superimposed on the functions (2.7) for uniform shear in the sheet, in the form

$$\varphi^*(z) = -\tfrac{1}{2}i\tau \sum_{k=-\infty}^{\infty} [\sqrt{(z + kz_0)^2 - c^2} - (z + kz_0)], \tag{2.13}$$

$$\psi^*(z) = \tfrac{1}{2}i\tau \sum_{k=-\infty}^{\infty} \left[\frac{c^2}{\sqrt{(z+kz_0)^2 - c^2}} + 2\sqrt{(z+kz_0)^2 - c^2} - 2(z+kz_0)\right]. \tag{2.14}$$

It is easily seen that these series, interpreted as principal values, are absolutely and uniformly convergent, except at the end points of the cracks. The stresses are now given by

$$\sigma_x + \sigma_y = -i\tau \sum_{k=-\infty}^{\infty} \left[\frac{z + kz_0}{\sqrt{(z + kz_0)^2 - c^2}} - \frac{\bar{z} + k\bar{z}_0}{\sqrt{(\bar{z} + k\bar{z}_0)^2 - c^2}}\right], \tag{2.15}$$

$$\sigma_y + i\tau_{xy} = \tfrac{1}{2}i\tau \sum_{k=-\infty}^{\infty} \left[-\frac{z + kz_0}{\sqrt{(z+kz_0)^2 - c^2}} + \frac{\bar{z} + k\bar{z}_0}{\sqrt{(\bar{z}+k\bar{z}_0)^2 - c^2}} - \right.$$
$$\left. - \frac{c^2\{z - \bar{z} + k(z_0 - \bar{z}_0)\}}{\{(z + kz_0)^2 - c^2\}^{\frac{3}{2}}} + 2\left\{\frac{z + kz_0}{\sqrt{(z + kz_0)^2 - c^2}} - 1\right\}\right]. \tag{2.16}$$

These series are absolutely and uniformly convergent. It is also easily seen that the stresses are periodic with a period z_0. Likewise, the discontinuity of displacements across a crack is the same for all cracks and is given by (2.10).

In order to evaluate (2.16) along a crack, we remember that the pitch of the cracks $|z_0| = 2b$ is larger than the crack width $2c$. Taking $|z| < c$, we may expand each term of (2.16) for $k \neq 0$ in an absolutely and uniformly convergent series, e.g.,

$$(2.17)\quad \frac{z + kz_0}{\sqrt{(z + kz_0)^2 - c^2}} = \frac{1}{\sqrt{1 - c^2(z + kz_0)^{-2}}} =$$

$$= 1 + \frac{1}{2}\frac{c^2}{(z + kz_0)^2} + \frac{3}{4}\frac{c^4}{(z + kz_0)^4} + \cdots =$$

$$= 1 + \frac{1}{2}\frac{c^2}{z_0^2}\frac{1}{k^2}\left[1 - \frac{z}{kz_0} + \frac{z^2}{k^2 z_0^2} - \cdots\right]^2 +$$

$$+ \frac{3}{4}\frac{c^4}{(z + kz_0)^4} + \cdots.$$

Applying a similar process to each term in (2.16), we obtain after some elementary algebra for $|z| < c$

$$(2.18)\quad \sigma_y + i\tau_{xy} = \tfrac{1}{2}i\tau\left[-2 + \frac{z}{\sqrt{z^2 - c^2}} + \frac{\bar{z}}{\sqrt{\bar{z}^2 - c^2}} + \frac{c^2(\bar{z} - z)}{(z^2 - c^2)^{\frac{3}{2}}} + \right.$$

$$\left. + \left\{-\frac{c^2}{z_0^2} + \frac{c^2}{\bar{z}_0^2} + 2\frac{c^2\bar{z}_0}{z_0^3}\right\}\sum_{k=1}^{\infty}\frac{1}{k^2} + O\left(\frac{c^4}{b^4}\right)\right],$$

where the last term indicates that the remainder of the series is of order c^4/b^4. Since

$$(2.19)\quad \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6},$$

the stresses along the crack $-c < x < c$ are now given by

$$(2.20)\quad \sigma_y + i\tau_{xy} = -i\tau\left[1 + \frac{\pi^2}{12}\left\{\frac{c^2}{z_0^2} - \frac{c^2}{\bar{z}_0^2} - 2\frac{c^2\bar{z}_0}{z_0^3}\right\} + O\left(\frac{c^4}{b^4}\right)\right].$$

In the case of *collinear cracks*, where $z_0 = 2b$, we obtain

$$(2.21)\quad \sigma_y + i\tau_{xy} = -i\tau\left[1 - \frac{\pi^2 c^2}{24b^2} + O\left(\frac{c^4}{b^4}\right)\right],$$

i.e., with an error of order c^4/b^4 only a *constant shear stress*. In order to reduce the shear stress along the crack to zero, the stress functions (2.13), (2.14) should in this approximation be multiplied by $(1 + \pi^2c^2/24b^2)$, and the corresponding discontinuity in the displacements is given by (2.10), multiplied by the same factor. Hence the *increase in elastic strain energy per crack*, and per unit thickness of the sheet, is given by (2.11), again multiplied by the same factor

$$\Delta A = \frac{\pi}{8}(\kappa + 1)\frac{\tau^2c^2}{G}\left(1 + \frac{\pi^2c^2}{24b^2}\right) = \tag{2.22}$$

$$= \pi\frac{\tau^2c^2}{E}\left(1 + \frac{\pi^2c^2}{24b^2}\right),$$

where E denotes Young's modulus, and the relative error of this formula is of the order c^4/b^4. This result is in agreement with the rigorous result for collinear cracks [1]

$$\Delta A = -\frac{\kappa + 1}{\pi}\frac{\tau^2b^2}{G}\log\left[\cos\frac{\pi c}{2b}\right], \tag{2.23}$$

if this general result is expanded in a series with respect to c/b.

In the case of *cracks perpendicular to the direction connecting homologous points on the various cracks* we have $z_0 = 2ib$, and we obtain for the stresses along the crack again (2.21). The increase in elastic strain energy per crack is therefore again given by (2.22). *For widely spaced cracks the increase in strain energy per crack is*, with an error of order c^4/b^4, *the same for collinear cracks and for cracks perpendicular to the direction of periodicity.*

§ 3. Approximate solution for closely spaced cracks

In the case of closely spaced cracks, perpendicular to the direction connecting homologous points on the various cracks, it is convenient to choose coordinates as in Fig. 1. For reasons of symmetry no normal stresses σ_y and no extensions ε_x will occur along the lines $y = (2k + 1)b$ for k any positive or negative integer, and $x < 0$ or $x > 2c$. Likewise, no normal stresses σ_x will occur along the line $x = c$. We can restrict our attention to a strip $-b < y < b$ indicated in Fig. 2, and we have to *superimpose the stress distribution in this strip, due to the constant tangential loads τ in the range $0 < x < 2c$, on the uniform shear stress distribution in the uncut sheet.*

The loads $t(x)$ for $x < 0$ and $x > 2c$ are to be determined from the

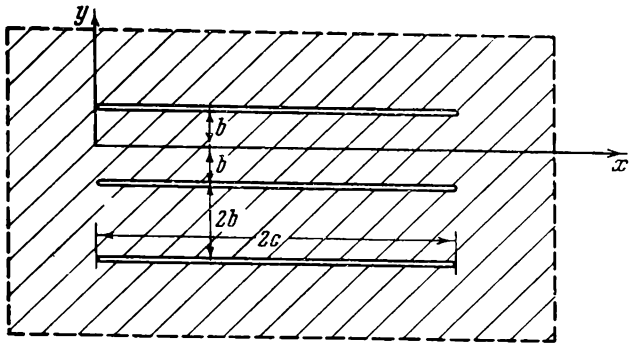

Fig. 1

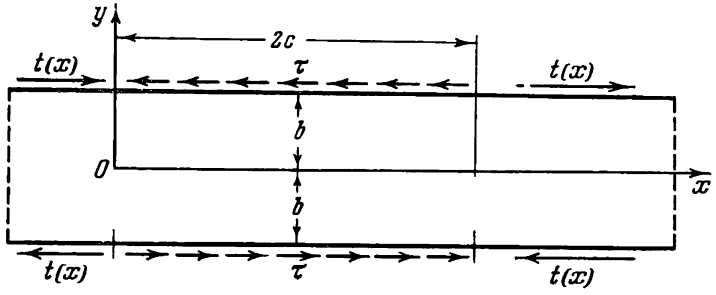

Fig. 2

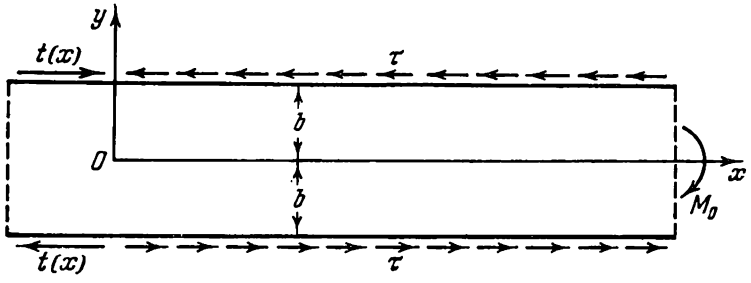

Fig. 3

condition that no displacements u occur along the lines $y = \pm b$, $x < 0$ or $x > 2c$. Finally, we have the condition

$$\int_{-\infty}^{0} t(x)\,dx = \int_{2c}^{\infty} t(x)\,dx = \tau c. \tag{3.1}$$

If the cracks are closely spaced, we have $c \gg b$, and the effect of the self-equilibrating loads on the strip in the region $x > c$ have a negligible effect on the stress distribution in the vicinity of $x = 0$. In order to investigate

this stress distribution, we may therefore replace our strip problem by the modified problem of Fig. 3, where the (infinite) moment M_0 at infinity has to be chosen in such a way that the resulting moment in $x = 0$ is $2\tau cb$ per unit thickness of the strip [cf. (3.1)].

The stresses in the strip are derived from an Airy stress-function $f(x, y)$ by means of

$$\sigma_x = \frac{\partial^2 f}{\partial y^2}, \quad \tau_{xy} = -\frac{\partial^2 f}{\partial x \partial y}, \quad \sigma_y = \frac{\partial^2 f}{\partial x^2}. \tag{3.2}$$

The Airy-function satisfies the biharmonic equation and the boundary conditions

$$\begin{aligned} y = \pm b : \sigma_y = 0, \quad \tau_{xy} &= -\tau \text{ for } x > 0, \\ \tau_{xy} &= t(x) \text{ for } x < 0, \end{aligned} \tag{3.3}$$

where $t(x)$ is as yet unknown. Along the boundaries $y = \pm b$ the displacement component u satisfies the equation

$$\frac{\partial u}{\partial x} = \frac{\sigma_x}{E}, \tag{3.4}$$

and the additional boundary condition is

$$y = \pm b : u = 0 \text{ for } x < 0. \tag{3.5}$$

We now assume that $e^{-\alpha x} f(x, y)$ is absolutely integrable with respect to x from $-\infty$ to $+\infty$ for $0 < \alpha < \beta$, that $e^{-\alpha x} u(x)$ for $y = b$ is absolutely integrable from 0 to $+\infty$ for $\alpha > 0$, and that $e^{-\alpha x} t(x)$ is absolutely integrable from $-\infty$ to 0 for $\alpha < \beta$. [1] The Fourier-transforms

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x, y) e^{i\lambda x}\, dx = F(\lambda, y), \text{ regular for } 0 < \operatorname{Im} \lambda < \beta, \tag{3.6}$$

$$\frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} u(x) e^{i\lambda x}\, dx = U_+(\lambda), \text{ regular for } \operatorname{Im} \lambda > 0, \tag{3.7}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} t(x) e^{i\lambda x}\, dx = T_-(\lambda), \text{ regular for } \operatorname{Im} \lambda < \beta, \tag{3.8}$$

$$\frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \tau e^{i\lambda x}\, dx = \frac{i\tau}{\sqrt{2\pi}} \frac{1}{\lambda}, \text{ regular for } \operatorname{Im} \lambda > 0, \tag{3.9}$$

[1] It is easily verified a posteriori that our solution satisfies these assumptions.

are then regular in the strip and half-planes indicated with each transform.

We apply a Fourier-transformation to the biharmonic equation, satisfied by the stress-function, and to the boundary conditions (3.3). The transformed equation reads [1]

$$\frac{d^4F}{dy^4} - 2\lambda^2\frac{d^2F}{dy^2} + \lambda^4F = 0, \tag{3.10}$$

and the transformed boundary conditions are

$$y = \pm b : -\lambda^2F = 0,\ i\lambda\frac{dF}{dy} = T_-(\lambda) - \frac{i\tau}{\sqrt{2\pi}}\frac{1}{\lambda}. \tag{3.11}$$

The solution of (3.10), satisfying the boundary conditions (3.11), is given by

$$F(\lambda, y) = \frac{i}{\lambda^2}\left[T_-(\lambda) - \frac{i\tau}{\sqrt{2\pi}}\frac{1}{\lambda}\right]\frac{\lambda b\cosh\lambda b\sinh\lambda y - \lambda y\sinh\lambda b\cosh\lambda y}{\sinh\lambda b\cosh\lambda b - \lambda b}. \tag{3.12}$$

Applying a Fourier-transformation to (3.4) and using (3.5), we obtain

$$y = b : \frac{1}{E}\frac{d^2F}{dy^2} = -i\lambda U_+(\lambda). \tag{3.13}$$

Substituting (3.12) into (3.13), we obtain the Wiener-Hopf type equation, connecting $T_-(\lambda)$ and $U_+(\lambda)$,

$$E\lambda U_+(\lambda) = \frac{2\sinh^2\lambda b}{\sinh\lambda b\cosh\lambda b - \lambda b}\left[T_-(\lambda) - \frac{i\tau}{\sqrt{2\pi}}\frac{1}{\lambda}\right], \tag{3.14}$$

holding in the strip $0 < \operatorname{Im}\lambda < \beta$. In view of (3.1) our solution must also satisfy the auxiliary condition

$$T_-(0) = \frac{\tau c}{\sqrt{2\pi}}. \tag{3.15}$$

In order to solve our Wiener-Hopf equation (3.14) we write

$$\frac{\sinh\lambda b\cosh\lambda b - \lambda b}{\sinh^2\lambda b} = \frac{\lambda b}{\sqrt{\lambda^2b^2 + 9/4}}H(\lambda b), \tag{3.16}$$

where the positive branch of the square root should be taken on the real axis. We note that $H(\lambda b)$ is regular and non-zero in a strip containing the

[1] It is again easily verified a posteriori that our solution satisfies the condition under which these transformations are valid.

real axis $-\frac{3}{2} < \operatorname{Im} \lambda b < \frac{3}{2}$. Moreover, $H(\lambda b)$ is an even function of λb, positive on the real axis, and we have $H(0) = 1$ and $H(\lambda b) \to 1 + O[(\lambda b)^{-2}]$ for $|\lambda b| \to \infty$ in its strip of regularity. Hence

$$\chi(\lambda b) = \log H(\lambda b), \tag{3.17}$$

where the real branch of the logarithm is taken on the real axis, is also regular in this strip. We may write

$$\chi(\lambda b) = \chi_+(\lambda b) - \chi_-(\lambda b), \tag{3.18}$$

where $\chi_+(\lambda b)$ and $\chi_-(\lambda b)$, defined by

$$\chi_+(\lambda b) = \frac{1}{2\pi i} \int_{-i\gamma b-\infty}^{-i\gamma b+\infty} \frac{\chi(w)}{w - \lambda b}\, dw, \quad 0 < \gamma b < \tfrac{3}{2}, \tag{3.19}$$

$$\chi_-(\lambda b) = \frac{1}{2\pi i} \int_{i\gamma b-\infty}^{i\gamma b+\infty} \frac{\chi(w)}{w - \lambda b}\, dw, \quad 0 < \gamma b < \tfrac{3}{2}, \tag{3.20}$$

are regular in the upper half-plane $\operatorname{Im} \lambda > -\gamma$ and the lower half-plane $\operatorname{Im} \lambda < \gamma$. We observe that $\chi_+(\lambda b)$ and $\chi_-(\lambda b)$ tend to zero for $|\lambda b| \to \infty$ in their respective half-planes of regularity. Writing

$$H_+(\lambda b) = e^{\chi_+(\lambda b)}, \quad H_-(\lambda b) = e^{\chi_-(\lambda b)}, \tag{3.21}$$

we rewrite our Wiener-Hopf equation in the form

$$\frac{E\lambda^3 b}{\sqrt{\lambda b + \frac{3}{2}i}} H_+(\lambda b) U_+(\lambda) = 2\sqrt{\lambda b - \tfrac{3}{2}i}\, H_-(\lambda b) \left[\lambda T_-(\lambda) - \frac{i\tau}{\sqrt{2\pi}}\right], \tag{3.22}$$

where the square roots are principal values.

The left-hand member of (3.22) is regular in the upper half-plane $\operatorname{Im} \lambda > 0$, whereas the right-hand member is regular in the lower half-plane $\operatorname{Im} \lambda < \gamma$. By the principle of analytic continuation both members then represent the same *integral* function. Because $H_-(\lambda b)$ tends to unity for $|\lambda b| \to \infty$ in the lower half-plane and because $T_-(\lambda)$, as a Fourier-transform, must tend to zero for $|\lambda b| \to \infty$ in the lower half-plane, this integral function must be a *linear* function $C\lambda + D$, where C and D are arbitrary complex constants. Hence we have

$$T_-(\lambda) = \frac{i\tau}{\sqrt{2\pi}} \frac{1}{\lambda} + \frac{C\lambda + D}{2\lambda\sqrt{(\lambda b - \frac{3}{2}i)} H_-(\lambda b)}, \tag{3.23}$$

$$U_+(\lambda) = \frac{(C\lambda + D)\sqrt{\lambda b + \frac{3}{2}i}}{E\lambda^3 b H_+(\lambda b)}. \tag{3.24}$$

Finally, we use our auxiliary condition (3.15); the regularity of $T_-(\lambda)$ at $\lambda = 0$ requires

$$D = -\frac{2i\tau}{\sqrt{2\pi}}\sqrt{-\tfrac{3}{2}i}\,H_-(0), \tag{3.25}$$

and the specified value of $T_-(0)$ yields

$$C = \frac{2\tau}{\sqrt{2\pi}}\sqrt{-\tfrac{3}{2}i}\,H_-(0)[c + b\{\tfrac{1}{3} - i\chi_-'(0)\}], \tag{3.26}$$

where $\chi_-'(\lambda b)$ is the derivative of (3.20) with respect to λb. Our solution is now completely determined by (3.23) to (3.26), and it is obviously unique.

The displacement component $u(x)$ along $y = b$ is obtained by the inverse transformation

$$u(x) = \frac{1}{\sqrt{2\pi}}\int_{i\gamma-\infty}^{i\gamma+\infty} U_+(\lambda)e^{-i\lambda x}\,d\lambda, \tag{3.27}$$

where again $0 < \gamma b < \frac{3}{2}$. By closing the contour in the upper half-plane it is easily verified that $u(x) = 0$ for negative x. The increase in elastic strain energy per crack and per unit thickness of the sheet is now given by (cf. Fig. 2)

$$\Delta A = -4\int_0^c \tfrac{1}{2}\tau u(x)\,dx = \frac{2i\tau}{\sqrt{2\pi}}\int_{i\gamma-\infty}^{i\gamma+\infty} U_+(\lambda)\frac{1 - e^{-i\lambda c}}{\lambda}\,d\lambda. \tag{3.28}$$

Again closing the contour in the upper half-plane, we obtain

$$\int_{i\gamma-\infty}^{i\gamma+\infty} \frac{U_+(\lambda)}{\lambda}\,d\lambda = 0. \tag{3.29}$$

For the second integral in (3.28) we shift the line of integration to $\operatorname{Im}\lambda = -\gamma$, taking due account of the pole at $\lambda = 0$. The integral along $\operatorname{Im}\lambda = -\gamma$ tends to zero as $e^{-\gamma c}$ as c increases, and it may therefore be neglected. Hence we obtain

$$\Delta A = -2\tau\sqrt{2\pi}\,\operatorname{Residue}\left\{U_+(\lambda)\frac{e^{-i\lambda c}}{\lambda}\right\}_{\lambda=0}. \tag{3.30}$$

Evaluation of (3.30) requires some straightforward, if somewhat lengthy algebra. *Our final result may be expressed in the following form:*

$$\Delta A = \frac{2\tau^2 c^3}{Eb}\left[1 + \alpha_1\frac{b}{c} + \alpha_2\left(\frac{b}{c}\right)^2 + \alpha_3\left(\frac{b}{c}\right)^3\right]. \tag{3.31}$$

The nondimensional coefficients α_1, α_2, α_3 are given by

$$\alpha_1 = 1 - 3i\chi_+'(0), \tag{3.32}$$

$$\alpha_2 = \tfrac{1}{3} - 2i\chi_+'(0) - 3\{\chi_+'(0)\}^2 = \tfrac{1}{3}[1 - 3i\chi_+'(0)]^2, \tag{3.33}$$

$$\alpha_3 = -\tfrac{1}{9} - \tfrac{1}{3}i\chi_+'(0) - \{\chi_+'(0)\}^2 - \tfrac{1}{2}i\chi_+'''(0) + i\{\chi_+'(0)\}^3, \tag{3.34}$$

where free use has been made of the identities

$$\chi_+(0) = \chi_-(0) = \log H_+(0) = \log H_-(0) = 0, \tag{3.35}$$

$$\chi_+'(0) = \chi_-'(0), \tag{3.36}$$

which follow from our definition of $H(\lambda b)$. The derivatives occurring in (3.32) to (3.34) are obtained by numerical integration

$$\chi_+'(0) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\chi(z)}{z^2}\, dz = -0.0468i, \tag{3.37}$$

$$\chi_+'''(0) = \frac{3}{\pi i} \int_{-\infty}^{\infty} \frac{\chi(z) - \frac{1}{2}\chi''(0)z^2}{z^4}\, dz = +0.1874i, \tag{3.38}$$

and our numerical results for the non-dimensional coefficients are

$$\alpha_1 = 0.8596, \tag{3.39}$$

$$\alpha_2 = 0.2463, \tag{3.40}$$

$$\alpha_3 = -0.0309. \tag{3.41}$$

The first two terms in (3.31) allow a simple physical interpretation. If we superimpose the uniform shear stress τ in the uncut sheet on the loading condition given in Fig. 2, we have strips of length $2c$, width $2b$ and thickness unity, which are elastically clamped at their ends and loaded by shear forces $2\tau b$ in their planes; the bending moment in each strip is zero in its mid-section $x = c$. The dominant part of the strain energy in the strips is due to bending. Neglecting the effect of elastic clamping at the ends, elementary bending theory gives a value

$$2 \cdot \tfrac{1}{2}(2\tau b)^2 \frac{c^3}{3E\frac{1}{12}(2b)^3} = \frac{2\tau^2 c^3}{Eb} \tag{3.42}$$

for this bending energy, i.e., the first term in (3.31). Likewise, the second term in (3.31) arises from the fact that the strips are elastically clamped. If the strip of Fig. 2 is loaded only by a given moment M_0 at infinity, the

increase in strain energy due to the fact that the strip is connected to the half-plane $x < 0$, may be calculated rigorously by an analysis similar to the analysis of the present paper; this increase in strain energy is given by

$$\Delta_1 A = \tfrac{1}{4}\alpha_1 \frac{M_0^2}{Eb^2}. \tag{3.43}$$

Substituting $M_0 = 2\tau bc$, and taking account of both ends, we obtain the second term in (3.31).

§ 4. Numerical results and discussion

The numerical evaluation of our asymptotic solutions for small c/b and large c/b is given in Table 1 and in Fig. 4. The two curves nearly coincide for c/b in the range from 0.3 to 0.6, and it is immediately obvious from Fig. 4 that *our two asymptotic solutions together describe the effect of the cracks on the elastic strain energy with adequate accuracy over the entire*

TABLE 1

Results for $\Delta A \cdot \dfrac{E}{\pi\tau^2 c^2}$

c/b	Asymptotic Solutions for	
	small c/b	large c/b
0	1.000	
0.1	1.004	0.212
0.2	1.016	0.967
0.3	1.037	1.042
0.4	1.066	1.071
0.6	1.148	1.136
0.8	1.263	1.222
1.0	1.411	1.321
1.2	1.592	1.428
1.4	1.806	1.540
1.6		1.656
1.8		1.774
2.0		1.894

range $0 < c/b < \infty$. *For* $c/b < 0.6$, *the solution for small* c/b, given by (2.22) and the second column in Table 1, *and for* $c/b > 0.6$ *the solution for large* c/b, given by (3.31) and the last column in Table 1, *give the required result with an error of at most one percent.*

It may cause some surprise that our asymptotic solutions are valid over so wide a range (even over the entire range for the two solutions together!). However, in many problems in the theory of elasticity, a similar situation occurs. It is indeed often possible to link asymptotic

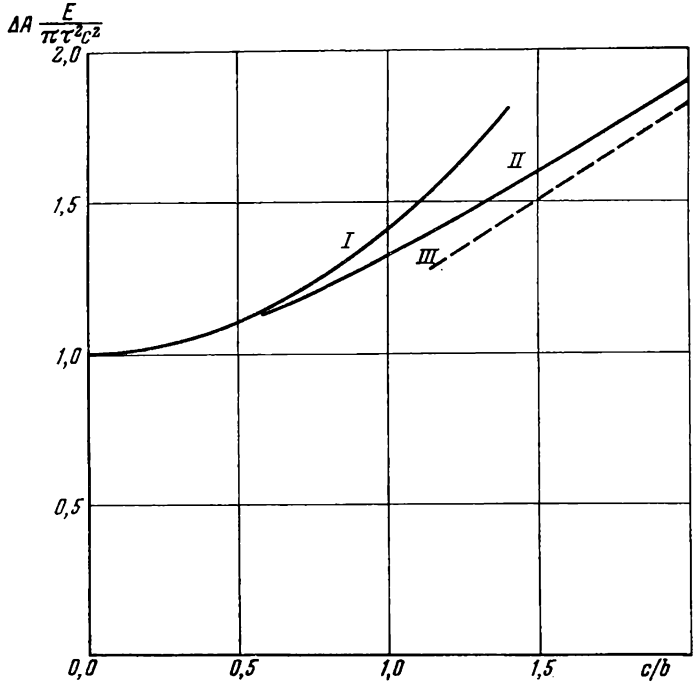

Fig. 4. Asymptotic solutions
(I) small c/b
(II) large c/b
(III) asymptote to curve (II)

solutions, each valid, in principle, in the vicinity of the ends of a range of a parameter, by an interpolated curve, and this situation has been exploited with considerable success in a variety of problems. The reason for the extremely large range of validity of the asymptotic solution for large values of c/b in the present problem is due to the fact that a self-equilibrating antisymmetric load, applied in one cross-section of an infinite strip, induces stresses that die out very rapidly with increasing distance to the cross-section of load application. At a distance equal to half the width of the strip, the resulting stresses are usually already negligible. The loads to the right of the cross-section $x = c$ in Fig. 2 are self-equilibrating, and their effect is therefore already negligible in the

vicinity of $x = 0$, if c is approximately equal to b. Hence it should indeed be expected that the asymptotic solution for large c/b is already valid with great accuracy for c/b equal to about unity.

Finally, it should be observed that the mutual interference between the cracks is relatively unimportant if the crack width is less than about half the pitch between cracks. As in the problem of collinear cracks [1], the increase in strain energy per crack for $c/b = \frac{1}{2}$ is only about ten percent higher than in the case of a single crack. It may safely be concluded that a similar result will hold in the case of a row of parallel cracks whose directions make an arbitrary (constant) angle with the direction connecting homologous points on the cracks.

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INTRODUCTION TO THE DISLOCATION THEORY OF ELASTO-PLASTIC TORSION

M. IA. LEONOV and N. IU. SHVAIKO
L'vov, U.S.S.R.

The problem of elasto-plastic deformation in torsion is considered when the stress-strain diagram has a peak. The dimensions of the first slip zone and the conditions under which further zones will occur are established.

§ 1. Screw dislocations and discontinuous displacements

Let the deformations of an unbounded body referred to a cylindrical coordinate system (r, φ, z) be characterized only by the displacement w of its points in the direction Oz, where

$$w = \frac{\lambda}{2\pi}\varphi, \quad -\pi < \varphi < \pi. \tag{1.1}$$

We obtain the above deformation if we cut the body in the half plane $\varphi = \pi$ and displace the surfaces of the cut relative to each other in the Oz direction by an amount λ. Then, points lying before deformation in planes $z = \lambda n$, $(n = 0, \pm 1, \pm 2, \cdots)$, lie after deformation on the screw surface $z = \frac{1}{2}\lambda\varphi/\pi$.

The state of stress corresponding to the deformation (1.1) is determined by the components

$$\tau_{\varphi z} = \frac{\lambda}{2\pi}\frac{G}{r}, \quad (\sigma_r = \sigma_\varphi = \sigma_z = \tau_{r\varphi} = \tau_{zr} = 0). \tag{1.2}$$

This state of stress satisfies the conditions of static equilibrium of an elastic body. It has on the axis Oz $(r = 0)$ a singularity, called a screw dislocation [1] of strength λ. In what follows we will say that the displacements (1.1) have been caused by the dislocations stated. They are characterized by the constant discontinuity λ in the plane $\varphi = \pi$ and continuity of the displacements in the remaining part of the body. Hence one may define the strength of screw dislocations as the increase of the discontinuity of the displacements at a given point. For the sake of clarity of the above, we will assume that there are given in the unbounded elastic body in the plane $x = 0$ the discontinuous displacements

$$w(\pm 0, y) = \pm \tfrac{1}{2}\delta(y), \quad |y| < h. \tag{1.3}$$

Figure 1 shows what is happening in the case of this deformation in the planes $y = 0$ and $z = 0$. The displacements (1.3) may be considered to be the result of the presence of screw dislocations continuously

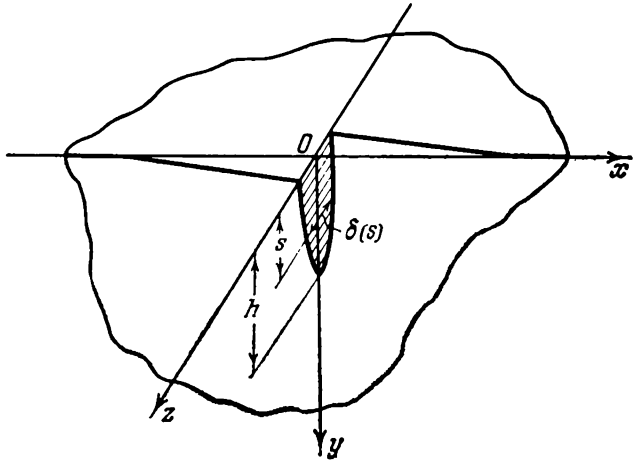

Fig. 1

distributed in the plane $x = 0$. The strength of the dislocations per length dy is equal to the increase in the discontinuity of the displacements over this segment, i.e.,

$$d\lambda = \delta'(y)dy. \tag{1.4}$$

The density of dislocations $\mu(y)$, distributed along the line of discontinuity of displacements, will be

$$\mu(y) = \frac{d\lambda}{dy} = \delta'(y). \tag{1.5}$$

On the basis of (1.1) the displacement dw at any point of the body, caused by the dislocation element $\mu(s)ds$, is equal to

$$dw = \frac{1}{2\pi} \arctan \frac{x}{y - s} \mu(s)ds. \tag{1.6}$$

The total displacement of the system of dislocations, distributed in the plane $x = 0$, $|y| < h$ is determined by the formula

$$w(x, y) = \frac{1}{2\pi} \int_{-h}^{h} \arctan \frac{x}{y - s} \mu(s)ds. \tag{1.7}$$

For the shear stresses we obtain from the last formula

$$\tau_{xz} - i\tau_{zy} = G\left(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\right) = -\frac{G}{2\pi i}\int_{-h}^{h}\frac{\mu(s)}{\zeta - is}\,ds, \tag{1.8}$$

$$\zeta = x + iy.$$

When the point ζ approaches $\zeta_0 = iy$, $(|y| < h)$, we have from (1.8) by the Sokhotskii-Plemelj formula [2]

$$\tau_{xz}(0, y) - i\tau_{zy}(\pm 0, y) = \frac{G}{2\pi}\int_{-h}^{h}\frac{\mu(s)}{y - s}\,ds \pm i\frac{G}{2}\mu(y), \tag{1.9}$$

i.e.,

$$\tau_{xz}(0, y) = \frac{G}{2\pi}\int_{-h}^{h}\frac{\mu(s)}{y - s}\,ds,$$

$$\tau_{zy}(\pm 0, y) = \mp\frac{G}{2}\mu(y), \tag{1.10}$$

$$|y| < h.$$

In the case when the function $\mu(y)$ is odd, we obtain from (1.8) that $\tau_{zy}(x, 0) = 0$, i.e., the plane $y = 0$ is free from stresses. Hence, the space may be cut on this plane without changing the state of stress. It follows from this that the displacements and stresses, caused by screw dislocations distributed with density $\mu(y)$ in the space $y > 0$, may be determined from the formulae (1.7) and (1.8), assuming in these formulae that the function $\mu(y)$ is odd.

§ 2. The elasto-plastic problem for a material having a peak in the stress-strain diagram

In the case of an ideal periodic structure of atomic planes one has to apply for their displacement shear stresses which are ten times less than the shear modulus. After overcoming the potential barrier against shear displacement of the atomic planes, deformation may take place by inertia without external loading.

Disturbances in an ideally regular structure, to one or the other degree always being true in real crystals, very considerably distort the above relationship between the stresses and strains, lowering by ten and a hundred times the maximum shear stress τ_m required for displacement, but increasing the resistance to establishing shear flow τ_c. In the existing

theories of plasticity it is assumed that $\tau_c \geq \tau_m$. This condition is not fulfilled by material having a peak in the stress-strain diagram.

The simplest hard body, having a peak in the stress-strain diagram, is that which obeys Hooke's Law for $\tau_{\max} \leq \tau_m$ and experiences a constant stress τ_c on the slip surfaces. The study of the deformation of such a body reduces to the following problem.

Find the solution of the linear problem of the theory of elasticity, continuous everywhere in the region occupied by the body, except over certain surfaces. The tangential components of the displacements at these surfaces experience discontinuities and the tangential stresses there are constant in magnitude and directed in the opposite directions to the displacements.

§ 3. Elementary plastic displacement in torsion

It will be assumed that the maximum stress attains the value τ_m for which plastic displacements take place, at first only at a single point of the boundary of the cross-section of the rod. The depth h of the distributed plastic shear displacement (Fig. 2) is assumed to be small in comparison with the dimensions of the cross-section of the body which

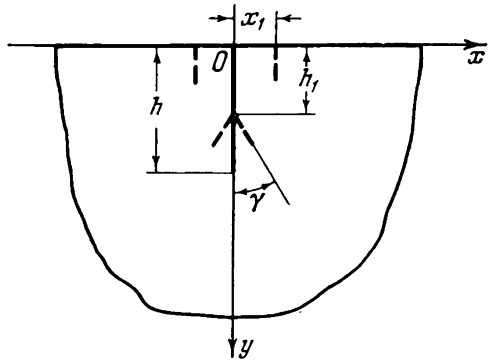

Fig. 2

therefore may be considered to be a half-space. The coordinate axes will be chosen so that the displacements take place in the plane $x = 0$ for $y \leq h$. In what follows we will denote by $w^0(x, y)$, τ_{xz}^0, τ_{zy}^0 respectively the displacement and the tangential stresses, determined without consideration of plastic deformations.

By the strength of what has been said in the preceding section regarding the surfaces experiencing plastic displacements, there will act constant shear stresses $\tau_c < \tau_m$, directed in the opposite direction to the

displacements, and the displacement w will suffer a jump in the continuity. The discontinuous displacements are considered to be the result of the presence in the strip $x = 0$, $y \leq h$ of screw dislocations, distributed with density $\mu(y)$.

If we denote by $\tau_{xz}^1(x, y)$ and $\tau_{zy}^1(x, y)$ the stresses caused by the distributed screw dislocations $\mu(y)$, then one may write down on the line of discontinuous displacements the condition

$$\tau_{xz}^0(0, y) + \tau_{xz}^1(0, y) = \tau_c, \quad y < h. \tag{3.1}$$

Determining the stress $\tau_{xz}^1(0, y)$ from (1.10) and substituting it in (3.1), we obtain the equation for the determination of the function $\mu(y)$

$$\frac{G}{2\pi} \int_{-h}^{h} \frac{\mu(s)}{y - s}\, ds = \tau_c - \tau_{xz}^0(0, y), \tag{3.2}$$

$$\mu(-s) = -\mu(s).$$

The general solution of the equation (3.2) is given in [2] by N. I. Muskhelishvili.

In the present case we obtain

$$\mu(y) = \frac{2}{\pi G\sqrt{h^2 - y^2}} \int_{-h}^{h} \frac{\sqrt{h^2 - s^2}}{s - y} f(s)ds + \frac{c}{\sqrt{h^2 - y^2}}, \tag{3.3}$$

where

$$f(s) = \tau_c - \tau_{xz}^0(0, s). \tag{3.4}$$

For $s < 0$ one must understand by the expression $f(s)$ the mirror image of the function $\tau_c - \tau_{xz}^0(0, y)$ in the plane $y = 0$. The constant c and the depth h of penetration of plastic displacement is to be determined from the condition of boundedness of the stresses at the point $x = 0$, $y = h$.

In the particular case when one may write

$$\tau_{xz}^0(0, y) = \alpha\tau_m - by, \quad y \leq h, \alpha \geq 1, \tag{3.5}$$

we obtain from (3.3)

$$\mu(y) = \frac{4by}{\pi G} \log \frac{h + \sqrt{h^2 - y^2}}{y} - \frac{2[2hb - \pi(\alpha\tau_m - \tau_c)]y + c}{\pi G\sqrt{h^2 - y^2}}. \tag{3.6}$$

Using the condition of boundedness of the stresses, we obtain

$$\mu(y) = \frac{4by}{\pi G}\log\frac{h+\sqrt{h^2-y^2}}{y}, \qquad h = \pi\frac{\alpha\tau_m - \tau_c}{2b}. \tag{3.7}$$

After plastic displacements have occurred the displacement and the tangential stresses are determined by the formulae

$$w(x,y) = w^0(x,y) + \frac{1}{2\pi}\int_{-h}^{h}\mu(s)\arctan\frac{x}{y-s}\,ds,$$
$$\tau_{xz} - i\tau_{zy} = \tau_{xz}^0 - i\tau_{zy}^0 - \frac{G}{2\pi i}\int_{-h}^{h}\frac{\mu(s)}{\zeta - is}\,ds. \tag{3.8}$$

For the case (3.5) we have from the last formula

$$\tau_{xz} - i\tau_{zy} = \tau_{xz}^0 - i\tau_{zy}^0 - (\alpha\tau_m - \tau_c)\left[1 - \frac{\zeta}{h}\log\frac{h+\sqrt{h^2+\zeta^2}}{\zeta}\right]. \tag{3.9}$$

As the parameter α increases the tangential stresses may attain the limiting value τ_m at new points of the cross-section of the rod, i.e., there arises the condition for the formation of new plastic displacements. The points at which new slip regions occur may be found either on the side of the first slip zone or near this zone on the surface $y = 0$. Having this in mind we will write for small $|x|$

$$\tau_{xz}^0(x, 0) = \alpha\tau_m - ax^2. \tag{3.10}$$

Then the stress $\tau_{xz}(x, 0)$, determined from (3.9), takes the value τ_m at the points

$$x = \pm x,\ y = 0$$

for

$$\alpha = \alpha_1 = \frac{\tau_c}{\tau_m} + \frac{2bx_1(\tau_m - \tau_c - ax_1^2)}{\tau_m\sqrt{4b^2x_1^2 - \pi^2(\tau_m - \tau_c - ax_1^2)^2}}, \tag{3.11}$$

where x_1 is determined from the equation

$$\log\frac{2bx_1 + \pi(\tau_m - \tau_c - ax_1^2)}{2bx_1 - \pi(\tau_m - \tau_c - ax_1^2)} - \frac{\pi}{bx_1}(\tau_m - \tau_c + ax_1^2) = 0. \tag{3.12}$$

If for given a and b the quantity

$$\alpha_1 > \alpha_2 = \frac{\tau_c}{\tau_m} + \frac{1}{k}\sqrt{1 - \left(\frac{\tau_c}{\tau_m}\right)^2}, \tag{3.13}$$

where

$$k = \max\left[\frac{y}{h}\log\frac{h + \sqrt{h^2 - y^2}}{y}\right] = 0.662\cdots, \tag{3.14}$$

then the stress τ_m is attained in the first place on planes, making an angle $\gamma = \arccos \tau_c/\tau_m$ with the axis Oy at the points $x = \pm 0$, $y = (0.552\cdots)h$.

In the case $\alpha_1 < \alpha_2$, the following displacement takes place at the points $x = \pm x_1$, $y = 0$ perpendicular to the axis Ox. In the second case ($\alpha_1 > \alpha_2$) displacement occurs at depth $y = (0.552\cdots)h$ at an angle γ with the Oy axis. In Fig. 2, the possible following displacements are shown by dotted lines.

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ON THE STATIC GEOMETRIC ANALOGUE OF SHELL THEORY

A. I. LUR'E
Leningrad, U.S.S.R.

§ 1. In his well-known work N. I. Muskhelishvili gives the formulation and deduces the general solution of the problems of the plane theory of elasticity when the state of stress in a region is being sought for given external forces or displacements on its boundary (first and second boundary problems). The methods of N. I. Muskhelishvili have found application in many problems of mathematical physics, which lead to the biharmonic problem; in particular, a large number of papers is devoted to the problem of bending of plates. Using the terminology above, we will call the first boundary problem of this theory the search for the state of stress of a bent plate when forces and moments distributed along its edge are given, and the second when the displacement and the angle of rotation of the edges are specified. In fact, it is well-known that the Airy-stress function and its normal derivative at a point of the edge are equal to the resultant moment about this point and the projection on the local tangent to the edge of the resultant vector of the external forces, distributed along the section of the edge between a fixed point and the point under consideration. But this same Airy-function also represents the deflection w of the plate [1] for the boundary values of the deflection and angle of rotation. The first plane boundary problem becomes entirely the second bending boundary problem. Less obvious and of greater interest is a second property: the solution of the second plane boundary problem corresponds to the first bending boundary problem. It turns out that replacement in the expressions for the displacements u_1, u_2 of Poisson's number m by $-m$ leads to the stress functions B_1, B_2 solving the bending problem[2]; then the boundary values of the functions u_i and B_i are identical, and the B_i represent the projections on the corresponding directions of the resultant moment about the edge point under consideration of the forces and moments, distributed over the part of the edge (measured from a fixed point). It is quite obvious what is the advantage of this method of solution of the first bending boundary

[1] It is assumed that body forces are absent in the plane problem and surface loading in the bending problem.

[2] We speak here of the state of plane stress.

problem over the generally accepted method for finding the deflection function w which leads to boundary conditions containing second and third derivatives of this function.

All that has been said here is a very particular development of a known static geometrical analogue in the theory of shells [1]. The objective of this note is to give certain properties of this analogue which, while known to the author, have not been stated in the literature.

§ 2. The equilibrium equations of a shell element in the absence of surface loading may be described in the form of the two vector equations

$$\frac{\partial}{\partial\alpha_1}\mathbf{K}_1A_2 + \frac{\partial}{\partial\alpha_2}\mathbf{K}_2A_1 = 0, \tag{2.1}$$

$$\frac{\partial}{\partial\alpha_1}\mathbf{M}_1A_2 + \frac{\partial}{\partial\alpha_2}\mathbf{M}_2A_1 + \frac{\partial\mathbf{r}}{\partial\alpha_1}\times A_2\mathbf{K}_1 + \frac{\partial\mathbf{r}}{\partial\alpha_2}\times A_1\mathbf{K}_2 = 0, \tag{2.2}$$

where $\mathbf{r}(\alpha_1, \alpha_2)$ is the radius vector of a point on the middle surface, α_1, α_2 are orthogonal Gaussian coordinates on this surface, and A_1, A_2 are the Lamé coefficients. The symbols $\mathbf{K}_i$, $\mathbf{M}_i$ denote the resultant vector and moment (about a point on the middle surface) of the stresses in the normal section of the shell referred to unit length of the arc of the coordinate lines $\alpha_i = \text{const.}$:

$$\mathbf{K}_1 = T_1\boldsymbol{\tau}_1 + S_1\boldsymbol{\tau}_2 + Q_1\mathbf{n},\ \mathbf{K}_2 = S_2\boldsymbol{\tau}_1 + T_2\boldsymbol{\tau}_2 + Q_2\mathbf{n}, \tag{2.3}$$

$$\mathbf{M}_1 = -H_1\boldsymbol{\tau}_1 + G_1\boldsymbol{\tau}_2,\ \mathbf{M}_2 = -G_2\boldsymbol{\tau}_1 + H_2\boldsymbol{\tau}_2. \tag{2.4}$$

The ten quantities

$$T_i,\ S_i,\ Q_i,\ G_i,\ H_i,\quad (i = 1, 2) \tag{2.5}$$

represent the direct, tangential and shear forces, and the bending and twisting moments. The symbols

$$\boldsymbol{\tau}_i = \frac{1}{A_i}\frac{\partial\mathbf{r}}{\partial\alpha_i},\quad \mathbf{n} = \boldsymbol{\tau}_1\times\boldsymbol{\tau}_2 \tag{2.6}$$

denote the unit vectors of the orthogonal trihedron of the tangents to the coordinate lines and of the normal to the surface. If one selects as coordinate lines the lines of curvature, the formulae for the derivatives of these vectors have the form

$$\begin{aligned}
&\frac{\partial\boldsymbol{\tau}_1}{\partial\alpha_1} = -\frac{1}{A_2}\frac{\partial A_1}{\partial\alpha_2}\boldsymbol{\tau}_2 - \frac{A_1}{R_1}\mathbf{n},\\
&\frac{\partial\boldsymbol{\tau}_2}{\partial\alpha_1} = \frac{1}{A_2}\frac{\partial A_1}{\partial\alpha_2}\boldsymbol{\tau}_1,\quad \frac{\partial\mathbf{n}}{\partial\alpha_1} = \frac{A_1}{R_1}\boldsymbol{\tau}_1,\quad (1\to 2)\ ^{3}
\end{aligned} \tag{2.7}$$

[3] This sign is to indicate that the second group of equations is obtained from that written down by interchange of the subscripts 1 and 2.

Applying these formulae and substituting the values of the vectors (2.3) and (2.4) into the equations (2.1) and (2.2), we arrive at the six equations of statics – the three equations for the forces

$$\frac{\partial}{\partial\alpha_1} T_1A_2 - \frac{\partial A_2}{\partial\alpha_1} T_2 + \frac{\partial}{\partial\alpha_2} A_1S_2 + \tag{2.8}$$

$$+ S_1 \frac{\partial A_1}{\partial\alpha_2} + \frac{A_1A_2}{R_1} Q_1 = 0, \quad (1 \to 2),$$

$$\frac{\partial}{\partial\alpha_1} Q_1A_2 + \frac{\partial}{\partial\alpha_2} Q_2A_1 - A_1A_2\left(\frac{T_1}{R_1} + \frac{T_2}{R_2}\right) = 0, \tag{2.9}$$

and the three equations for the moments

$$A_1A_2Q_1 = \frac{\partial}{\partial\alpha_1} G_1A_2 - G_2 \frac{\partial A_2}{\partial\alpha_1} + \frac{\partial}{\partial\alpha_2} A_1H_2 + H_1 \frac{\partial A_1}{\partial\alpha_2}, \quad (1 \to 2), \tag{2.10}$$

$$S_1 - \frac{H_2}{R_2} = S_2 - \frac{H_1}{R_1} = S, \tag{2.11}$$

where S denotes the general value of the stated differences.

After elimination of the shear forces in the remaining three static equations one has eight quantities, interrelated by the conditions (2.11). However, a formal transformation of these equations may be achieved to arrive at a version involving only the six quantities

$$T_1,\ T_2,\ S,\quad G_1,\ G_2,\ H, \tag{2.12}$$

where

$$H = \tfrac{1}{2}(H_1 + H_2). \tag{2.13}$$

In order to verify this result, we introduce into the consideration the "deviating" shear forces

$$Q_1 = N_1 + \frac{1}{2A_2} \frac{\partial}{\partial\alpha_2} (H_2 - H_1), \quad (1 \to 2). \tag{2.14}$$

Then, replacing in accordance with (2.11) the shear forces S_i by the quantities

$$S_1 = S + \frac{H_2}{R_2}, \quad (1 \to 2), \tag{2.15}$$

and using the known Codazzi formulae

$$\frac{\partial}{\partial\alpha_1} \frac{A_2}{R_2} = \frac{1}{R_1} \frac{\partial A_2}{\partial\alpha_1}, \quad (1 \to 2), \tag{2.16}$$

we obtain, after elementary manipulations, the three equations of statics

$$(2.17)\quad L_1(T_1, T_2, S, G_1, G_2, H) = \frac{\partial}{\partial\alpha_1} T_1A_2 - T_2\frac{\partial A_2}{\partial\alpha_1} + \\ + \frac{\partial}{\partial\alpha_2} A_1\left(S + \frac{H}{R_1}\right) + \left(S + \frac{H}{R_2}\right)\frac{\partial A_1}{\partial\alpha_2} + \frac{A_1A_2}{R_1} N_1 = 0, \quad (1 \to 2),$$

$$(2.18)\quad L_3(T_1, T_2, S, G_1, G_2, H) = \frac{\partial}{\partial\alpha_1} N_1A_2 + \frac{\partial}{\partial\alpha_2} N_2A_1 - \\ - A_1A_2\left(\frac{T_1}{R_1} + \frac{T_2}{R_2}\right) = 0,$$

in which N_1, N_2 must be replaced by the expressions

$$(2.19)\quad A_1A_2N_1 = \frac{\partial}{\partial\alpha_1} G_1A_2 - G_2\frac{\partial A_2}{\partial\alpha_1} + \frac{\partial}{\partial\alpha_2} HA_1 + \frac{\partial A_1}{\partial\alpha_2} H, \quad (1\to 2).$$

The deduction of the equations of static equilibrium in this form was given in [2]; they were obtained by the help of the principle of virtual displacements in [3]. Right from the outset, it is clear from the above that the number of unknowns in the equations of statics must be equal to the number of quantities characterizing the deformation of the shell

$$(2.20)\qquad \varepsilon_1,\ \varepsilon_2,\ \gamma,\ \kappa_1,\ \kappa_2,\ \omega$$

(the relative extensions and shear, changes of curvature and twist) and that the static quantities (2.12) represent the generalized forces, referred to these parameters.

§ 3. In the papers [1] and [4] the statement has been made that the equations of statics (2.1), (2.2) are satisfied identically by the introduction of two vector stress functions in terms of which the vectors $\mathbf{K}_i$, $\mathbf{M}_i$ are expressed by the relations

$$(3.1)\qquad \mathbf{K}_1A_2 = \frac{\partial\mathbf{R}}{\partial\alpha_2},\ \mathbf{K}_2A_1 = -\frac{\partial\mathbf{R}}{\partial\alpha_1},$$

$$(3.2)\quad \mathbf{M}_1A_2 = \frac{\partial\mathbf{B}}{\partial\alpha_2} + \frac{\partial\mathbf{r}}{\partial\alpha_2}\times\mathbf{R},\ \mathbf{M}_2A_1 = -\left(\frac{\partial\mathbf{B}}{\partial\alpha_1} + \frac{\partial\mathbf{r}}{\partial\alpha_1}\times\mathbf{R}\right).$$

However, the vectors $\mathbf{M}_i$ do not involve the components normal to the middle surface; this permits writing down the two equalities

$$(3.3)\qquad \mathbf{M}_1\cdot\mathbf{n} = 0, \quad \text{or}\ \frac{1}{A_2}\frac{\partial\mathbf{B}}{\partial\alpha_2}\cdot\mathbf{n} = \mathbf{R}\cdot\boldsymbol{\tau}_1, \quad (1\to 2).$$

Introducing the notation

$$\tilde{\vartheta}_i = \frac{1}{A_i}\frac{\partial \mathbf{B}}{\partial \alpha_i}\cdot\mathbf{n}, \quad (i = 1, 2), \tag{3.4}$$

we obtain

$$\mathbf{R} = \tilde{\vartheta}_2\boldsymbol{\tau}_1 - \tilde{\vartheta}_1\boldsymbol{\tau}_2 + \mathbf{R}\cdot\mathbf{nn}. \tag{3.5}$$

One may now construct with the help of the formulae (3.1), (3.2) and (2.3), (2.4) expressions for the quantities (2.5) in terms of the four stress functions

$$\mathbf{B}\cdot\boldsymbol{\tau}_i = B_i, \ \mathbf{B}\cdot\mathbf{n} = B_3, \ \mathbf{R}\cdot\mathbf{n}.$$

After this the quantities S and H are determined from the formulae (2.11) and (2.13). As a result, we arrive at the formulae

$$T_1 = \tilde{\kappa}_2, \ T_2 = \tilde{\kappa}_1, \ S = -\tilde{\omega}, \ G_1 = \tilde{\varepsilon}_2, \ G_2 = \tilde{\varepsilon}_1, \ 2H = -\tilde{\gamma}, \tag{3.6}$$

where the quantities with the $\sim$ sign are expressed in terms of the vector $\mathbf{B}$ exactly by the help of the same relations by which the deformations (2.20) are expressed in terms of the displacement vector $\mathbf{u}$ of points of the middle surface. It is important to note that only the vector $\mathbf{B}$ enters into the basic relations of the static geometric analogue (3.6); the fourth stress function $\mathbf{R}\cdot\mathbf{n}$ vanishes from the consideration during the transition from the forces and moments (2.5) to the necessary and sufficient number of static quantities (2.12).

The equations of statics (2.17), (2.18), written in the form

$$L_k(\tilde{\kappa}_2, \tilde{\kappa}_1, -\tilde{\omega}, \tilde{\varepsilon}_2, \tilde{\varepsilon}_1, -\tfrac{1}{2}\tilde{\gamma}) = 0, \quad (k = 1, 2, 3), \tag{3.7}$$

are satisfied identically by substitution of the expressions of their arguments in terms of the stress functions B_1, B_2, B_3. Therefore the equations

$$L_k(\kappa_2, \kappa_1, -\omega, \varepsilon_2, \varepsilon_1, -\tfrac{1}{2}\gamma) = 0, \quad (k = 1, 2, 3), \tag{3.8}$$

being identities with regard to the projections u_1, u_2, w of the displacement vector $\mathbf{u}$, represent the three strain compatibility conditions. Substituting in them the expressions for the deformations in terms of the forces and moments

$$\begin{gathered}\varepsilon_1 = \frac{1}{Eh}\left(\tilde{\kappa}_2 - \frac{\tilde{\kappa}_1}{m}\right), \ \kappa_1 = -\frac{12}{Eh^3}\left(\tilde{\varepsilon}_2 - \frac{\tilde{\varepsilon}_1}{m}\right), \quad (1 \to 2) \\ \gamma = -\frac{2(m+1)}{mEh}\tilde{\omega}, \ \omega = \frac{12}{Eh^3}\cdot\frac{m+1}{2m}\tilde{\gamma},\end{gathered} \tag{3.9}$$

we arrive at three differential equations for the stress functions; after

division by the factor $-12/h^2$, they take the form

$$(3.10)\quad L_k\Big(\tilde{\varepsilon}_1-\frac{\tilde{\varepsilon}_2}{m},\ \tilde{\varepsilon}_2-\frac{\tilde{\varepsilon}_1}{m},\ \frac{m+1}{2m}\tilde{\gamma},\ -\frac{h^2}{12}\Big(\tilde{\kappa}_1-\frac{\tilde{\kappa}_2}{m}\Big),$$
$$-\frac{h^2}{12}\Big(\tilde{\kappa}_2-\frac{\tilde{\kappa}_1}{m}\Big),\ -\frac{m+1}{m}\frac{h^2}{12}\tilde{\omega}\Big)=0,\quad (k=1,2,3).$$

We find the three differential equations of the theory of elastic shells in terms of displacements by replacing the forces and moments in the equations of statics (3.7) by the deformations, expressed in terms of the projections of **u**,

$$\tilde{\kappa}_2=\frac{Em^2h}{m^2-1}\Big(\varepsilon_1+\frac{\varepsilon_2}{m}\Big),\ \tilde{\varepsilon}_2=-\frac{Em^2h^3}{12(m^2-1)}\Big(\kappa_1+\frac{\kappa_2}{m}\Big),\ (1\to 2),$$
$$(3.11)\qquad \tilde{\omega}=-\frac{Em^2h}{m^2-1}\cdot\frac{m-1}{2m}\gamma,\ \tilde{\gamma}=\frac{Em^2h^3}{12(m^2-1)}\cdot\frac{2(m-1)}{m}\omega.$$

We obtain

$$(3.12)\quad L_k\Big(\varepsilon_1+\frac{\varepsilon_2}{m},\ \varepsilon_2+\frac{\varepsilon_1}{m},\ \frac{m-1}{2m}\gamma,\ -\frac{h^2}{12}\Big(\kappa_1+\frac{\kappa_2}{m}\Big),$$
$$-\frac{h^2}{12}\Big(\kappa_2+\frac{\kappa_1}{m}\Big),\ -\frac{m-1}{m}\frac{h^2}{12}\omega\Big)=0,\quad (k=1,2,3).$$

The groups of equations (3.10) and (3.12) go over into each other after replacement of Poisson's number m by $-m$. It is not known to the author whether this simple consequence of the static geometric analogue has been noted previously.

§ 4. We will still derive one property of the static geometric analogue in an attempt to clarify the mechanical meaning of the vector of the

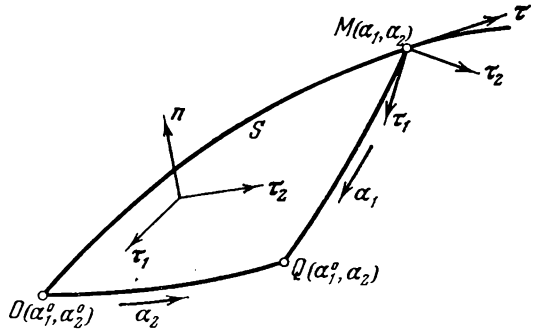

Fig. 1.

stress functions. For this purpose we will evaluate the resultant vector $\mathbf{V}_{OM}$ and the resultant moment $\mathbf{m}^C_{OM}$ about an arbitrary point C of the forces and moments distributed along the curve OM (Fig. 1); its origin O is fixed, while M is a moving end point. We will assume (this is, however, inessential) that the curve OM is composed of arc sections of coordinate lines; the point M may be situated either on the same coordinate line $\alpha_1 = \alpha_1^0$ as O, or on the coordinate line $\alpha_2 = \text{const}$. We will present the evaluation for the second case, as being more general, where Q denotes the point of intersection of the above coordinate lines. We have, from (3.1),

$$\mathbf{V}_{OM} = \int_O^Q \mathbf{K}_1 A_2\, d\alpha_2 + \int_M^Q \mathbf{K}_2 A_1\, d\alpha_1 = \int_O^Q \frac{\partial \mathbf{R}}{\partial \alpha_2}\, d\alpha_2 + \int_Q^M \frac{\partial \mathbf{R}}{\partial \alpha_1}\, d\alpha_1,$$

and further, from (3.1) and (3.2),

$$\mathbf{m}^C_{OM} = \int_O^Q (\mathbf{M}_1 + \mathbf{r} \times \mathbf{K}_1) A_2\, d\alpha_2 + \int_M^Q (\mathbf{M}_2 + \mathbf{r} \times \mathbf{K}_2) A_1\, d\alpha_1 =$$

$$= \int_O^Q \frac{\partial}{\partial \alpha_2} (\mathbf{B} + \mathbf{r} \times \mathbf{R})\, d\alpha_2 + \int_Q^M \frac{\partial}{\partial \alpha_1} (\mathbf{B} + \mathbf{r} \times \mathbf{R})\, d\alpha_1.$$

Thus

$$\text{(4.1)} \qquad \mathbf{V}_{OM} = \mathbf{R}_M - \mathbf{R}_0,\ \mathbf{m}^C_{OM} = (\mathbf{B} + \mathbf{r} \times \mathbf{R})_M - (\mathbf{B} + \mathbf{r} \times \mathbf{R})_0.$$

The forces and moments will not change when the vector stress functions $\mathbf{B}$ and $\mathbf{R}$ are augmented by the terms

$$\text{(4.2)} \qquad \mathbf{B}_0 + \mathbf{r} \times \mathbf{R}_0,\ \mathbf{R}_0,$$

where $\mathbf{B}_0$ and $\mathbf{R}_0$ are constant vectors. Correspondingly, the addition to the vector $\mathbf{u}$ of a small displacement of the shell as a rigid body is not accompanied by a deformation.

Thus, the formulae (4.1), by aligning the moment center with the point M, may likewise be written in the form

$$\text{(4.3)} \qquad \mathbf{V}_{OM} = \mathbf{R}_M,\ \mathbf{m}^M_{OM} = \mathbf{B}_M.$$

It is also clear that the curve OQM may be replaced by any curve S on the middle surface which contains the points O and M, since the part $OQMSO$ of the shell is in equilibrium. We note further that the projection of the resultant vector $\mathbf{V}_{OM}$ on the tangent $\boldsymbol{\tau}$ to the curve S at the point M (in the direction of increasing arc, measured from O to M) is also

determined by the vector **B**. In fact, we have by (3.5)

$$\mathbf{V}_{OM}\cdot\boldsymbol{\tau} = \tilde{\vartheta}_2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau} - \tilde{\vartheta}_1\boldsymbol{\tau}_2\cdot\boldsymbol{\tau} = \tilde{\vartheta}_M, \tag{4.4}$$

where $\tilde{\vartheta}_i$ is determined by (3.4). We note that the analogous quantity

$$\vartheta_2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau} - \vartheta_1\boldsymbol{\tau}_2\cdot\boldsymbol{\tau} = \vartheta_M \tag{4.5}$$

represents the angle of rotation of the edge of the shell, formed by the curve S (measured about $\boldsymbol{\tau}$).

Thus, the vector stress function **B** is equal to the resultant moment about the end point M of the forces and moments distributed along the section of the curve S which connects a fixed point with M. The projection on the tangent to the curve S at the point M of the resultant vector of these forces is given by (4.4) which is completely determined by the projection of the vector **B** [5].

The second boundary problem of shell theory is concerned with the determination of the displacement vector from the system of differential equations (3.12) when at each point M of the edge the values of this vector and of the angle of rotation

$$\mathbf{u} = \mathbf{u}_M, \ \vartheta = \vartheta_M \tag{4.6}$$

are given.

The first boundary problem is concerned with the same system of differential equations (with m replaced by $-m$) and the same formulation of the boundary conditions. Therefore the solution of one of the problems automatically leads to that of the other. The simplest example illustrating this property of the static geometric analogue is represented by the link between the plane problem and the bending problem of plates, mentioned in § 1.

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ON THE PROBLEM OF THE INFLUENCE OF CAPILLARITY ON THE FLOW OF LIQUID IN A POROUS MEDIUM TO A RECTILINEAR SLOPE OF A POROUS MASS

G. K. MIKHAILOV
Moscow, U.S.S.R.

§ 1. In the present study we will assume that the motion of the liquid takes place in accordance with Darcy's law for full saturation everywhere in the homogeneous porous medium. Under these conditions the discharge velocity is known to have the potential

$$\varphi = -\kappa\left(\frac{p}{\gamma} + y\right) + \text{const.},$$

where κ is the coefficient of permeability (a physical characteristic of the medium and of the moving liquid), p is the pressure, γ is the weight per unit volume of liquid, y is a coordinate, directed vertically upward. We will consider two dimensional flow in the vertical plane.

We introduce the complex potential $f = \varphi + i\psi$, the complex coordinate of the flow region $z = x + iy$ and the complex discharge velocity $w = df/dz = u - iv$; φ and ψ are harmonic functions.

At first we will make several remarks regarding the flow of a liquid to the inclined slope 1′–4 on the impermeable basis 1–4 (Fig. 1). Obviously, on the boundaries of the flow, we will have conditions in the form:

on 1–2: $\psi = Q, \quad p = -\gamma h_c;$

on 2–3: $\psi = Q, \quad y = -x \tan \alpha\pi;$

on 3–4: $p = 0, \quad y = -x \tan \alpha\pi;$

on 4–1: $\psi = 0, \quad y = x \tan \beta\pi.$

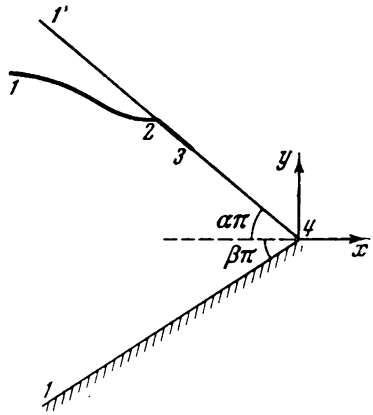

Fig. 1

Here Q is the discharge (per unit length) flowing to the slope, h_c is the height of capillary rise of the liquid in the porous medium. For the sake of simplicity, we consider the flow to the slope from infinity, digressing from the con-

ditions on the boundary on which the liquid enters the mass of the porous medium.[1]

The formulated problem is mainly of practical interest for the case $\beta = 0$, but in this particular case its solution encounters a difficulty. Therefore we will analyze certain particular cases.

First of all, we will consider the magnitude of the ratio ν of the length l of the seepage segment 3–4 to the intensity of the seepage $q = Q/\kappa$. For $\beta = 0$ and $h_c = 0$, S. V. Fal'kovich [1] in a development of a more general solution of P. Ia. Polubarinova-Kochina found

$$\nu_0' = \frac{4}{\pi^2} \int_0^{\pi/2} (\cot t)^{2\alpha-1} \log \cot \frac{t}{2}\, dt. \tag{1.1}$$

From the solution of V. V. Vedernikov, corrected by Iu. D. Sokolov [2], we obtain, for the flow to a triangular canal ($\beta = \frac{1}{2}$) in the absence of capillarity ($h_c = 0$), the formula

$$\nu_0'' = \frac{\Gamma(\alpha) - \sqrt{\pi}\,\Gamma(\alpha + \frac{1}{2})}{\sqrt{\pi}\,\Gamma(\alpha + \frac{1}{2}) \cos \alpha\pi}. \tag{1.2}$$

In the case $\beta = 0$ and $h_c \to \infty$, we have in accordance with B. K. Rizenkampf's work [3]

$$\nu_\infty' = \frac{\Gamma(\alpha)}{\sqrt{\pi}\,\Gamma(\alpha + \frac{1}{2}) \sin \alpha\pi}. \tag{1.3}$$

Finally, it is not difficult to obtain for the case $\beta = \frac{1}{2}$ and $h_c \to \infty$ by use of B. K. Rizenkampf's method the formula

$$\nu_\infty'' = \frac{\Gamma(\alpha + \frac{1}{2})}{\sqrt{\pi}\,\Gamma(\alpha + 1) \sin \alpha\pi}. \tag{1.4}$$

In the formulae (1.1)–(1.4) for ν the subscripts 0 and ∞ correspond to the values $h_c = 0$ and $h_c \to \infty$ respectively; the superscripts refer to the value of β (0 or $\frac{1}{2}$).

We will now compare the dependence of $\eta \equiv \nu_\infty$: $\nu_0 = \eta(\alpha)$ for the case $\beta = 0$ (′) and $\beta = \frac{1}{2}$ (″). As α changes from 0 to 1 the ratio η''/η' varies only from 1 to 1.081. This shows that the angle β practically does not affect the character of the function $\eta(\alpha)$. Therefore, when deducing a first estimate of the influence of capillarity on the magnitude of the

[1] The fact that it is unnecessary to take the actual shape of the upstream part of an earth dam into account in a study of the flow of water to the lower slope has been remarked upon in many papers (cf., for example, [4]).

seepage segment, it is sufficient to study the problem for any value of β. We will choose here the value $\beta = \frac{1}{2}$. The dependence of $1/\eta''$ on α is shown in Fig. 2.

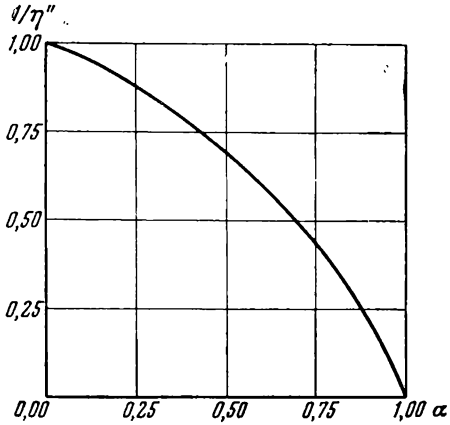

Fig. 2

We consider in the following sections the case of flow of liquid to a vertical slope ($\alpha = \beta = \frac{1}{2}$) for an arbitrary magnitude of the height of capillary rise.

§ 2. In the particular case $\alpha = \beta = \frac{1}{2}$ the problem formulated, according to the classification of B. K. Rizenkampf [5], concerns the sixth type of problem, solvable with the help of the method of conformal mapping. This case is the only one for which the problem of the influence of the magnitude of the seepage segment may be solved in closed form [6] with consideration being given to capillarity.

Let the region of flow 1–2–3–4–1 be mapped onto the upper half-plane of the auxiliary complex variable ζ, with the following correspondence of points: $\zeta_2 = -\zeta_3 = 1$, $\zeta_1 = -\zeta_4 = 1/k$, where k is a parameter which must be found from the solution. We introduce the Zhukovskii function $G = f - i\kappa z$. For the function G we have on the boundaries of the flow region the conditions

$$\left.\begin{array}{ll} \text{on 1–2:} & \operatorname{Re} G = h_c, \\ \text{on 2–3:} & \operatorname{Im} G = q, \\ \text{on 3–4:} & \operatorname{Re} G = 0, \\ \text{on 4–1:} & \operatorname{Im} G = 0. \end{array}\right\} \tag{2.1}$$

We conclude then from the obvious conditions

$$\left.\frac{\partial \psi}{\partial y}\right|_{3-4} > 0, \quad \left. v \right|_{2-3} > -\kappa, \tag{2.2}$$

that the region of motion in the plane of the function G represents an irregular quadrangle, since it does not contain a cut at the point 3. Hence we obtain

$$G = iq \frac{E(\theta, k') - kF(\theta, k') - k\sqrt{\zeta^2 - 1}\sqrt{\frac{1 + kz}{1 - kz}}}{E(k') - kK(k')}, \tag{2.3}$$

$$\sin\theta = \sqrt{\frac{1 - k^2\zeta^2}{1 - k^2}}, \quad k' = \sqrt{1 - k^2},$$

where $F(\theta, k')$ and $E(\theta, k')$ are elliptic integrals of the first and second kind with modulus k'; $K(k')$ and $E(k')$ are complete elliptic integrals. The modulus k is determined by the equation

$$\frac{h_c}{q} = 2\frac{E(k) - (1 - k)K(k)}{E(k') - kK(k')}, \tag{2.4}$$

obtained from the boundary conditions (2.1).

Further, we consider the function

$$\frac{dz}{dG} = \frac{1}{\frac{df}{dz} - i\kappa}. \tag{2.5}$$

We obtain for this function by a study of the region of the velocity hodograph the following boundary conditions

$$\left.\begin{aligned} &\text{on 1–2:} \quad \operatorname{Im} dz/dG = 1/\kappa, \\ &\text{on 2–3:} \quad \operatorname{Re} dz/dG = 0, \\ &\text{on 3–4:} \quad \operatorname{Im} dz/dG = 0, \\ &\text{on 4–1:} \quad \operatorname{Re} dz/dG = 0. \end{aligned}\right\} \tag{2.6}$$

Thus, in the plane of the variable dz/dG, the flow region represents a polygon with the five corners 1–m–2–3–4–1, where the angle of the corner m is 2π. Let the point m, at which the curved free surface 1–2 has the largest slope, correspond to the point $\zeta_m = \mu$ $(1 \leq \mu \leq 1/\kappa)$ in the ζ plane. Then one obtains readily for the function dz/dG, taking into consideration (2.6), the expression

$$\frac{dz}{dG} = \frac{E(k') - k^2K(k')}{\pi} F(\theta, k') + \frac{K(k')}{\pi}\left[k^2F(\theta, k') - E(\theta, k') + \right. \tag{2.7}$$

$$\left. + k\sqrt{\frac{(\zeta^2 - 1)(1 + k\zeta)}{1 - k\zeta}} + (1 + k)\sqrt{\frac{1 - \zeta}{-1 - \zeta}\frac{1 + k\zeta}{1 - k\zeta}}\right],$$

where

$$\mu = \frac{E(k') - K(k')}{E(k') - k^2K(k')}. \tag{2.8}$$

These representations of the functions G and dz/dG are convenient on the segment 3–4 of the boundary of the flow region. For the segment 2–3, we obtain correspondingly

$$G = iq + \frac{h_c}{E(k) - (1-k)K(k)}\left[E(\omega, k) - (1-k)F(\omega, k) - \right.$$

$$\left. - k\sqrt{\frac{1-\zeta^2}{1-k^2\zeta^2}} + \frac{k'}{2}\sqrt{\frac{1-\zeta}{1+\zeta}} - \frac{1-k}{2}\sqrt{\frac{1-\zeta}{1+\zeta}\frac{1-k\zeta}{1+k\zeta}}\right], \tag{2.9}$$

$$\sin\omega = \frac{\sqrt{2(1+\zeta)}}{\sqrt{(1+k)(1+k\zeta)} + \sqrt{(1-k)(1-k\zeta)}}$$

and

$$\frac{dz}{dG} = i + \frac{2i}{\pi}[K(k') - E(k')] - \frac{2iK(k')}{E(k') - k^2K(k')}\left[E(\omega, k) + \right.$$

$$+ \frac{k'}{2}\sqrt{\frac{1-\zeta}{1+\zeta}} - \frac{1-k}{2}\sqrt{\frac{1-\zeta}{1+\zeta}\frac{1-k\zeta}{1+k\zeta}} - \tag{2.10}$$

$$\left. - \frac{1+k}{2}\sqrt{\frac{1+\zeta}{1-\zeta}\frac{1-k\zeta}{1+k\zeta}}\right].$$

The final solution of the problem may be presented in the form

$$f = G(\zeta) + i\kappa z(\zeta),$$

$$z = \int_{-1/\kappa}^{\zeta} \frac{dz}{dG}(\zeta)\frac{dG(\zeta)}{d\zeta}\,d\zeta. \tag{2.11}$$

§ 3. The magnitude h of the seepage segment 3–4 and the magnitude Δ of the segment 2–3, which are of interest to us, are determined in accordance with the formulae (2.11) in the form

$$\frac{h}{q} = \frac{1-k}{\pi}\frac{K(k')}{E(k') - kK(k')}\left[\log\frac{2}{1+k} - \varepsilon\int_0^1 \frac{Z(\theta, k')dt}{\sqrt{1-\varepsilon^2t^2}}\right],$$

$$\varepsilon = \frac{1-k}{1+k}, \quad \sin\theta = \frac{2t}{(1+k)(1+\varepsilon t^2)} \tag{3.1}$$

and

$$(3.2)\quad \frac{\Delta - h_c}{q} = \frac{k(1-k)}{E(k') - kK(k')}\left\{\frac{2K(k')}{\pi\varepsilon}\left[1 - \frac{2\log(1+k)}{1+k}\right] - \right.$$
$$\left. - \int_{-1}^{+1} \frac{\Lambda_0(\omega, k')\sqrt{1+\zeta}\, d\zeta}{\sqrt{1-\zeta}\sqrt{1-k^2\zeta^2}\,(1-k\zeta)}\right\},$$

where

$$(3.3)\quad Z(\theta, k') = \frac{k'^2\sqrt{1-k'^2\sin^2\theta}\sin 2\theta}{2K(k')}\int_0^{K(k')} \frac{\mathrm{sn}^2 u\, du}{1-k'^2\sin^2\theta\,\mathrm{sn}^2 u}$$

is the Jacobi function and

$$(3.4)\quad \Lambda_0(\omega, k') = \frac{k^2\sin 2\omega}{\pi\sqrt{1-k^2\sin^2\omega}}\int_0^{K(k')} \frac{du}{1 - \dfrac{k'^2\,\mathrm{sn}^2 u}{1-k^2\sin^2\omega}}$$

is the Heumann function [7].

Results calculated from the formulae (3.1) and (3.2) are shown in Fig. 3, where all quantities refer to the dimension h_0 of the magnitude of the seepage segment, evaluated for $h_c = 0$ and given discharge in accordance with (1.2).

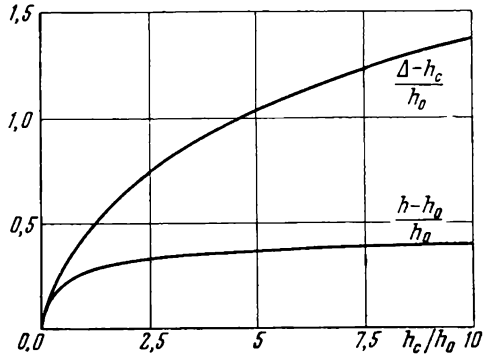

Fig. 3

Attention must still be directed at the rapid growth of the magnitude of the moistened part of the slope $h + \Delta$ even for an increase of the height of capillary rise h_c, insignificant compared with zero. This had to be anticipated, since the theoretical free surface curve for the flow

to the vertical slope in the absence of capillarity has infinite curvature at the outlet point.

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ON THE EQUATIONS OF MOTION OF PIEZOELECTRIC CRYSTALS

R. D. MINDLIN
New York, USA

§ 1. Introduction

In this paper, a systematic derivation is given of the linear equations governing small vibrations of piezoelectric crystals, including the coupling between mechanical, electrical and thermal fields. The treatment of the thermal variables is based on Biot's formulation of irreversible thermodynamics of continua [1]. Constitutive equations and equations of motion are established and a solution of the latter is exhibited, illustrating the attenuation of plane waves. In addition, a proof of a theorem of uniqueness of solution of the equations of motion is indicated and appropriate boundary conditions are enumerated.

§ 2. Cauchy's and Maxwell's equations

We take, as already established, the following definitions and equations.

a. The definition of small strain, S_{ij} $(i, j = 1, 2, 3)$, and its expression in terms of displacement:

$$2S_{ij} = u_{i,j} + u_{j,i}. \tag{2.1}$$

b. The definition of stress, T_{ij}, in terms of traction, t_i, across a surface whose unit outward normal is $\boldsymbol{\nu}$:

$$\nu_i T_{ij} = t_j. \tag{2.2}$$

c. The stress-equations of motion, omitting body forces,

$$T_{ij,i} = \rho \ddot{u}_j, \quad T_{ij} = T_{ji}. \tag{2.3}$$

d. The equations of the electromagnetic field:

$$\varepsilon_{ijk} H_{k,j} = \dot{D}_i + J_i, \quad \varepsilon_{ijk} E_{j,k} = \dot{B}_i. \tag{2.4}$$

As we shall be concerned with non-conductors at frequencies far below optical frequencies, the conduction current, J_i, and rate of change of magnetic induction, $\dot{B}_i$, may be neglected and (2.4) reduce to the equations of the quasi-stationary electric field:

$$\dot{D}_{i,i} = 0, \quad E_i = -\phi_{,i} \tag{2.5}$$

where D_i, E_i and ϕ are the electric displacement, field and potential, respectively.

§ 3. Energetic and thermodynamic considerations

We begin with the principle of conservation of energy in the form: In any portion V of a body, bounded by a surface S with unit outward normal $\boldsymbol{\nu}$, the rate of increase of energy is equal to the rate at which work is done by the tractions across S less the flux of electrical and thermal energy outward across S. Thus,

$$\int_V (\dot{K} + \dot{U})\, dV = \int_S \nu_i (T_{ij}\dot{u}_j - \phi \dot{D}_i - \dot{q}_i)\, dS, \tag{3.1}$$

where

$$K = \tfrac{1}{2}\rho \dot{u}_i \dot{u}_i \tag{3.2}$$

is the kinetic energy-density, U is the internal energy-density, $\dot{q}_i$ is the heat flux vector and $\phi \dot{D}_i$ is the form taken by the Poynting vector $\varepsilon_{ijk} E_j H_k$ in a quasi-stationary electric field.

Applying the divergence theorem to the right hand side of (3.1) and using (2.1), (2.3), (2.5) and (3.2), we have

$$\int_V \dot{U}\, dV = \int_V (T_{ij}\dot{S}_{ij} + E_i \dot{D}_i - \dot{q}_{i,i})\, dV. \tag{3.3}$$

Since this must hold for any V,

$$\dot{U} = T_{ij}\dot{S}_{ij} + E_i \dot{D}_i - \dot{q}_{i,i}, \tag{3.4}$$

which is the first law of thermodynamics for the piezoelectric medium.

By the definition of entropy density, σ, we have

$$\dot{q}_{i,i} = -\Theta \dot{\sigma}, \tag{3.5}$$

where Θ is the absolute temperature. Hence

$$\dot{U} = T_{ij}\dot{S}_{ij} + E_i \dot{D}_i + \Theta \dot{\sigma}. \tag{3.6}$$

Now, let

$$\Theta = \Theta_0 + \theta, \tag{3.7}$$

where Θ_0 is a constant, positive, reference temperature.

Biot's generalized free energy is defined as

$$B = U - \Theta_0\sigma \tag{3.8}$$

instead of the usual free energy $U - \Theta\sigma$. Hence

$$\dot{B} = T_{ij}\dot{S}_{ij} + E_i\dot{D}_i + \theta\dot{\sigma}. \tag{3.9}$$

At this point it is convenient to introduce the abbreviated indicial notation in which a pair of indices, ij or kl, is replaced by a single index p or q according to the scheme

ij or kl	11	22	33	23,32	31,13	12,21
p or q	1	2	3	4	5	6

and, for the strains only,

$$S_p = \begin{cases} S_{ij}, & i = j, \ (p = 1, 2, 3), \\ 2S_{ij}, & i = j, \ (p = 4, 5, 6). \end{cases}$$

Then (3.9) becomes

$$\dot{B} = T_p\dot{S}_p + E_i\dot{D}_i + \theta\dot{\sigma}. \tag{3.10}$$

The independent variables of Biot's potential are taken to be the strain, electric displacement and entropy

$$B = B(S_p, D_i, \sigma).$$

Hence

$$\dot{B} = \frac{\partial B}{\partial S_p}\dot{S}_p + \frac{\partial B}{\partial D_i}\dot{D}_i + \frac{\partial B}{\partial \sigma}\dot{\sigma}. \tag{3.11}$$

Then, from (3.10) and (3.11),

$$T_p = \frac{\partial B}{\partial S_p}, \quad E_i = \frac{\partial B}{\partial D_i}, \quad \theta = \frac{\partial B}{\partial \sigma}. \tag{3.12}$$

The second law of thermodynamics may be written as [2]

$$\dot{\sigma} + (\dot{q}_i/\Theta)_{,i} \geq 0, \tag{3.13}$$

and this becomes, with (3.5),

$$-(\Theta_{,i}\dot{q}_i/\Theta^2) \geq 0. \tag{3.14}$$

Equation (3.14) is satisfied by the Fourier law of heat conduction

$$-\dot{q}_i = \kappa_{ij}\Theta_{,j} = \kappa_{ij}\theta_{,j}, \tag{3.15}$$

if $\kappa_{ij}\Theta_{,i}\Theta_{,j}$ is a positive definite quadratic form. By Onsager's principle, the heat conduction coefficient, κ_{ij}, is symmetric; and we then require that the three discriminants of $\kappa_{ij}\Theta_{,i}\Theta_{,j}$, obtained by omission of none, one and two of the $\Theta_{,i}$ be greater than zero, i.e.,

$$\text{(3.16)} \qquad \begin{vmatrix} \kappa_{11} & \kappa_{12} & \kappa_{13} \\ \kappa_{21} & \kappa_{22} & \kappa_{23} \\ \kappa_{31} & \kappa_{32} & \kappa_{33} \end{vmatrix} > 0, \quad \begin{vmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{vmatrix} > 0, \quad \kappa_{11} > 0.$$

Now, rewrite (3.5) as

$$\dot{\sigma} = -\frac{\dot{q}_{i,i}}{\Theta_0 + \theta} = -\left(1 - \frac{\theta}{\Theta_0} + \frac{\theta^2}{\Theta_0{}^2} - \cdots\right)\frac{\dot{q}_{i,i}}{\Theta_0}.$$

Hence, if the change of temperature from the reference temperature is small

$$\text{(3.17)} \qquad \theta \ll \Theta_0,$$

we have, approximately,

$$\text{(3.18)} \qquad \dot{q}_{i,i} = -\Theta_0\dot{\sigma}$$

which, with (3.15), gives

$$\text{(3.19)} \qquad \kappa_{ij}\theta_{,ij} = \Theta_0\dot{\sigma}.$$

Also, from (3.18),

$$\theta\dot{\sigma} = -\frac{\theta\dot{q}_{i,i}}{\Theta_0} = -\left(\frac{\theta\dot{q}_i}{\Theta_0}\right)_{,i} + \frac{\theta_{,i}\dot{q}_i}{\Theta_0},$$

and this becomes, with (3.15),

$$\text{(3.20)} \qquad \theta\dot{\sigma} = -\left(\frac{\theta q_i}{\Theta_0}\right)_{,i} - \kappa_{ij}^{-1}\Theta_0\left(\frac{\dot{q}_i\dot{q}_j}{\Theta_0{}^2}\right),$$

where κ_{ij}^{-1}, the reciprocal of κ_{ij}, is the heat resistance coefficient.

We now introduce Biot's definitions of entropy displacement

$$\text{(3.21)} \qquad s_i = q_i/\Theta_0$$

and dissipation function

$$\text{(3.22)} \qquad F = \tfrac{1}{2}\Theta_0\kappa_{ij}^{-1}\dot{s}_i\dot{s}_j,$$

so that (3.20) becomes

$$\text{(3.23)} \qquad \theta\dot{\sigma} = -(\theta\dot{s}_i)_{,i} - 2F.$$

With this result, we can rewrite the expression (3.1) of the principle

of conservation of energy in a more useful form. The thermal flux term in (3.1) is

$$\int_S \nu_i \dot{q}_i \, dS = \int_V \dot{q}_{i,i} \, dV =$$

$$= -\int_V (\Theta_0 \dot{\sigma} + \theta \dot{\sigma}) \, dV =$$

$$= -\int_V [\Theta_0 \dot{\sigma} - (\theta \dot{s}_i)_{,i} - 2F] \, dV =$$

$$= \int_V (2F - \Theta_0 \dot{\sigma}) \, dV + \int_S \nu_i \theta \dot{s}_i \, dS =$$

$$= \int_V (2F + \dot{B} - \dot{U}) \, dV + \int_S \nu_i \theta \dot{s}_i \, dS.$$

Upon reinsertion of this last expression into (3.1), we have, finally,

$$\int_V (\dot{K} + \dot{B} + 2F) \, dV = \int_S \nu_i (T_{ij} \dot{u}_j - \phi \dot{D}_i - \theta \dot{s}_i) \, dS. \tag{3.24}$$

If the time derivatives were replaced by variations, (3.24) would be Biot's thermoelastic variational principle extended to include the electric field.

§ 4. Constitutive equations

The form of Biot's potential which leads to the usual linear constitutive equations is

$$B = \tfrac{1}{2} c_{ijkl}^{D\sigma} S_{ij} S_{kl} + \tfrac{1}{2} \beta_{ij}^{\sigma S} D_i D_j + \tfrac{1}{2} b^{SD} \sigma^2 - h_{kij}^{\sigma} D_k S_{ij} - q_i^S \sigma D_i - \gamma_{ij}^D \sigma S_{ij}.$$

$$c_{ijkl}^{D\sigma} = c_{jikl}^{D\sigma} = c_{ijlk}^{D\sigma} = c_{klij}^{D\sigma}, \; \beta_{ij}^{\sigma S} = \beta_{ji}^{\sigma S}, \; h_{kij}^{\sigma} = h_{kji}^{\sigma}, \; \gamma_{ij}^{D} = \gamma_{ji}^{D},$$

or

$$B = \tfrac{1}{2} c_{pq}^{D\sigma} S_p S_q + \tfrac{1}{2} \beta_{ij}^{\sigma S} D_i D_j + \tfrac{1}{2} b^{SD} \sigma^2 - \\ - h_{ip}^{\sigma} D_i S_p - q_i^S \sigma D_i - \gamma_p^D \sigma S_p, \tag{4.1}$$

in which the symbols are those employed by Mason [3] with the exception of

$$b^{SD} = \Theta_0 / \rho C^{SD},$$

where C^{SD} is the specific heat at constant volume and constant electric displacement [1]. Then, from (3.12),

$$
\begin{aligned}
T_p &= c_{pq}^{D\sigma}\, S_q - h_{ip}^{\sigma}\, D_i - \gamma_p{}^{D}\, \sigma, \\
E_j &= -\, h_{jq}^{\sigma}\, S_q + \beta_{ij}^{\sigma S}\, D_i - q_j{}^{S}\, \sigma, \\
\theta &= -\, \gamma_q{}^{D}\, S_q - q_i{}^{S}\, D_i + b^{SD}\, \sigma.
\end{aligned}
\tag{4.2}
$$

These relations may be written in eight forms, all of which are represented on the octahedron shown in Figure 1. For example, interchanging

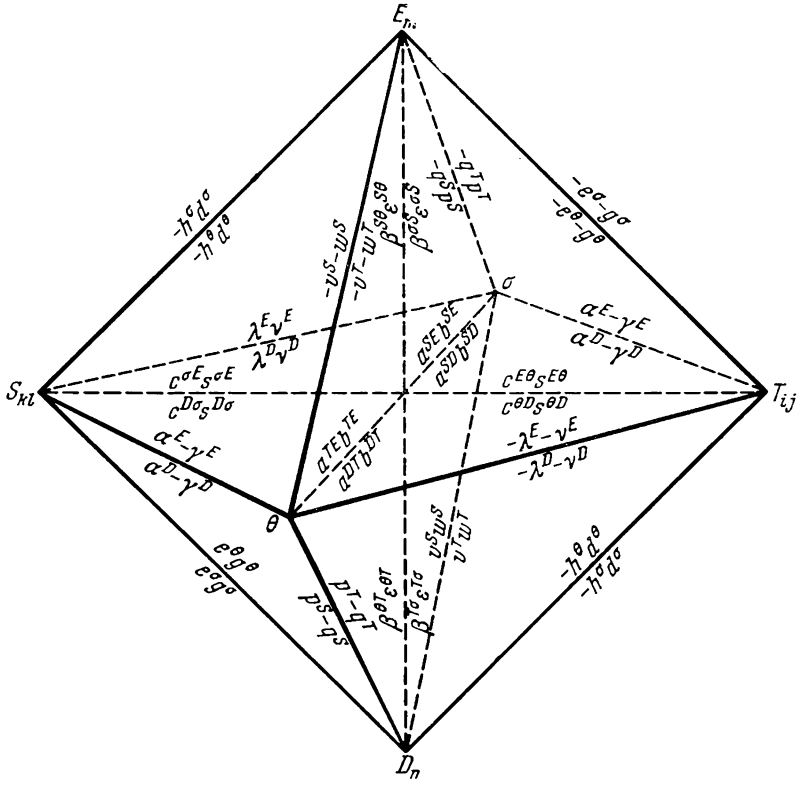

Fig. 1

the positions of θ and σ, (4.2) become

$$
\begin{aligned}
T_p &= c_{pq}^{D\theta}\, S_q - h_{ip}^{\theta}\, D_i - \lambda_p{}^{D}\, \theta, \\
E_j &= -\, h_{jq}^{\theta}\, S_q + \beta_{ij}^{\theta S}\, D_i - v_j{}^{S}\, \theta, \\
\sigma &= \lambda_q{}^{D}\, S_q + v_i{}^{S}\, D_i + a^{SD}\, \theta,
\end{aligned}
$$

[1] The symbol for the pyroelectric constant, $q_i{}^{s}$, should not be confused with the symbol for the heat vector q_i.

where the isothermal constants are expressed in terms of the adiabatic constants by

$$c_{pq}^{D\theta} = c_{pq}^{D\sigma} - a^{SD}\,\gamma_p{}^D\,\gamma_q{}^D, \qquad a^{SD}b^{SD} = 1,$$
$$h_{ip}^{\theta} = h_{ip}^{\sigma} + a^{SD}\,q_i{}^S\,\gamma_p{}^D, \qquad \lambda_p{}^D = a^{SD}\,\gamma_p{}^D,$$
$$\beta_{ij}^{\theta S} = \beta_{ij}^{\sigma S} - a^{SD}\,q_i{}^S\,q_j{}^S, \qquad v_i{}^S = a^{SD}\,q_i{}^S.$$

A further interchange, of E_i and D_i, gives the form convenient for the establishment of equations of motion

$$T_p = c_{pq}^{E\theta}\,S_q - e_{ip}^{\theta}\,E_i - \lambda_p{}^E\,\theta,$$
$$D_j = e_{jq}^{\theta}\,S_q + \varepsilon_{ij}^{\theta S}\,E_i + p_j{}^S\,\theta, \tag{4.4}$$
$$\sigma = \lambda_q{}^E\,S_q + p_i{}^S\,E_i + a^{SE}\,\theta,$$

where the new constants are given in terms of the preceding ones by

$$c_{pq}^{E\theta} = c_{pq}^{D\theta} - h_{ip}^{\theta}\,e_{iq}^{\theta}, \qquad \varepsilon_{ij}^{\theta S}\,\beta_{ij}^{\theta S} = \delta_{ij},$$
$$\lambda_p{}^E = \lambda_p{}^D + v_i{}^S\,e_{ip}^{\theta}, \qquad e_{ip}^{\theta} = \varepsilon_{ij}^{\theta S}\,h_{jp}^{\theta},$$
$$a^{SE} = a^{SD} + p_i{}^S\,v_i{}^S, \qquad p_i{}^S = \varepsilon_{ij}^{\theta S}\,v_j{}^S.$$

§ 5. Equations of motion and boundary conditions

To obtain the equations of motion, we replace the indices p and q in (4.4) by ij and kl and substitute in (2.3), (2.5) and (3.19), with the result

$$c_{ijkl}^{E\theta}\,u_{k,li} + e_{kij}^{\theta}\,\phi_{,ki} - \lambda_{ij}^{E}\,\theta_{,i} = \rho\ddot{u}_j,$$
$$e_{kij}^{\theta}\,u_{i,jk} - \varepsilon_{ij}^{\theta S}\,\phi_{,ij} + p_i{}^S\,\theta_{,i} = 0, \tag{5.1}$$
$$\lambda_{ij}^{E}\,\dot{u}_{i,j} - p_i{}^S\,\dot{\phi}_{,i} + a^{SE}\,\dot{\theta} = \Theta_0{}^{-1}\kappa_{ij}\,\theta_{,ij}.$$

Initial and boundary conditions sufficient for a unique solution of these equations may be established readily with the aid of the form of the principle of conservation of energy given in (3.24) in which S is now to be interpreted as the boundary of the body. The kinetic energy-density K is positive definite and, by (3.22) and (3.16), so is the dissipation function F. It is necessary, for stability, to require Biot's potential B to be positive definite and this may be done by specifying that its ten discriminants, obtained by omission of $0,1\cdots 9$ of its ten variables (six T_p or S_p three E_i or D_i and θ or σ), be greater than zero. Then the theorem of uniqueness and its proof may follow those given by J. Weiner [4] for the isotropic, thermoelastic case, requiring only the addition of the

electrical terms. For the boundary conditions, inspection of the right hand side of (3.24) shows that at each point of the boundary S, where the unit normal is $\boldsymbol{\nu}$, any one of four mechanical conditions and either of two electrical conditions and either of two thermal conditions are sufficient as follows:

I_M : u_i (displacement),
or II_M: $\nu_i T_{ij}$ (traction),

or III_M: $\mu_i u_i$ and $\nu_j T_{ij} - \mu_i \mu_j \nu_k T_{jk}$ (the component of displacement in any direction $\boldsymbol{\mu}$ and the resultant traction in the plane perpendicular to $\boldsymbol{\mu}$),

or IV_M: $\nu_i \mu_j T_{ij}$ and $u_i - \mu_i \mu_j u_j$ (the component of traction in any direction $\boldsymbol{\mu}$ and the resultant displacement in the plane perpendicular to $\boldsymbol{\mu}$),

and I_E: ϕ (surface potential),
or II_E: $\nu_i D_i$ (surface charge)
and I_T: θ (surface temperature change),
or II_T: $\nu_i s_i$ (heat flux across surface).

In addition, conditions may be included such as the thermal radiation condition or elastic supports or the connection of the crystal to an electric circuit, if the appropriate energies are added to (3.24).

§ 6. Attenuation of plane waves

As an example of solutions of the equations of motion, we consider the case of plane waves in an infinite crystal. There is no loss of generality by taking a wave with its normal in the direction of the axis x_1

$$
\begin{aligned}
u_i &= A_i \cos \xi x_1 e^{i\omega t}, \quad i = 1, 2, 3, \\
\phi &= A_4 \cos \xi x_1 e^{i\omega t}, \qquad (6.1) \\
\theta &= A_5 \sin \xi x_1 e^{i\omega t}.
\end{aligned}
$$

Substituting (6.1) into (5.1) and setting the determinant of the coefficients of the constants A equal to zero, we obtain the secular equation

$$
\begin{vmatrix}
c_{11}^{E\theta} - c & c_{16}^{E\theta} & c_{15}^{E\theta} & e_{11}^{\theta} & \lambda_1^E \\
c_{16}^{E\theta} & c_{66}^{E\theta} - c & c_{56}^{E\theta} & e_{16}^{\theta} & \lambda_6^E \\
c_{15}^{E\theta} & c_{56}^{E\theta} & c_{55}^{E\theta} - c & e_{15}^{\theta} & \lambda_5^E \\
e_{11}^{\theta} & e_{16}^{\theta} & e_{15}^{\theta} & -\varepsilon_{11}^{\theta S} & -p_1^S \\
\lambda_1^E & \lambda_6^E & \lambda_5^E & -p_1^S & -\Omega
\end{vmatrix} = 0, \qquad (6.2)
$$

where

$$c = \frac{\rho\omega^2}{\xi^2}, \quad \Omega = a^{SE} + \frac{\kappa_{11}\xi^2}{i\omega\Theta_0}.$$

The roots c of (6.2) give the three complex stiffnesses.

Equation (6.2) may be condensed to the form

$$\omega\Delta^{D\theta} + i\omega_m\Delta^{D\sigma} = 0, \tag{6.3}$$

where

$$\omega_m = cC^{SE}\,\varepsilon_{11}^{\sigma S}/\kappa_{11}\,\varepsilon_{11}^{\theta S},$$

$$\Delta^{D\theta} = \begin{vmatrix} c_{11}^{D\theta} - c & c_{16}^{D\theta} & c_{15}^{D\theta} \\ c_{16}^{D\theta} & c_{66}^{D\theta} - c & c_{56}^{D\theta} \\ c_{15}^{D\theta} & c_{56}^{D\theta} & c_{55}^{D\theta} - c \end{vmatrix},$$

$$\Delta^{D\sigma} = \begin{vmatrix} c_{11}^{D\sigma} - c & c_{16}^{D\sigma} & c_{15}^{D\sigma} \\ c_{16}^{D\sigma} & c_{66}^{D\sigma} - c & c_{56}^{D\sigma} \\ c_{15}^{D\sigma} & c_{56}^{D\sigma} & c_{55}^{D\sigma} - c \end{vmatrix}.$$

Thus, $\Delta^{D\theta} = 0$ is the equation whose roots c give the three real isothermal stiffnesses, and $\Delta^{D\sigma} = 0$ is the equation whose roots give the three real adiabatic stiffnesses. Equation (6.3) shows that, as $\omega \to 0$, the stiffnesses approach their adiabatic values, and, as $\omega \to \infty$, the stiffnesses approach their isothermal values. An alternative form of (6.3) is

$$\Delta^{D\sigma} = (\Delta^{D\sigma} - \Delta^{D\theta})\frac{\omega^2 - i\omega\omega_m}{\omega^2 + \omega^2{}_m}. \tag{6.4}$$

Since the attenuation is small, the imaginary part of the quantity in parentheses may be neglected. Then the right hand side of (6.4) can be regarded as a correction on the equation for the adiabatic stiffnesses. The imaginary part gives the attenuations and has its maxima at $\omega = \omega_m$.

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ON SOME DYNAMICAL PROBLEMS OF THERMOELASTICITY

W. NOWACKI
Warsaw, Poland

§ 1. General relations and equations

Beginning in 1956, as the result of papers by M. A. Biot [1], M. Lessen [2], and P. Chadwick and I. N. Sneddon [3], a new topic of investigation in dynamical thermoelasticity was initiated which concerned the coupling between the temperature and strain fields. In 1958, P. Chadwick and I. N. Sneddon [3] made a detailed examination of the influence of thermal volume changes in a body on the form of plane harmonic waves. E. J. Lockett [4] considered the influence of temperature and strain fields on the velocity of propagation of Rayleigh surface waves. I. N. Sneddon [5] investigated the propagation of thermal stresses in thin metallic rods, produced by periodic forces and an impulse, or by heat sources situated at the end of the rod. In two papers [6, 7], H. Zorski examined the propagation of stresses in an infinite space, produced by the action of a thermal impulse.

In this paper, we shall be concerned with the propagation of elastic spherical, cylindrical and plane waves, due to the action of heat sources or centers of pressure which vary harmonically in time. We deal also with the generation of longitudinal waves in an infinite space with a spherical or cylindrical cavity, and with the propagation of thermal stresses produced by heating the plane boundary of a half-space.

The equations for a thermoelastic medium have the form

$$\mu \nabla^2 \mathbf{u} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{u} - \rho \ddot{\mathbf{u}} = \gamma_0 \operatorname{grad} \theta, \tag{1.1}$$

$$\nabla^2 \theta - \frac{1}{\kappa} \dot{\theta} + \eta \operatorname{div} \dot{\mathbf{u}} = -\frac{Q}{\kappa}, \tag{1.2}$$

where $\mathbf{u}$ is the displacement vector, θ is the temperature (under the assumption that $T + \theta$ is the absolute temperature and the state $\theta = 0$ is free of stresses and displacements), λ, μ are the Lamé constants, ρ is the density of the material and κ is the thermal conductivity. In addition, $Q = W/(\rho c)$, $\eta = \gamma_0 T/(\rho c)$, $\gamma_0 = \alpha_t(3\lambda + 2\mu)$, where W is the amount of heat generated in a unit volume of the body per unit time, c is the specific heat capacity and α_t is the coefficient of linear thermal expansion.

Introducing into the equations (1.1) and (1.2) the displacement function

$$\mathbf{u} = \operatorname{grad}\phi + \operatorname{rot}\boldsymbol{\psi}, \tag{1.3}$$

we reduce the system of equations (1.1)—(1.2) to the system of three equations

$$\nabla^2\phi - \frac{1}{c_1^2}\ddot{\phi} = \vartheta_0\theta, \tag{1.4}$$

$$\nabla^2\psi_i - \frac{1}{c_2^2}\ddot{\psi}_i = 0, \qquad (i = 1, 2, 3) \tag{1.5}$$

$$\nabla^2\theta - \frac{1}{\kappa}\dot{\theta} - \eta\nabla^2\dot{\phi} = -\frac{Q}{\kappa}. \tag{1.6}$$

In the above formulae $c_1 = \sqrt{\lambda + 2\mu}/\sqrt{\rho}$ is the velocity of propagation of the elastic longitudinal wave and $c_2 = \sqrt{\mu/\rho}$ is the velocity of the transverse wave. Moreover

$$\vartheta_0 = \frac{3\lambda + 2\mu}{\lambda + 2\mu}\alpha_t.$$

In the quasi-steady treatment of the problem the inertia forces are neglected; this assumption is admissible, if the temperature varies slowly in time. In this case we have the system of equations

$$\nabla^2\phi = \vartheta_0\theta, \tag{1.7}$$

$$\nabla^2\theta - \frac{1}{\kappa}\dot{\theta} - \eta\nabla^2\dot{\phi} = -\frac{Q}{\kappa}. \tag{1.8}$$

We can eliminate the function ϕ from the second equation. Then the conduction equation in the infinite space takes the form

$$\nabla^2\theta - \frac{1}{\kappa'}\dot{\theta} = -\frac{Q}{\kappa}, \quad \frac{1}{\kappa'} = \frac{1}{\kappa} + \eta\vartheta_0. \tag{1.9}$$

For a stationary field temperature the coupling of the fields vanishes, and in the infinite space we have the system

$$\nabla^2\theta = -\frac{Q}{\kappa}, \quad \nabla^2\phi = \vartheta_0\theta. \tag{1.10}$$

§ 2. Stresses due to the action of heat sources in an infinite body

We proceed to the investigation of thermoelastic waves in an infinite space. We shall successively consider the effects of point, linear and plane heat sources. We assume that the heat sources vary harmonically in time, i.e., $Q(P, t) = Q_0(P)e^{i\omega t}$, the frequency ω of the vibrations of the source being real and positive. Since the effect of heat sources results only in generation of longitudinal waves, we consider the equations (1.4) and (1.6). Setting

$$\phi(P, t) = e^{i\omega t}\phi^*(P), \quad \theta(P, t) = e^{i\omega t}\theta^*(P),$$

we arrive at the system of equations

$$(\nabla^2 + \sigma^2)\phi^* = \vartheta_0\theta^*, \tag{2.1}$$

$$(\nabla^2 - q)\theta^* - q\eta'\nabla^2\phi^* = -\frac{Q_0}{\kappa}, \tag{2.2}$$

where

$$\sigma^2 = \frac{\omega^2}{{c_1}^2}, \quad q = i\frac{\omega}{\kappa}, \quad \eta' = \eta\kappa.$$

Eliminating from the equations (2.1) and (2.2) first the function ϕ^*, and then the function θ^*, we obtain

$$(\nabla^2 - q)(\nabla^2 + \sigma^2)\theta^* - q\varepsilon\nabla^2\theta^* = -\frac{1}{\kappa}(\nabla^2 + \sigma^2)Q_0(P), \tag{2.3}$$

$$(\nabla^2 - q)(\nabla^2 + \sigma^2)\phi^* - q\varepsilon\nabla^2\phi^* = -\frac{\vartheta_0}{\kappa}Q_0(P), \tag{2.4}$$

where

$$\varepsilon = \eta\kappa\vartheta_0.$$

Let a concentrated heat source be situated at the origin of the coordinate system. The solution of the equation (2.4) in cylindrical coordinates can be represented in the integral form

$$\phi^* = -\frac{Q_0\vartheta_0}{2\pi^2\kappa}\int_0^\infty\int_0^\infty \frac{\alpha J_0(\alpha r)\cos\gamma r}{F(\alpha, \gamma)}\,d\alpha\,d\gamma, \tag{2.5}$$

where

$$F(\alpha, \gamma) = (\alpha^2 + \gamma^2 + {k_1}^2)(\alpha^2 + \gamma^2 + {k_2}^2),$$

$${k_1}^2 + {k_2}^2 = q(1 + \varepsilon) - \sigma^2, \quad {k_1}^2{k_2}^2 = -q\sigma^2.$$

After the indicated integration has been carried out the function ϕ^* takes the closed form

$$\phi^* = -\frac{Q_0\vartheta_0}{4\pi\kappa R(k_1^2 - k_2^2)}(e^{-k_1R} - e^{-k_2R}), \tag{2.6}$$

where

$$R = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}, \quad k_{1,2} = a_{1,2} + ib_{1,2}, \quad a_{1,2} > 0.$$

Solving the equation (2.3) we have

$$\theta^* = \frac{Q_0}{4\pi\kappa R(k_1^2 - k_2^2)}[(\sigma^2 + k_1^2)e^{-k_1R} - (\sigma^2 + k_2^2)e^{-k_2R}]. \tag{2.7}$$

Let us now move the heat source to the point (ξ_1, ξ_2, ξ_3) and set $Q_0 = 1$. Then $\phi^*(x_r, \xi_r)$ and $\theta^*(x_r, \xi_r)$ are the Green's functions for the infinite space.

The stresses corresponding to the temperature field θ are given by the relations

$$\sigma_{ij}(x_r, \xi_r, t) = 2\mu[\phi_{,ij} - \delta_{ij}\phi_{,kk}] + \rho\ddot{\phi}\delta_{ij}. \tag{2.8}$$

Neglecting the coupling between the temperature and strain fields (i.e., for $\varepsilon = 0$, $k_1 = \sqrt{q}$, $k_2 = i\sigma$) we obtain the familiar result [9]

$$\begin{aligned} \phi^* &= \frac{Q_0\vartheta_0}{4\pi\kappa R(\sigma^2 + q)}(e^{-R\sqrt{q}} - e^{-Ri\sigma}), \\ \theta^* &= \frac{Q_0}{4\pi\kappa R}e^{-R\sqrt{q}}. \end{aligned} \tag{2.9}$$

Let us now consider the axi-symmetric problem. Suppose that on the surface of a cylinder of radius ρ the heat sources $Q(r, z) = e^{i\omega t}Q_0\delta(r - \rho)$ are uniformly distributed. Solving the equation (2.4) we arrive at the integral

$$\phi^* = -\frac{Q_0\vartheta_0\rho}{\kappa}\int_0^\infty \frac{\alpha J_0(\alpha\rho)J_0(\alpha r)}{F(\alpha)}\,d\alpha, \tag{2.10}$$

where

$$F(\alpha) = (\alpha^2 + k_1^2)(\alpha^2 + k_2^2), \quad r = (x_1^2 + x_2^2)^{\frac{1}{2}}.$$

The function ϕ^* can be represented in the form

$$\phi^* = \frac{Q_0\vartheta_0\rho}{\kappa(k_1^2 - k_2^2)}\begin{cases} I_0(k_1r)K_0(k_1\rho) - I_0(k_2r)K_0(k_2\rho) & \text{for } 0 < r < \rho, \\ I_0(k_1\rho)K_0(k_1r) - I_0(k_2\rho)K_0(k_2r) & \text{for } \rho < r < \infty. \end{cases} \tag{2.11}$$

In the particular case of a linear heat source, assuming that

$$\lim_{\rho \to 0} 2\pi\rho Q_0 = q_0,$$

we obtain from (2.10)

$$\phi^* = -\frac{q_0\vartheta_0}{2\pi\kappa}\int_0^\infty \frac{\alpha J_0(\alpha r)}{F(\alpha)}\,d\alpha, \tag{2.12}$$

or

$$\phi^* = \frac{q_0\vartheta_0}{2\pi\kappa(k_1^2 - k_2^2)}[K_0(k_1 r) - K_0(k_2 r)]. \tag{2.13}$$

It can readily be proved that

$$\theta^* = \frac{q_0}{2\pi\kappa(k_1^2 - k_2^2)}[(\sigma^2 + k_1^2)K_0(k_1 r) - (\sigma^2 + k_2^2)K_0(k_2 r)]. \tag{2.14}$$

In the above formulae $I_0(z)$ denotes the modified Bessel function of the first kind and $K_0(z)$ that of the third kind.

If the coupling between the temperature and strain fields is neglected, the formulae (2.13) and (2.14) yield the familiar results [9]

$$\phi^* = \frac{q_0\vartheta_0}{2\pi\kappa(\sigma^2 + q)}[K_0(r\sqrt{q}) - K_0(i\sigma r)], \tag{2.15}$$

$$\theta^* = \frac{q_0}{2\pi\kappa}K_0(r\sqrt{q}). \tag{2.16}$$

The stresses corresponding to the axi-symmetric function ϕ^* are given by the relations

$$\begin{aligned} \sigma_{rr} &= -e^{i\omega t}(2\mu r^{-1}\phi^*_{,r} + \rho\omega^2\phi^*), \\ \sigma_{\varphi\varphi} &= -e^{i\omega t}(2\mu\phi^*_{,rr} + \rho\omega^2\phi^*). \end{aligned} \tag{2.17}$$

Assume now that there acts in the plane $x_1 = 0$ a plane heat source $Q(P, t) = e^{i\omega t}Q_0\delta(x_1)$. Solving the equations (2.3) and (2.4) we obtain

$$\phi^* = \frac{Q_0\vartheta_0}{2\kappa(k_1^2 - k_2^2)}\left[\frac{e^{-k_1 x_1}}{k_1} - \frac{e^{-k_2 x_1}}{k_2}\right], \tag{2.18}$$

$$\theta^* = \frac{Q_0}{2\kappa(k_1^2 - k_2^2)}\left[(\sigma^2 + k_1^2)\frac{e^{-k_1 x_1}}{k_1} - (\sigma^2 + k_2^2)\frac{e^{-k_2 x_1}}{k_2}\right]. \tag{2.19}$$

The stresses produced by a plane heat source are

$$\sigma_{11} = -e^{i\omega t}\rho\omega^2\phi^*, \qquad \sigma_{22} = \sigma_{33} = -e^{i\omega t}(2\mu\phi^*_{,11} + \rho\omega^2\phi^*). \tag{2.20}$$

§ 3. Effect of a center of pressure in an infinite space

The coupled equations of thermoelasticity enable us to determine the temperature field associated with the effect of forces varying in time. Consider the simplest three-dimensional problem in which only longitudinal waves occur.

Let there be given a center of pressure, i.e., a system of three force doublets acting in the directions x_1, x_2, x_3 respectively, and assume that it acts at the origin of the coordinate system. The displacement equations in this case have the form

$$\mu\nabla^2 u_i + (\lambda + \mu)u_{k,ki} + [\delta(x_r)]_{,i} = \rho\ddot{u}_i + \gamma_0\theta_{,i}, \qquad (i = 1, 2, 3) \tag{3.1}$$

where $\delta(x_r) = \delta(x_1)\delta(x_2)\delta(x_3)$. In the case of a center of pressure varying harmonically in time we take

$$u_i = e^{i\omega t}\phi^*_{,i}. \tag{3.2}$$

Thus the system of equations (3.1) is reduced to the equation

$$(\nabla^2 + \sigma^2)\phi^* = \vartheta_0\theta^* - \frac{1}{\lambda + 2\mu}\delta(x_r). \tag{3.3}$$

Assuming that there are no heat sources inside the body we supplement (3.3) by the equation

$$(\nabla^2 - q)\theta^* - q\eta'\nabla^2\phi^* = 0. \tag{3.4}$$

The solution of the system (3.3)—(3.4) yields for a point center of pressure

$$\phi^* = \frac{1}{4\pi\beta R(k_1^2 - k_2^2)}[(k_1^2 - q)e^{-k_1 R} - (k_2^2 - q)e^{-k_2 R}], \tag{3.5}$$

$$\theta^* = \frac{\eta' q}{4\pi\beta R(k_1^2 - k_2^2)}[k_1^2 e^{-k_1 R} - k_2^2 e^{-k_2 R}]. \tag{3.6}$$

Knowing the function ϕ^* we can determine the stresses by means of the formulas (2.8). Since the temperature field has a singularity at the origin, the point center of pressure plays a role analogous to that of the heat source in the preceding considerations.

If the coupling between the temperature and strain fields be neglected ($\eta' = 0$), we obtain the known result [10]

$$\phi = \frac{1}{4\pi\beta R} e^{i(\omega t - R\sigma)}, \qquad \theta = 0. \tag{3.7}$$

In the case of the effect of a line center of pressure (along the x_3-axis) we have

$$\phi^* = \frac{1}{2\pi\beta(k_1^2 - k_2^2)} [(k_1^2 - q)K_0(k_1 r) - (k_2^2 - q)K_0(k_2 r)], \tag{3.8}$$

$$\theta^* = \frac{\eta' q}{2\pi\beta(k_1^2 - k_2^2)} [k_1^2 K_0(k_1 r) - k_2^2 K_0(k_2 r)]. \tag{3.9}$$

Taking $\eta' = 0$ (no coupling) we arrive at the familiar result

$$\phi = \frac{1}{2\pi\beta} K_0(i\sigma r) e^{i\omega t}, \quad \theta = 0. \tag{3.10}$$

Finally, for a plane center of pressure in the plane $x_1 = 0$, we have

$$\phi^* = \frac{1}{2\beta(k_1^2 - k_2^2)} \left(\frac{k_1^2 - q}{k_1} e^{-k_1 x_1} - \frac{k_2^2 - q}{k_2} e^{-k_2 x_1} \right), \tag{3.11}$$

$$\theta^* = \frac{\eta' q}{2\beta(k_1^2 - k_2^2)} (k_1^2 e^{-k_1 x_1} - k_2^2 e^{-k_2 x_1}), \tag{3.12}$$

$$x_1 > 0.$$

If $\eta' = 0$, then

$$\phi = \frac{1}{2\beta\sqrt{i\sigma}} \exp [i(\omega t - \sigma x_1)], \qquad \theta = 0. \tag{3.13}$$

§ 4. State of stress in an infinite space with a spherical or cylindrical cavity

Consider first a spherical cavity of radius a in an infinite space. The solution of the homogeneous equations (2.3)—(2.4) can be represented in the form

$$\phi^* = \frac{1}{R} (A e^{-k_1 R} + B e^{-k_2 R}), \tag{4.1}$$

$$\theta^* = \frac{1}{R\vartheta_0} [A(k_1^2 + \sigma^2) e^{-k_1 R} + B(k_2^2 + \sigma^2) e^{-k_2 R}]. \tag{4.2}$$

The constants A and B are to be determined from the boundary conditions for $R = a$. If we assume that

$$\theta^*(a) = \theta_0,$$

$$\sigma^*_{RR}(a) = -\left[\frac{4\mu}{R}\phi^*_{,R} + \rho\omega^2\phi^*\right]_{R=a} = 0, \tag{4.3}$$

then

$$A = \theta_0 a\vartheta_0 \frac{m_2 e^{k_1 a}}{\Delta}, \quad B = -\theta_0 a\vartheta_0 \frac{m_1 e^{k_2 a}}{\Delta}, \tag{4.4}$$

where

$$\Delta = (k_1^2 + \sigma^2)m_2 - (k_2^2 + \sigma^2)m_1,$$

$$m_1 = 4\mu(1 + ak_1) - a^2\omega\rho, \quad m_2 = 4\mu(1 + ak_2) - a^2\omega\rho.$$

Setting now

$$\theta^*(a) = 0, \quad \sigma^*_{RR} = -p_0, \tag{4.5}$$

we find

$$A = \frac{p_0 a^3 e^{k_1 a}}{\Delta}, \quad B = \frac{-p_0 a^3 e^{k_1 a}}{\Delta} \cdot \frac{k_1^2 + \sigma^2}{k_2^2 + \sigma^2}. \tag{4.6}$$

Knowledge of the function ϕ^* makes it possible to determine the stresses in accordance with the formulae

$$\sigma_{RR} = -\frac{4\mu}{R}\phi_{,R} + \rho\ddot{\phi},$$

$$\sigma_{\varphi\varphi} = \sigma_{\theta\theta} = -2\mu(\phi_{,RR} + R^{-1}\phi_{,R}) + \rho\ddot{\phi}. \tag{4.7}$$

If the coupling between the temperature and strain fields is neglected ($\varepsilon = 0$), we obtain from the formulae (4.1) and (4.2), taking into account the boundary conditions (4.5), the solutions

$$\phi^* = \frac{p_0 a^3}{\bar{m}_2 R} e^{-(R-a)i\sigma}, \qquad \theta^* = 0, \tag{4.8}$$

where

$$\bar{m}_2 = 4\mu(1 + a\sqrt{q}) - a^2\omega^2\rho.$$

In the case of a cylindrical cavity the solution of the homogeneous

equations (2.3) and (2.4) has the form

$$\phi^* = AK_0(k_1 r) + BK_0(k_2 r), \tag{4.9}$$

$$\theta^* = \frac{1}{\vartheta_0}[A(k_1^2 + \sigma^2)K_0(k_1 r) + B(k_2^2 + \sigma^2)K_0(k_2 r)]. \tag{4.10}$$

Assuming that

$$\theta^*(a) = \theta_0,$$

$$\sigma_{rr}^*(a) = -\left[\frac{2\mu}{r}\phi_{,r}^* + \rho\omega^2\phi^*\right]_{r=a} = 0, \tag{4.11}$$

we obtain

$$A = \frac{Q_0\vartheta_0 n_2}{\Delta_1}, \quad B = -\frac{Q_0\vartheta_0 n_1}{\Delta_1}, \tag{4.12}$$

where

$$n_1 = 2\mu k_1 K_1(k_1 a) - \rho\omega^2 a K_0(k_1 a),$$

$$n_2 = 2\mu k_2 K_1(k_2 a) - \rho\omega^2 a K_0(k_2 a),$$

$$\Delta_1 = n_2(\sigma^2 + k_1^2)K_0(k_1 a) - n_1(\sigma^2 + k_2^2)K_0(k_2 a).$$

In an analogous manner we determine the constants A and B for other boundary conditions. The stresses in the elastic body are given by the relations

$$\sigma_{rr} = -\frac{2\mu}{r}\phi_{,r} + \rho\ddot{\phi},$$

$$\sigma_{\varphi\varphi} = -2\mu\phi_{,rr} + \rho\ddot{\phi}. \tag{4.13}$$

We have now to investigate the roots of the equations

$$(k^2 + k_1^2)(k^2 + k_2^2) = 0,$$

$$k_1^2 + k_2^2 = q(1 + \varepsilon) - \sigma^2,$$

$$k_1^2 k_2^2 = -q\sigma^2, \tag{4.14}$$

which enter into all the results of §§ 2–4. These roots can be represented in the form

$$k_{1,2}^2 = \frac{a^2}{2\kappa^2}[-\eta^2 + i\eta(1 + \varepsilon) \pm \Delta],$$

$$\Delta = \sqrt{\eta^2[\eta^2 - (1 + \varepsilon)^2] + 2i\eta^3(1 - \varepsilon)},$$

where $\eta = \omega/\omega^*$ is a dimensionless quantity and $\omega^* = c_1^2/\kappa$ is a characteristic quantity of the thermoelastic medium which was introduced by P. Chadwick and I. N. Sneddon [3].

The root k_1 corresponds to the modified thermal wave, and k_2 to the modiefied elastic longitudinal wave. In fact, $k_1 = \sqrt{q}$, $k_2 = i\sigma$ for $\varepsilon = 0$. In the problems under consideration we are interested in roots the real parts of which are positive, since they describe the thermal and elastic waves diverging from the center of excitation into infinity.

P. Chadwick and I. N. Sneddon [3] investigated in detail the behavior of the roots k_1 and k_2 in terms of the parameter η, and determined their approximate values for $\eta \ll 1$ and $\eta \gg 1$. They proved that for $\eta \ll 1$

$$\begin{aligned} \frac{\kappa}{c_1} k_1 &= \sqrt{\frac{1+\varepsilon}{2}}\,\eta\left[(1+i) - \frac{\eta\varepsilon}{2(1+\varepsilon)^2}(1-i)\right], \\ \frac{\kappa}{c_1} k_2 &= \frac{\eta^2\varepsilon}{2(1+\varepsilon)^{\frac{1}{2}}} + i\frac{\eta}{(1+\varepsilon)^{\frac{1}{2}}}, \end{aligned} \tag{4.15}$$

and for $\eta \gg 1$

$$\begin{aligned} \frac{\kappa}{c_1} k_1 &= \sqrt{\frac{\eta}{2}}\left[\left(1 - \frac{\varepsilon}{2\eta}\right) + i\left(1 + \frac{\varepsilon}{2\eta}\right)\right], \\ \frac{\kappa}{c_1} k_2 &= \tfrac{1}{2}\varepsilon + i\eta. \end{aligned} \tag{4.16}$$

In the same paper the authors analysed in detail the influence of the parameter ε on the velocity of longitudinal waves and on the dispersion coefficient. The results of Chadwick and Sneddon concern the influence of the temperature field on plane harmonic waves. It is clear, however, that these results are also true for spherical and cylindrical waves, since in this case the roots k_1 and k_2 are the same. The velocity of propagation of the modified elastic wave for $\eta \ll 1$ is given by the relation

$$c_1' = (1+\varepsilon)c_1,$$

where c_1 denotes the velocity of propagation of the elastic wave in the problem in which no coupling occurs, i.e., for $\varepsilon = 0$. The dispersion coefficient has the form

$$\chi = \frac{\varepsilon}{2}(2 - 5\varepsilon)\eta^2.$$

Since the quantity ε is small (for aluminum $\varepsilon = 3.56 \times 10^{-2}$, for steel $\varepsilon = 2.97 \times 10^{-4}$ and for lead $\varepsilon = 7.33 \times 10^{-2}$) it is evident that the influence of the coupling between the temperature and strain fields on the velocity of propagation of elastic plane, cylindrical and spherical waves due to the action of heat sources or centers of pressure is in-

significant. Similarly, the differences in the magnitudes of the stresses are very small. Nevertheless, it is important to observe that by taking account of the coupling, it is possible to determine the temperature field generated by the effect of forces varying harmonically in time.

Finally, let us observe that the results deduced may prove useful for the determination of the stresses produced by instantaneous sources of heat or force. Denoting by $\sigma_{ij}(P, t)$ the stresses produced by the action of such sources we have

$$\sigma_{ij}(P, t) = \int_0^\infty \sigma_{ij}^*(P, \omega) e^{i\omega t}\, d\omega, \tag{4.17}$$

where $\sigma_{ij}^*(P, \omega)$ is the stress due to the sources of heat or forces which vary harmonically in time.

§ 5. Stresses due to heating of the plane boundary of an elastic half-space

Suppose that on the plane $z = 0$, bounding the elastic half-space, the temperature $Q(r, 0, t) = \theta_0^*(r) e^{i\omega t}$ is prescribed. This heating causes in the elastic semi-space an axi-symmetric temperature field and axi-symmetric state of stress. Assume that the surface $z = 0$ is free of tractions.

Let us construct the solution in two parts: the solution of the system of equations (2.1)—(2.2) for the infinite space the boundary condition $\theta(r, 0, t) = \theta_0^*(r) e^{i\omega t}$ being satisfied on the plane $z = 0$, and the additional solution for the half-space ensuring the fulfilment of the remaining boundary conditions. To deduce the first solution consider the equations

$$(\nabla^2 - q)\theta^* - q\eta' \nabla^2 \phi^* = 0, \tag{5.1}$$

$$(\nabla^2 + \sigma^2)\phi^* = \vartheta_0 \theta^*. \tag{5.2}$$

Applying the Fourier-Hankel integral transform we arrive at the system of algebraic equations

$$\begin{aligned} -(\alpha^2 + \gamma^2 - q)\tilde{\theta}^* + q\eta'(\alpha^2 + \gamma^2)\tilde{\phi}^* + \gamma \tilde{\theta}_0^* = 0, \\ -(\alpha^2 + \gamma^2 - \sigma^2)\tilde{\phi}^* = \vartheta_0 \tilde{\theta}^*, \end{aligned} \tag{5.3}$$

This problem was solved by a different method by Eason and Sneddon [11], and Paria [12].

where

$$\tilde{\theta}^*(\alpha,\gamma)=\sqrt{\frac{2}{\pi}}\int_0^\infty\int_0^\infty \theta^*(r,z)rJ_0(\alpha r)\sin\gamma z\,dr\,dz,$$

$$\tilde{\phi}^*(\alpha,\gamma)=\sqrt{\frac{2}{\pi}}\int_0^\infty\int_0^\infty \phi^*(r,z)rJ_0(\alpha r)\sin\gamma z\,dr\,dz, \tag{5.4}$$

$$\tilde{\theta}_0^*(\alpha)=\int_0^\infty rJ_0(\alpha r)\theta_0^*(r)\,dr.$$

Solving the system (5.3) we obtain

$$\tilde{\phi}^*=-\frac{\gamma\vartheta_0\tilde{\theta}_0^*}{F(\alpha,\gamma)},\quad \tilde{\theta}^*=\frac{\gamma\tilde{\theta}^*}{F(\alpha,\gamma)}(\alpha^2+\gamma^2-\sigma^2), \tag{5.5}$$

where

$$F(\alpha,\gamma)=(\alpha^2+\gamma^2+k_1^2)(\alpha^2+\gamma^2+k_2^2),$$

$$k_1^2+k_2^2=q(1+\varepsilon)-\sigma^2, \tag{5.6}$$

$$k_1^2k_2^2=-q\sigma^2.$$

Inverting the transform we have

$$\phi^*(r,z)=-\vartheta_0\sqrt{\frac{2}{\pi}}\int_0^\infty\int_0^\infty \frac{\tilde{\theta}_0^*(\alpha)\gamma\alpha J_0(\alpha r)\sin\gamma z}{F(\alpha,\gamma)}\,d\alpha\,d\gamma, \tag{5.7}$$

or

$$\phi^*(r,z)=-\sqrt{\frac{2}{\pi}}\frac{\vartheta_0}{k_1^2-k_2^2}\int_0^\infty \tilde{\theta}_0^*(\alpha)\alpha(e^{-k_1z}-e^{-k_2z})J_0(\alpha r)\,d\alpha. \tag{5.8}$$

Knowing the function ϕ^* we calculate the stresses

$$\sigma_{rr}^{(1)}=-e^{i\omega t}\left[2\mu\left(\frac{\phi_{,r}^*}{r}+\phi_{,zz}^*\right)+\rho\omega^2\phi^*\right],$$

$$\sigma_{\varphi\varphi}^{(1)}=-e^{i\omega t}[2\mu(\phi_{,rr}^*+\phi_{,zz}^*)+\rho\omega^2\phi^*], \tag{5.9}$$

$$\sigma_{zz}^{(1)}=-e^{i\omega t}\left[2\mu\left(r^{-1}\frac{\phi_{,r}^*}{r}+\phi_{,rr}^*\right)+\rho\omega^2\phi^*\right],$$

$$\sigma_{rz}^{(1)}=e^{i\omega t}\cdot 2\mu\phi_{,rz}^*.$$

In the plane $z = 0$ we have $\sigma_{zz}^{(1)} = 0$ and $\sigma_{rz}^{(1)} \neq 0$,

$$\sigma_{rz}^{(1)}(r, 0, t) = \sqrt{\frac{2}{\pi}} \frac{\vartheta_0 e^{i\omega t}}{k_1 + k_2} \int_0^\infty \alpha^3 \tilde{\theta}_0^*(\alpha) J_0(\alpha r) d\alpha. \tag{5.10}$$

The second stage of the solution consists in completing the state $\sigma_{ij}^{(1)}$ by a state $\sigma_{ij}^{(2)}$ such that in the plane $z = 0$ the following boundary conditions are satisfied:

$$\begin{gathered} \sigma_{zz}^{(2)}(r, 0, t) = 0, \\ \sigma_{rz}^{(1)}(r, 0, t) + \sigma_{rz}^{(2)}(r, 0, t) = 0, \\ \theta^{(2)}(r, 0, t) = 0. \end{gathered} \tag{5.11}$$

In the elastic half-space occur both longitudinal and transverse waves. Hence, in the second stage, we have to consider the system of equations

$$(\nabla^2 - q)(\nabla^2 + \sigma^2)\phi^{*(2)} - q\varepsilon\nabla^2\phi^{*(2)} = 0, \tag{5.12}$$

$$(\nabla^2 - \tau^2)\psi^* = 0. \tag{5.13}$$

The displacements are related to the functions $\phi^{(2)}$ and ψ by the formulae

$$\begin{aligned} u_r^{(2)} &= \phi_{,r}^{(2)} + \psi_{,rz}, \\ u_z^{(2)} &= \phi_{,z}^{(2)} - \psi_{,rr} - \frac{\psi_{,r}}{r} = \phi_{,z}^{(2)} + \psi_{,zz} - \frac{\psi}{c_2^2}. \end{aligned} \tag{5.14}$$

Hence

$$\begin{aligned} u_r^{*(2)} &= \phi_{,r}^{*(2)} + \psi_{,rz}^*, \\ u_z^{*(2)} &= \phi_{,z}^{*(2)} + \psi_{,zz}^* + \tau^2\psi^*. \end{aligned} \tag{5.15}$$

Expressing the stresses by the displacements we obtain

$$\begin{aligned} \sigma_{zz}^{*(2)} &= 2\mu(\partial_z^2 - \nabla^2)\phi^{*(2)} - \rho\omega^2\phi^{*(2)} + 2\mu(\partial_z^2 + \tau^2)_{,z}\psi^*, \\ \sigma_{rz}^{*(2)} &= 2\mu\phi_{,rz}^{*(2)} + \mu(2\partial_z^2 + \tau^2)_{,r}\psi^*. \end{aligned} \tag{5.16}$$

The solution of the equation (5.12) can be represented by the Hankel integral

$$\phi^{*(2)} = \int_0^\infty (Ae^{-\lambda_1 z} + Be^{-\lambda_2 z}) J_0(\alpha z)\, d\alpha, \tag{5.17}$$

where λ_1, λ_2 are the roots of the equation

$$\lambda^4 + [\sigma^2 - q(1 + \varepsilon) - 2\alpha^2]\lambda^2 + \alpha^4 - \alpha^2[\sigma^2 - q(1 + \varepsilon)] - q\sigma^2 = 0 \tag{5.18}$$

the real parts of which are positive.

The solution of the equation (5.13) is assumed to have the form

$$\psi^* = \int_0^\infty c(\alpha) e^{-\nu z} J_0(\alpha z) d\alpha, \qquad \nu = \sqrt{\alpha^2 - \tau^2}. \tag{5.19}$$

The amplitude of the temperature is given by the relation

$$\theta^{*(2)} = \frac{1}{\vartheta_0} (\nabla^2 + \sigma^2) \phi^{*(2)}. \tag{5.20}$$

Substituting (5.17) into (5.20) we have

$$\theta^{*(2)} = \frac{1}{\vartheta_0} \int_0^\infty [A(\lambda_1^2 + \sigma^2 - \alpha^2) e^{-\lambda_1 z} + B(\lambda_2^2 + \sigma^2 - \alpha^2) e^{-\lambda_2 z}] J_0(\alpha z) d\alpha. \tag{5.21}$$

The quantities A, B and C are to be determined in accordance with the boundary conditions (5.11). Thus we have

$$A = -\sqrt{2\pi} \frac{\vartheta_0 \mu \alpha^3 \tilde{\theta}_0^*(\alpha)}{k_1 + k_2} \cdot \frac{\nu n_2}{\Delta_1},$$

$$B = -A \frac{n_1}{n_2}, \quad C = \frac{2\mu\alpha^2 - \rho\omega^2}{2\mu\nu\alpha^2} \cdot \frac{\lambda_2^2 - \lambda_1^2}{n_2} A, \tag{5.22}$$

$$\Delta_1 = (\lambda_1 n_2 - \lambda_2 n_1) 4\mu\nu\alpha^2 - (2\mu\alpha^2 - \rho\omega^2)(\nu^2 + \alpha^2)(\lambda_2^2 - \lambda_1^2),$$

$$n_{1,2} = \lambda_{1,2}^2 + \sigma^2 - \alpha^2.$$

Knowledge of the functions $\phi^{*(2)}$ and $\psi^{(*)}$ enables us to determine the displacements $u_r^{*(2)}$, $u_z^{*(2)}$, and then also the stresses $\sigma_{ij}^{*(2)}$. Adding the stresses $\sigma_{ij}^{(1)}$ and $\sigma_{ij}^{(2)}$ we obtain the final stresses σ_{ij}.

The second stage of the solution is identical with the Lamb-problem for the elastic half-space.

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A NON-LINEAR EIGEN-VALUE PROBLEM IN THE THEORY OF CREEP

F. K. G. ODQVIST
Stockholm, Sweden

Consider the problem of creep of a circular membrane, plane or almost plane from the outset, under the action of constant lateral pressure p. The radius of the membrane may be a and the boundary condition is that the membrane be hinged at $r = a$ for all values of the time t. Inertia forces are neglected. Rotational symmetry is supposed to prevail, implying that only displacements in the radial and lateral directions u, w are considered. Then the strains in the radial and tangential directions are (to second order quantities in the derivatives of the lateral displacement w)

$$\varepsilon_r = \frac{du}{dr} + \tfrac{1}{2}\left(\frac{dw}{dr}\right)^2, \quad \varepsilon_t = \frac{u}{r}, \tag{1}$$

and the corresponding strain rates

$$\frac{d\varepsilon_r}{dt} = \dot{\varepsilon}_r = \frac{d\dot{u}}{dr} + \frac{dw}{dr}\frac{d\dot{w}}{dr}, \quad \dot{\varepsilon}_t = \frac{\dot{u}}{r}. \tag{2}$$

Neglecting elastic deformation as well as so-called primary (transient) creep deformation, we have the following relationships between the strain rates (2) and the stress components σ_r and σ_t in the radial and tangential directions [1]

$$\begin{aligned} \dot{\varepsilon}_r &= F(\sigma_e)(\sigma_r - \sigma_t/2), \\ \dot{\varepsilon}_t &= F(\sigma_e)(\sigma_t - \sigma_r/2) \end{aligned} \tag{3}$$

with

$$\sigma_e^2 = \sigma_r^2 - \sigma_r\sigma_t + \sigma_t^2 \tag{4}$$

and

$$F(\sigma_e) = (\sigma_e/\sigma_c)^{n-1}/\sigma_c. \tag{5}$$

The quantities n and σ_c are material constants at a given temperature for a number of structural metals.

The equations of equilibrium are

$$\frac{d}{dr}(r\sigma_r) = \sigma_t, \tag{6}$$

$$\frac{1}{r}\frac{d}{dr}\left(r\sigma_r\frac{dw}{dr}\right) = -\frac{p}{h}, \tag{7}$$

where h is the membrane thickness, supposed to be constant. Here bending creep has been neglected. Equations (2) through (7) form a system of differential equations for the determination of the unknown functions. The boundary conditions are $u = 0$, $w = 0$ for $r = a$. The initial conditions shall be specified later. Taking account of the fact that $\sigma_r dw/dr$ should be finite for $r = 0$, equation (7) may be integrated once, *viz.*,

$$\sigma_r \frac{dw}{dr} = -\frac{pr}{2h}. \tag{7'}$$

We introduce dimensionless variables R, T, U, W, ρ instead of σ_r, σ_t, u, w and r, writing

$$\begin{aligned} &\sigma_r = R\sigma_c, \quad \sigma_t = T\sigma_c, \\ &u = U\sigma_c h/p, \quad w = W\sigma_c h/p, \quad r = \rho\sigma_c h/p, \end{aligned} \tag{8}$$

and obtain the system

$$\frac{d}{d\rho}(\rho R) = T, \tag{9}$$

$$R\frac{dW}{d\rho} = -\frac{\rho}{2}, \tag{10}$$

$$\frac{d\dot{U}}{d\rho} + \frac{dW}{d\rho}\frac{d\dot{W}}{d\rho} = [R^2 - RT + T^2]^{(n-1)/2}(R - T/2), \tag{11}$$

$$\frac{\dot{U}}{\rho} = [R^2 - RT + T^2]^{(n-1)/2}(T - R/2) \tag{12}$$

with the boundary conditions $U = 0$, $W = 0$ for

$$\rho = \rho_1 = \frac{ap}{\sigma_c h}. \tag{13}$$

The general appearance of the system (9)–(12) suggests a solution of the form

$$R = R(\rho, t) = S(t)\bar{R}(\rho), \tag{14}$$

$$T = T(\rho, t) = S(t)\bar{T}(\rho), \tag{15}$$

where S is a function of t only and $\bar{R}$, $\bar{T}$ are functions of ρ only. $\bar{R}$ and $\bar{T}$ will fulfill equation (9). From (12) we obtain

$$\frac{\dot{U}}{\rho} = S^n[\bar{R}^2 - \bar{R}\bar{T} + \bar{T}^2]^{(n-1)/2}(\bar{T} - \bar{R}/2), \tag{16}$$

and from (10)

$$\frac{dW}{d\rho} = -\frac{\rho}{2\bar{R}S}, \quad \frac{d\dot{W}}{d\rho} = \frac{\rho\dot{S}}{2\bar{R}S^2}. \tag{17}$$

Then the quantity $d\dot{U}/d\rho + dW/d\rho \cdot d\dot{W}/d\rho$ may be formed in two different ways, once using (11) directly and once (16) and (17). Equating the two expressions and rearranging the terms we obtain

$$\frac{4\bar{R}^2}{\rho^2}\left\{\frac{d}{d\rho}\rho[\bar{R}^2 - \bar{R}\bar{T} + \bar{T}^2]^{(n-1)/2}(\bar{T} - \bar{R}/2) - \right. \tag{18}$$

$$\left. -[\bar{R}^2 - \bar{R}\bar{T} + \bar{T}^2]^{(n-1)/2}(\bar{R} - \bar{T}/2)\right\} = \frac{\dot{S}}{S^{n+3}}.$$

By definition the two sides of equation (18) should be functions of ρ and t only. Hence we may put them both equal to a real constant $= -\lambda$, this giving rise to the eigenvalue problem

$$\frac{d}{d\rho}\rho[\bar{R}^2 - \bar{R}\bar{T} + \bar{T}^2]^{(n-1)/2}(\bar{T} - \bar{R}/2) - \tag{19}$$

$$-[\bar{R}^2 - \bar{R}\bar{T} + \bar{T}^2]^{(n-1)/2}(\bar{R} - \bar{T}/2) + \frac{\lambda\rho^2}{4\bar{R}^2} = 0,$$

$$\frac{d}{d\rho}(\rho\bar{R}) - \bar{T} = 0,$$

with the boundary condition

$$\bar{T} - \bar{R}/2 = 0 \text{ for } \rho = \rho_1. \tag{20}$$

This problem being solved, we have the additional relation

$$\frac{\dot{S}}{S^{n+3}} = -\lambda \tag{21}$$

which may be integrated with the initial condition

$$S = S_0 \quad \text{for} \quad t = 0; \tag{22}$$

hence

$$\frac{1}{S^{n+2}} - \frac{1}{S_0^{n+2}} = (n+2)\lambda t. \tag{23}$$

In order to solve the eigenvalue problem (19), (20), we introduce a new variable

$$\bar{\rho} = \rho/\rho_1 \tag{24}$$

and put

$$\frac{\lambda \rho_1^2}{8(n+3)} = C. \tag{25}$$

Thus, after some calculations, we arrive at the system

$$\frac{d\bar{R}}{d\bar{\rho}} = \frac{\bar{T} - \bar{R}}{\bar{\rho}}, \tag{26}$$

$$\frac{d\bar{T}}{d\bar{\rho}} = \frac{\bar{R} - \bar{T}}{\bar{\rho}} \cdot \frac{2(3-n)(\bar{R}^2 + \bar{T}^2) - (9-5n)\bar{R}\bar{T}}{4n(\bar{T}^2 - \bar{R}\bar{T}) + (3+n)\bar{R}^2} - \frac{8(n+3)C\bar{\rho}}{\bar{R}^2[\bar{R}^2 - \bar{R}\bar{T} + \bar{T}^2]^{(n-3)/2}[4n(\bar{T}^2 - \bar{R}\bar{T}) + (3+n)\bar{R}^2]} \tag{27}$$

to be integrated with the boundary condition

$$\bar{T} = 2\bar{R} \quad \text{for} \quad \bar{\rho} = 1. \tag{28}$$

The solution must be regular for $\bar{\rho} = 0$. Further, we may without restriction put $\bar{R}(0) = 1$. Then, from (26) and (27) follows

$$\bar{R} = 1 - C\bar{\rho}^2 + O(\bar{\rho}^4), \quad \bar{T} = 1 - 3C\bar{\rho}^2 + O(\bar{\rho}^4). \tag{29}$$

Hence we are interested in a solution of the system (26), (27) for which

$$\bar{R}(0) = \bar{T}(0) = 1 \text{ and } \lim_{\bar{\rho}=0}(\bar{R} - \bar{T})/\bar{\rho} = 0. \tag{30}$$

Further, the boundary condition (28) must be satisfied, which will determine the constant C.

The integration may be performed, using the Runge-Kutta method [2] starting from the condition (30) and using different values of C so as to satisfy (28). Linear interpolation then yields the C-values of the first row below:

$n = 1$	3	5	7	10	
$C = 0.159$	0.0961	0.0687	0.0536	0.0404	acc. to BESK,
$C = 0.16$	0.096	0.0695	0.0540	0.0410	acc. to [3].

Knowing the value of C, the eigenvalue λ will be immediately obtained from (25) and thus the solution is completely determined.

The numerical calculations were carried out on the digital machine BESK, Stockholm. The coding and operation is due to Mr. H. O. Kreiss. The work was supported by the *Swedish Technical Research Council*. Free machine time was generously put at our disposal by the *Swedish Board for Computing Machinery*.

An independent check is obtained, actually calculating the coefficients of the power series, the first terms of which are given by (29). This was done by the writer in an earlier paper [3], the results of which are also given in the above table. The agreement is seen to be satisfactory.

From a physical point of view, the eigenvalue problem (19)–(20) should admit only one solution. There seems to exist but one real eigenvalue λ. This may possibly be understood from the fact that the function R may not have a zero between $\rho = 0$ and $\rho = 1$, as seen from (10), at least as long as terms of higher than second order are neglected in (1).

From a mathematical point of view one would expect similar eigenvalue problems to exist also for other forms of the membrane, e.g., elliptic, quadratic etc. It is easy to see that this will not be so in the general case, if stating our problem in ordinary Cartesian coordinates x, y of the plane, *viz.*, besides two additional equilibrium equations

$$\frac{\partial}{\partial t}\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] = \frac{\partial^2 \dot{\varepsilon}_x}{\partial y^2} + \frac{\partial^2 \dot{\varepsilon}_y}{\partial x^2} - 2\frac{\partial^2 \dot{\varepsilon}_{xy}}{\partial x \partial y},$$

$$\dot{\varepsilon}_x = \left(\frac{\sigma_e}{\sigma_c}\right)^{n-1}(\sigma_x - \sigma_y/2)/\sigma_c,$$

$$\dot{\varepsilon}_y = \left(\frac{\sigma_e}{\sigma_c}\right)^{n-1}(\sigma_y - \sigma_x/2)/\sigma_c,$$

$$\dot{\varepsilon}_{xy} = \frac{3}{2}\left(\frac{\sigma_e}{\sigma_c}\right)^{n-1}\frac{\tau_{xy}}{\sigma_c},$$

$$\sigma_x \frac{\partial^2 w}{\partial x^2} + 2\tau_{xy}\frac{\partial^2 w}{\partial x \partial y} + \sigma_y \frac{\partial^2 w}{\partial y^2} = -\frac{p}{h},$$

$$\sigma_e{}^2 = \sigma_x{}^2 - \sigma_x\sigma_y + \sigma_y{}^2 + 3\tau_{xy}^2.$$

No separation of variables seems possible. However, it may be possible for special coordinate systems such as the polar one used above.

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THE NOTION OF NON-HOMOGENEITY AS APPLIED TO A PROBLEM OF CONTINUED PLASTIC FLOW

W. OLSZAK and S. ZAHORSKI
Warsaw, Poland

§ 1. Introductory remarks

The notion of mechanical non-homogeneity may sometimes be used in such a way that its assumption will serve as an auxiliary method for effectively solving problems concerning elasto-plastic equilibrium and plastic flow. One of the main difficulties of the theory of plasticity is the non-linear character of its fundamental equations which can only rarely be integrated in a direct way. On the other hand, types of non-homogeneity can often be introduced such that the fundamental equations may be solved in an exact manner. If, in addition, certain free parameters are introduced into these functions, they can be fixed afterwards in such a way as to approximate, as closely as possible, actual conditions, that is, for instance, homogeneity or non-homogeneity of a specified type.

It has been stressed by other authors (cf., e.g., [2], Chap. IV) that it is often more desirable to know the solution in closed form for a sufficiently approximate physical model than to introduce an accurate model for which the corresponding solution can only be obtained for particular conditions by elaborate numerical methods.

Thus the solution may be obtained with sufficient accuracy; instead of rigorous physical assumptions and approximate formal results, we introduce approximate physical assumptions for which rigorous formal solutions are obtained. [1]

Such an approach is possible in problems of elasto-plastic equilibrium (as was shown, for instance, in a paper by one of the present authors [7*b*]) and also in problems of continued plastic flow (cf., for example, our paper [8]).

The object of the present paper is to discuss an appropriate choice of types of plastic non-homogeneity adapted to the analysis of the problem of plastic flow of eccentric rings (Fig. 1) subjected to the action of internal

[1] Such an approach is often encountered in other branches of mechanics of deformable bodies, as, for instance, in problems of the theory of finite deformations (cf. [10]), in recent studies of mechanical phenomena in granular bodies (cf. [1]) and in basic theoretical research on plasticity (cf. [4]). In an analogous manner, some of the approximate solutions obtained previously have been verified (cf., for example, [12]).

and external pressures p and q respectively, and, in the case of a ring with a cut, to the action of a normal force P in the cross-section.

The geometry of the ring is determined either by the inner and outer radii c and d respectively and the eccentricity e, or by the parameters $\rho = c/d$ and $\eta = e/d$. Every pair of values of ρ and η corresponds to one of the possible forms of the ring. The problem of plastic flow, the changes of geometry of the system being taken into account, is considered in a moving reference system, the variability of which is determined by the variable geometry of the material system in question.

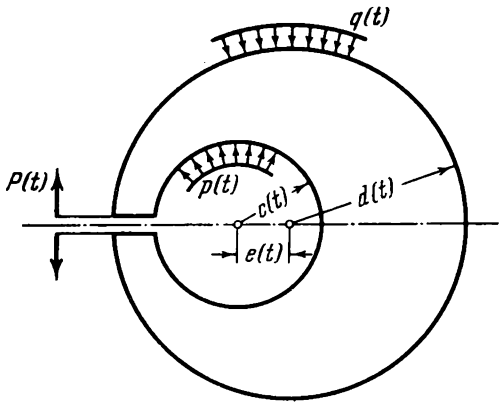

Fig. 1

The problem deals with continued plastic flow of an incompressible, perfectly plastic material between the smooth surfaces of eccentric circular cylinders under conditions of plane strain. In addition, it is assumed that the material satisfies the "quadratic" yield condition adapted to the case of a non-homogeneous material [7*a*].

If the external loads vary in time in a known manner, we are concerned with two-parameter or one-parameter (depending on whether both or one of the parameters ρ and η vary) continued plastic flow, as discussed in some detail in our paper, [8]. The statical elasto-plastic problem for an eccentric ring of a material characterized by one of the types of plastic non-homogeneity discussed in the present paper was analyzed by the first of the authors in [7*b*].

The types of plastic non-homogeneity discussed below are directly connected with the geometry of the ring under consideration and are readily represented, if the method of inversion mapping is used.

The development and wide application of the method of conformal mapping to plane problems of the theory of elasticity is mainly due to

N. I. Muskhelishvili, [6]. In the theory of plasticity, this method was used in [3], [9], [11]. The particular type of mapping "by inverted radii" was applied in [7*b*], [8].

§ 2. The inversion mapping

The following function of the complex variable will be used in our considerations:

$$Z_1 = \frac{C^2}{\bar{z}_1} = \frac{C^2}{\bar{z} \mp h}, \tag{2.1}$$

where $C > 0$ is the radius of the inversion circle, $h \geq 0$ denotes a parameter (depending on the geometry of the eccentric ring) and $\bar{z}_1$ or $\bar{z}$ are the conjugate complex to z_1 or z.

Letting

$$Z_1 = R_1 e^{i\Phi_1}, \quad z_1 = r_1' e^{i\varphi_1'}, \quad z = re^{i\varphi}, \tag{2.2}$$

one finds from (2.1)

$$R_1 = \frac{C^2}{r_1'} = \frac{C^2}{\sqrt{r^2 \mp 2hr\cos\varphi + h^2}}, \tag{2.3}$$

$$\Phi_1 = \varphi_1' = \arctan \frac{r\sin\varphi}{r\cos\varphi \mp h}.$$

The inverse relations are readily found to be

$$r = \sqrt{r_1'^2 \pm 2hr_1'\cos\varphi_1' + h^2}, \quad \varphi = \arctan \frac{r_1'\sin\varphi_1'}{r_1'\cos\varphi_1' \pm h}, \tag{2.4}$$

where the upper sign corresponds to the minus sign (—) and the lower to the plus sign (+) in equation (2.1).

Thus, the function (2.1) maps the interior of the circle of radius C in the original system O (R_1, Φ_1) into the exterior of the circle of radius C in the inverse system I (r_1', φ_1'), and vice versa.

If a function F (the stress function, for instance) satisfies the biharmonic equation in the (R_1, Φ_1) system

$$\nabla^2\nabla^2 F(R_1, \Phi_1) = 0, \tag{2.5}$$

where

$$\nabla^2 = \frac{1}{R_1}\frac{\partial}{\partial R_1} + \frac{1}{R_1^2}\frac{\partial^2}{\partial \Phi_1^2} + \frac{\partial^2}{\partial R_1^2}, \tag{2.6}$$

the corresponding biharmonic function F' in the (r_1', φ_1') system is related

to F by the expression

$$F'(r_1', \varphi_1') = \frac{C^2}{R_1^2} F(R_1, \Phi_1) = \frac{r_1'^2}{C^2} F\left(\frac{C^2}{r_1^1}, \varphi_1'\right). \tag{2.7}$$

§ 3. Generalized "interior" and "exterior" inversion

The eccentric ring represented in the original system O (R_1, Φ_1) in Figs. 2 and 3 may be transformed, by means of the function (2.1), into a

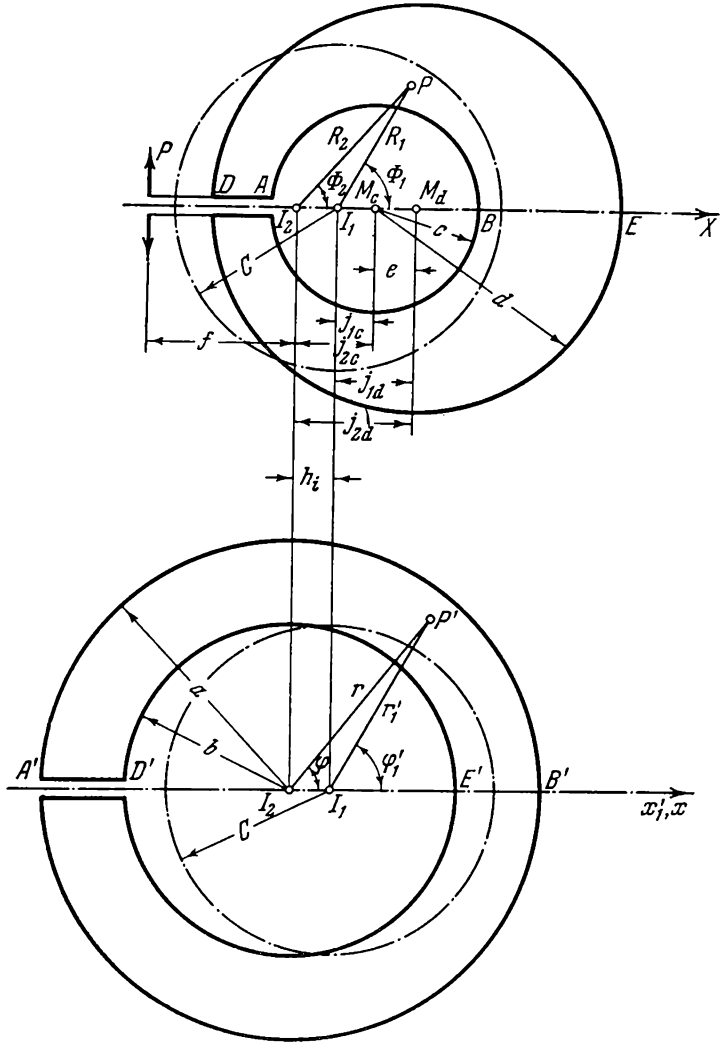

Fig. 2

concentric ring in the inverted system $I(r_1', \varphi_1')$ or (r, φ) in two different ways: either (1) by an "interior" inversion corresponding to the upper sign (—) in the equations (2.1) and (2.3) (cf. Fig. 2), or (2) by an "exterior" inversion corresponding to the lower sign (+) in the equations (2.1) and (2.3) (cf. Fig. 3).

If h denotes the distance between the inversion pole I_1 and the pole I_2 coinciding with the origin of coordinates (r, φ), then in the case of the "interior" inversion (I_1 being always located inside the eccentric ring)

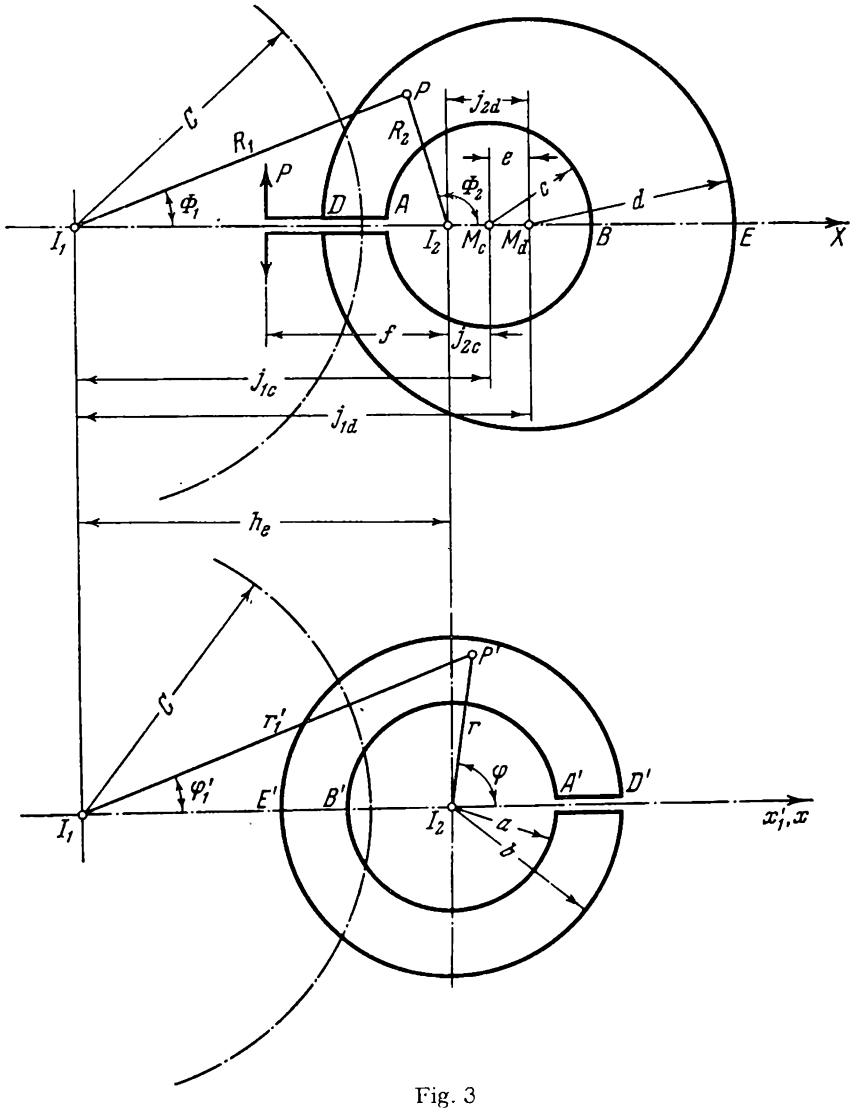

Fig. 3

the corresponding outer circumference of the eccentric ring transforms into the inner circumference of the concentric ring, and vice-versa.

In the case of "exterior" inversion (I_1 being always located outside the eccentric ring), the outer and the inner circumferences of the eccentric ring remain so also for the concentric ring.

The following geometrical relations follow from Figs. 2 and 3:

$$
\begin{gathered}
c = \pm C^2 \frac{a}{a^2 - h^2}, \quad d = \pm C^2 \frac{b}{b^2 - h^2}, \\
e = \pm h \frac{C^2(a^2 - b^2)}{(a^2 - h^2)(b^2 - h^2)}, \\
j_{1c} = \pm C^2 \frac{h}{a^2 - h^2}, \quad j_{1d} = \pm C^2 \frac{h}{b^2 - h^2}, \\
j_{2c} = \pm h\left[1 + \frac{C^2}{a^2 - h^2}\right], \quad j_{2d} = \pm h\left[1 + \frac{C^2}{b^2 - h^2}\right],
\end{gathered} \tag{3.1}
$$

where the upper (+) sign is to be taken for the "interior" inversion (Fig. 2), and the lower (—) sign for the "exterior" inversion (Fig. 3).

Introducing the dimensionless parameters ρ and η, we obtain additional expressions relating these parameters to the geometrical quantities in the inverted system I:

$$
\eta = \frac{h}{b} \frac{(a^2 - b^2)}{(a^2 - h^2)}, \quad \rho = \frac{a}{b} \frac{(b^2 - h^2)}{(a^2 - h^2)}. \tag{3.2}
$$

For every eccentric ring, the quantity h (different for "interior" and "exterior" inversion) is determined in a unique manner in terms of c, d, e, or of the parameters ρ and η respectively. In the case of "interior" inversion, $h = 0$ corresponds to a concentric ring in the original system O; in the case of "exterior" inversion, the concentric ring will be obtained for $h = \infty$.

If the quantities c, d and e (or ρ and η) are considered to be functions of the time t, then the quantities a, b and h will also depend on the time, in agreement with the relations (3.1) and (3.2).

It should be mentioned that the assumption $C = 1$ in our considerations does not restrict their generality; it only fixes in a particular manner the scale of the inversion mapping.

§ 4. Relations between stress fields. The yield condition

If the stress field in the eccentric ring under consideration is determined by means of a biharmonic stress function $F(R_1, \Phi_1)$, the associated bi-

harmonic functions $F'(r_1', \varphi_1')$ or $F'(r, \varphi)$ [corresponding to the transformation (2.7)] determine the stress field in the inverted concentric ring.

Making use of the relations (2.7) and the geometrical relations (2.3) and (2.4) and taking into account the known formulae for the stresses as expressed by the corresponding stress function in a polar coordinate system, we obtain the following relations between the stress fields in the original system O and those in the inverted system I (cf., for example, [8])

$$\sigma_R = \frac{1}{C^2}\left[\sigma_r'(r^2 \mp 2hr\cos\varphi + h^2) + \right.$$
$$\left. + 2F' - 2(r \mp h\cos\varphi)\frac{\partial F'}{\partial r} \mp 2\frac{h}{r}\frac{\partial F'}{\partial \varphi}\sin\varphi\right],$$

$$\sigma_T = \frac{1}{C^2}\left[\sigma_t'(r^2 \mp 2hr\cos\varphi + h^2) + \right. \tag{4.1}$$
$$\left. + 2F' - 2(r \mp h\cos\varphi)\frac{\partial F'}{\partial r} \mp 2\frac{h}{r}\frac{\partial F'}{\partial \varphi}\sin\varphi\right],$$

$$\tau_{RT} = -\frac{1}{C^2}\tau_{rt}'(r^2 \mp 2hr\cos\varphi + h^2).$$

The upper (—) sign indicates, as before, "interior" inversion, and the lower (+) sign "exterior" inversion. It may be mentioned that the relations (4.1), in view of their specific structure, preserve their validity for any two corresponding (by means of the inversion) systems of co-ordinates, for instance, R_1, Φ_1, R, Φ and r_1', φ_1', r, φ, respectively.

With the particular assumption $h = 0$ and $C = 1$ (the inversion pole I_1 thus coinciding with I_2), we obtain J. H. Michell's equations [5], known in the literature; they constitute one of the possible particular cases of our more general equations:

$$\sigma_R = \sigma_r'r^2 + 2F' - 2r\frac{\partial F'}{\partial r},$$

$$\sigma_T = \sigma_t'r^2 + 2F' - 2r\frac{\partial F'}{\partial r}, \tag{4.2}$$

$$\tau_{RT} = -\tau_{rt}'r^2.$$

In what follows, in view of the symmetry of the boundary conditions, we shall base our considerations on a biharmonic stress function F' (in the inverted system I) in the following form which depends only on the

variable r:

$$F'(r, t) = A_0(t) + B_0(t) \log r + C_0(t) r^2 + D_0(t) r^2 \log r, \tag{4.3}$$

for which the equations (4.1) undergo further significant simplifications. A_0, B_0, C_0, D_0 are quantities depending on the external loads $[p(t), q(t), P(t)]$, variable in time t, and on the yield limit also variable in time and space.

For the stress function in the form (4.3) the shear stresses τ_{RT} are identically zero.

If the eccentric ring has a cut at the section of least thickness (cf. Fig. 1), the corresponding resultant normal force in this cross-section will be obtained from the expression

$$-P = \int_A^D (\sigma_T)_{\varphi=\pi} \, dR_2 = \frac{\partial F}{\partial R_2}\Big|_A^D = \left(\frac{\partial F'}{\partial r} - \frac{2F'}{r+h}\right)\Big|_b^a \tag{4.4}$$

for "interior" inversion, and the expression

$$-P = \int_A^D (\sigma_T)_{\varphi=0} \, dR_2 = \frac{\partial F}{\partial R_2}\Big|_A^D = \left(\frac{\partial F'}{\partial r} - \frac{2F'}{r+h}\right)\Big|_a^b \tag{4.5}$$

for "exterior" inversion.

The position of the force P can be determined, for example, when its moment about the pole I_2 is known. We obtain, respectively,

$$-Pf = \int_A^D (\sigma_T)_{\varphi=\pi} R_2 \, dR_2 = \left(\frac{\partial F}{\partial R_2} R_2 - F\right)\Big|_A^D = \tag{4.6}$$

$$= \left[\frac{F'}{(r+h)^2} - \frac{2hF'}{r+h} + \left(h - \frac{1}{r+h}\right)\frac{\partial F'}{\partial r}\right]_b^a,$$

and

$$-Pf = \int_A^D (\sigma_T)_{\varphi=0} R_2 \, dR_2 = \left(\frac{\partial F}{\partial R_2} R_2 - F\right)\Big|_A^D = \tag{4.7}$$

$$= \left[\frac{F'}{(r+h)^2} - \frac{2hF'}{r+h} + \left(h - \frac{1}{r+h}\right)\frac{\partial F'}{\partial r}\right]_a^b.$$

Under the present conditions, the "quadratic" yield criterion will be written in the form (cf. [7a])

$$(\sigma_R - \sigma_T)^2 + 4\tau^2_{RT} = 4[K(P, t)]^2, \tag{4.8}$$

and hence, bearing in mind that $\tau_{RT} \equiv 0$,

$$|\sigma_R - \sigma_T| = 2K(P, t). \tag{4.9}$$

We postulate the following form of the function $K(P, t)$

$$K(P, t) = k(t)\bar{K}(P), \tag{4.10}$$

where $\bar{K}(P)$ is a function of the co-ordinates of the point $P(R, \Phi)$, and $k(t)$ is a function of the time t which is independent of the co-ordinates of the point P.

For the sake of clarity, all the geometrical quantities pertaining to the "interior" inversion will in the following be denoted by the subscript "i", and those pertaining to the "exterior" inversion by the subscript "e".

§ 5. Types (I) and (III) of non-homogeneity connected with the inversion pole I_1

For the first (I) and the third (III) types of non-homogeneity we assume the non-homogeneity connected with the pole I_1 in the system O (cf. Figs. 2 and 3) and expressed for "interior" and "exterior" inversion by

$$\begin{aligned} K_i(P, t) &= k(t)\frac{1}{R_{1i}^2}, \qquad \text{(I)} \\ K_e(P, t) &= k(t)\frac{1}{R_{1e}^2}, \qquad \text{(III)} \end{aligned} \tag{5.1}$$

or, in the co-ordinates of the point $P'(r, \varphi)$ in the inverted system I, by

$$\begin{aligned} K_i(P', t) &= k(t)r_{1i}'^2 = k(t)(r_i^2 - 2h_i r_i \cos\varphi_i + h_i^2), \\ K_e(P', t) &= k(t)r_{1e}'^2 = k(t)(r_e^2 + 2h_e r_e \cos\varphi_e + h_e^2). \end{aligned} \tag{5.2}$$

For $h_i = 0$, i.e., for a concentric ring, the pole I_1 coincides with the centre of the ring, and the corresponding $\bar{K}(P)$ decreases with the square of the radius R. For $h_e \to \infty$, i.e., also for a concentric ring ($e \to 0$), the material tends to homogeneity; however, the associated $\bar{K}(P)$ connected with the point at infinity tends gradually to zero, the material thus becoming perfectly "soft".

It can easily be observed that the biharmonic function

$$F'(r_j, t) = A_0(t) + D_0(t)r_j^2 \log r_j \quad (j = i \text{ or } e) \tag{5.3}$$

may constitute the point of departure for the determination of the stress

components satisfying the assumed boundary conditions and the yield criterion in the form (4.9) for the non-homogeneity determined by (5.1) or (5.2).

In view of the singular term of the biharmonic function (5.3), it is necessary to introduce a cut (Fig. 1). With a transformation of the type (2.7), this singularity is preserved. As a consequence, it would lead to multi-valued circumferential displacement components (cf. [7*b*]).

The stress components take the form

$$\begin{aligned} \sigma_R &= 2A_0 + D_0(2h^2 \log r - r^2 + h^2), \\ \sigma_T &= 2A_0 + D_0(2h^2 \log r + r^2 + 3h^2 \mp 4hr \cos\varphi), \\ \tau_{RT} &= 0, \end{aligned} \tag{5.4}$$

where the upper (—) sign refers to the case of "interior" inversion and the lower (+) sign to that of "exterior" inversion.

It can also be seen that

$$2A_0 = D_0(a^2 - h^2 - 2h^2 \log a) - p = D_0(b^2 - h^2 - 2h^2 \log b) - q \tag{5.5}$$

and

$$\begin{aligned} D_0 &= k(t) \quad \text{for } p > q, \\ D_0 &= -k(t) \quad \text{for } p < q; \end{aligned} \tag{5.6}$$

the corresponding pressure difference $\Pi(t)$, necessary for maintaining the material in the plastic state, is

$$\Pi(t) = p(t) - q(t) = D_0\left[(a^2 - b^2) - 2h^2 \log \frac{a}{b}\right]. \tag{5.7}$$

In agreement with (4.4) and (4.5), the resultant normal force in the cut is expressed by the formula

$$-P(t) = \pm \frac{qb}{(b+h)h} \mp \frac{pa}{(a+h)h} \pm D_0 \frac{a^2 - b^2}{h}, \tag{5.8}$$

whereas its moment with respect to I_2, in agreement with (4.6) and (4.7), is given by

$$\begin{aligned} -P(t)f(t) = \tfrac{1}{2}\Bigg\{ &\pm D_0 \frac{(a^2 - b^2)(2h^2 - 1)}{h^2} + \frac{\Pi(t)(2h^2 - 1)}{h^2} \pm \\ &\pm p\left[\frac{2h}{a+h} - \frac{1}{(a+h)^2}\right] \mp q\left[\frac{2h}{b+h} - \frac{1}{(b+h)^2}\right]\Bigg\}. \end{aligned} \tag{5.9}$$

The upper signs in (5.8) and (5.9) refer to "interior" inversion, the lower signs to "exterior" inversion.

§ 6. Types (II) and (IV) of non-homogeneity connected with the two poles I_1 and I_2

For the second (II) and fourth (IV) types, we shall assume the non-homogeneity determined for "interior" and "exterior" inversion by the expressions

$$\begin{aligned} K_i(P,t) &= k(t)\,\frac{1}{R_{2i}^2+(h_i^2-1)(R_{1i}^2-1)}\,, \qquad \text{(II)}\\ K_e(P,t) &= k(t)\,\frac{1}{R_{2e}^2+(h_e^2-1)(R_{1e}^2-1)}\,, \qquad \text{(IV)} \end{aligned} \tag{6.1}$$

respectively, or in the co-ordinates of the point $P'(r,\varphi)$ in the inverted system I by

$$\begin{aligned} K_i(P',t) &= k(t)\,\frac{r_{1i}'^2}{r_i^2} = k(t)\,\frac{r_i^2-2h_ir_i\cos\varphi_i+h_i^2}{r_i^2}\,,\\ K_e(P',t) &= k(t)\,\frac{r_{1e}'^2}{r_e^2} = k(t)\,\frac{r_e^2+2h_er_e\cos\varphi_e+h_e^2}{r_e^2}\,. \end{aligned} \tag{6.2}$$

The curves $\bar{K}=$ const. correspond in this case to concentric circles with center at a distance $1/h_i$ from the pole I_1 in the direction of the pole I_2 for "interior" inversion, and at a distance $1/h_e$ for "exterior" inversion.

If in the statical problem the dimensions of the eccentric ring are chosen so that $h_i=1$ or $h_e=1$, then, in agreement with (6.1), the non-homogeneity will be connected with the pole I_2 in an analogous manner as in the preceding section.

The biharmonic function

$$F'(r_j,t) = A_0(t) + B_0(t)\log r_j \quad (j=i \text{ or } e) \tag{6.3}$$

permits determination of the stress components satisfying the boundary conditions and the yield criterion (4.9) with the required type of non-homogeneity (6.1) or (6.2).

It should be stressed that for the function (6.3) the introduction of the cut in the eccentric ring is no longer necessary (in contrast to the function of § 5).

The following results preserve their validity for the ring with and without slit.

The stress components take the form

$$\begin{aligned} \sigma_R &= 2A_0 + B_0\left(2\log r + \frac{h^2}{r^2} - 1\right),\\ \sigma_T &= 2A_0 + B_0\left(2\log r - \frac{h^2}{r^2} - 3 \pm 4\frac{h}{r}\cos\varphi\right),\\ \tau_{RT} &= 0, \end{aligned} \tag{6.4}$$

where the upper (+) sign refers to "interior" inversion and the lower (—) sign to "exterior" inversion. It can be shown that

$$2A_0 = B_0\left(1 - \frac{h^2}{a^2} - 2\log a\right) - p = B_0\left(1 - \frac{h^2}{b^2} - 2\log b\right) - q \tag{6.5}$$

and

$$\begin{aligned} B_0 &= -k(t) \quad \text{for } p > q, \\ B_0 &= k(t) \qquad \text{for } p < q, \end{aligned} \tag{6.6}$$

and that the corresponding pressure difference $\Pi(t)$, necessary for maintaining the material in the plastic state, is

$$\Pi(t) = p(t) - q(t) = B_0\left[\frac{a^2 - b^2}{a^2 b^2} h^2 - 2\log\frac{a}{b}\right]. \tag{6.7}$$

If the ring is slit, the resultant normal force in the cross-section and its moment about the pole I_2 are expressed by the equations

$$-P(t) = \pm\frac{p}{a+h} \mp \frac{q}{b+h} \pm B_0 h\frac{a^2 - b^2}{a^2 b^2}, \tag{6.8}$$

$$\begin{aligned} -P(t)f(t) = \tfrac{1}{2}\Bigg\{B_0\left[\pm\frac{a^2 - ah - 2}{(a+h)a} \mp \frac{b^2 - bh - 2}{(b+h)b} \mp 2h^2\frac{a^2 - b^2}{a^2 b^2}\right] \pm \\ \pm p\left[\frac{2h}{a+h} + \frac{1}{(a+h)^2}\right] \mp q\left[\frac{2h}{b+h} + \frac{1}{(b+h)^2}\right]\Bigg\}, \end{aligned} \tag{6.9}$$

where the upper signs refer to "interior" inversion and the lower signs to "exterior" inversion.

The case $h_i = 0$ is of interest. It corresponds, in agreement with (6.1) (for $h_i = 0$, $R_{1i} = R_{2i}$), to a concentric ring characterized by uniform distribution of the yield limit. Assuming, in addition, that $q = 0$, the relations (6.4) will take a form well-known from the literature on the analysis of homogeneous concentric rings (cylinders) subjected to internal pressure, viz.,

$$\begin{aligned} \sigma_R &= 2k\log\frac{R}{d}, \\ \sigma_T &= 2k\left(1 + \log\frac{R}{d}\right). \end{aligned} \tag{6.10}$$

These results are contained, as one of the possible particular cases, in our more general relations.

§ 7. A more general type of plastic non-homogeneity (V)

The investigation of various possible combinations of the terms of the biharmonic function in the form (4.3) shows that, besides the four non-homogenity types (I), (II), (III), and (IV) above, various linear combinations may be considered. This follows also from the linear character of the yield condition (4.9).

A more general type of non-homogeneity (V) may be written in the form

$$(7.1)\qquad K(P,t) = k(t)\left[\pm\frac{\kappa^2(t)}{R_{1i}^2} \pm \frac{\lambda^2(t)}{R_{2i}^2 + (h_i - 1)(R_{1i}^2 - 1)} \pm \right.$$
$$\left. \pm \frac{\mu^2(t)}{R_{1e}^2} \pm \frac{\nu^2(t)}{R_{2e}^2 + (h_e - 1)(R_{1e}^2 - 1)}\right],$$

or, in the inverted co-ordinates (the system I),

$$(7.2)\quad K(P',t) = k(t)\left[\pm\kappa^2(t)r_{1i}'^2 \pm \lambda^2(t)\frac{r_{1i}'^2}{r_i^2} \pm \mu^2(t)r_{1e}'^2 \pm \nu^2(t)\frac{r_{1e}'^2}{r_e^2}\right],$$

where $\kappa^2(t)$, $\lambda^2(t)$, $\mu^2(t)$ and $\nu^2(t)$ denote arbitrary positively determined functions of time t characterizing the "time relations" of the superposed types of non-homogeneity. In the statical problem, these functions become constants.

It should also be added that in view of the necessarily required positiveness of the function K, not every combination of the signs (+) and (−) in the Eqs. (7.1) or (7.2) is possible. The corresponding relations, for different (+) and (−) signs in (7.1) and (7.2), and the range of variation of the involved functions are listed in Table 1.

The stress components take the form

$$\sigma_R = 2\bar{A}_0 + \bar{B}_0\left(2\log r_i + \frac{h_i^2}{r_i^2} - 1\right) + \bar{D}_0(2h_i^2\log r_i - r_i^2 + h_i^2) +$$
$$+ 2\bar{\bar{A}}_0 + \bar{\bar{B}}_0\left(2\log r_e + \frac{h_e^2}{r_e^2} - 1\right) + \bar{\bar{D}}_0(2h_e^2\log r_e - r_e^2 + h_e^2),$$

$$(7.3)\quad \sigma_T = 2\bar{A}_0 + \bar{B}_0\left(2\log r_i - \frac{h_i^2}{r_i^2} - 3 + 4\frac{h_i}{r_i}\cos\varphi_i\right) +$$
$$+ \bar{D}_0(2h_i^2\log r_i + r_i^2 + 3h_i^2 - 4h_i r_i\cos\varphi_i) + 2\bar{\bar{A}}_0 +$$
$$+ \bar{\bar{B}}_0\left(2\log r_e - \frac{h_e^2}{r_e^2} - 3 - 4\frac{h_e}{r_e}\cos\varphi_e\right) +$$
$$+ \bar{\bar{D}}_0(2h_e^2\log r_e + r_e^2 + 3h_e^2 + 4h_e r_e\cos\varphi_e),$$

$$\tau_{RT} = 0.$$

TABLE 1

No	Type of non-homogeneity according to (7.2)	Range of variation of the functions $\kappa^2, \lambda^2, \mu^2, \nu^2$	$p > q$				$p < q$			
			$\bar{B}_0$	$\bar{\bar{B}}_0$	$\bar{D}_0$	$\bar{\bar{D}}_0$	$\bar{B}_0$	$\bar{\bar{B}}_0$	$\bar{D}_0$	$\bar{\bar{D}}_0$
1	$K = k(t)\left(\kappa^2 r_{1i}^2 + \lambda^2 \frac{r_{1i}^2}{r_i^2} + \mu^2 r_{1e}^2 + \nu^2 \frac{r_{1e}^2}{r_e^2}\right)$	$0 \leqslant \frac{\lambda^2}{\kappa^2} \leqslant \infty$ $0 \leqslant \frac{\nu^2}{\mu^2} \leqslant \infty$	$-\lambda^2 k$	$-\nu^2 k$	$\kappa^2 k$	$\mu^2 k$	$\lambda^2 k$	$\nu^2 k$	$-\kappa^2 k$	$-\mu^2 k$
2	$K = k(t)\left(-\kappa^2 r_{1i}^2 + \lambda^2 \frac{r_{1i}^2}{r_i^2} + \mu^2 r_{1e}^2 + \nu^2 \frac{r_{1e}^2}{r_e^2}\right)$	$a_i^2 \leqslant \frac{\lambda^2}{\kappa^2} \leqslant \infty$ $0 \leqslant \frac{\nu^2}{\mu^2} \leqslant \infty$	$-\lambda^2 k$	$-\nu^2 k$	$-\kappa^2 k$	$\mu^2 k$	$\lambda^2 k$	$\nu^2 k$	$\kappa^2 k$	$-\mu^2 k$
3	$K = k(t)\left(\kappa^2 r_{1i}^2 - \lambda^2 \frac{r_{1i}^2}{r_i^2} + \mu^2 r_{1e}^2 + \nu^2 \frac{r_{1e}^2}{r_e^2}\right)$	$0 \leqslant \frac{\lambda^2}{\kappa^2} \leqslant b_i^2$ $0 \leqslant \frac{\nu^2}{\mu^2} \leqslant \infty$	$\lambda^2 k$	$-\nu^2 k$	$\kappa^2 k$	$\mu^2 k$	$-\lambda^2 k$	$\nu^2 k$	$-\kappa^2 k$	$-\mu^2 k$
4	$K = k(t)\left(\kappa^2 r_{1i}^2 + \lambda^2 \frac{r_{1i}^2}{r_i^2} - \mu^2 r_{1e}^2 + \nu^2 \frac{r_{1e}^2}{r_e^2}\right)$	$0 \leqslant \frac{\lambda^2}{\kappa^2} \leqslant \infty$ $a_e^2 \leqslant \frac{\nu^2}{\mu^2} \leqslant \infty$	$-\lambda^2 k$	$-\nu^2 k$	$\kappa^2 k$	$-\mu^2 k$	$\lambda^2 k$	$\nu^2 k$	$-\kappa^2 k$	$\mu^2 k$
5	$K = k(t)\left(-\kappa^2 r_{1i}^2 + \lambda^2 \frac{r_{1i}^2}{r_i^2} - \mu^2 r_{1e}^2 + \nu^2 \frac{r_{1e}^2}{r_e^2}\right)$	$a_i^2 \leqslant \frac{\lambda^2}{\kappa^2} \leqslant \infty$ $a_e^2 \leqslant \frac{\nu^2}{\mu^2} \leqslant \infty$	$-\lambda^2 k$	$-\nu^2 k$	$-\kappa^2 k$	$-\mu^2 k$	$\lambda^2 k$	$\nu^2 k$	$\kappa^2 k$	$\mu^2 k$
6	$K = k(t)\left(\kappa^2 r_{1i}^2 - \lambda^2 \frac{r_{1i}^2}{r_i^2} - \mu^2 r_{1e}^2 + \nu^2 \frac{r_{1e}^2}{r_e^2}\right)$	$0 \leqslant \frac{\lambda^2}{\kappa^2} \leqslant b_i^2$ $a_e^2 \leqslant \frac{\nu^2}{\mu^2} \leqslant \infty$	$\lambda^2 k$	$-\nu^2 k$	$\kappa^2 k$	$-\mu^2 k$	$-\lambda^2 k$	$\nu^2 k$	$-\kappa^2 k$	$\mu^2 k$
7	$K = k(t)\left(\kappa^2 r_{1i}^2 + \lambda^2 \frac{r_{1i}^2}{r_i^2} + \mu^2 r_{1e}^2 - \nu^2 \frac{r_{1e}^2}{r_e^2}\right)$	$0 \leqslant \frac{\lambda^2}{\kappa^2} \leqslant \infty$ $0 \leqslant \frac{\nu^2}{\mu^2} \leqslant b_e^2$	$-\lambda^2 k$	$\nu^2 k$	$\kappa^2 k$	$\mu^2 k$	$\lambda^2 k$	$-\nu^2 k$	$-\kappa^2 k$	$-\mu^2 k$
8	$K = k(t)\left(-\kappa^2 r_{1i}^2 + \lambda^2 \frac{r_{1i}^2}{r_i^2} + \mu^2 r_{1e}^2 - \nu^2 \frac{r_{1e}^2}{r_e^2}\right)$	$a_i^2 \leqslant \frac{\lambda^2}{\kappa^2} \leqslant \infty$ $0 \leqslant \frac{\nu^2}{\mu} \leqslant b_e^2$	$-\lambda^2 k$	$\nu^2 k$	$-\kappa^2 k$	$\mu^2 k$	$\lambda^2 k$	$-\nu^2 k$	$\kappa^2 k$	$-\mu^2 k$
9	$K = k(t)\left(\kappa^2 r_{1i}^2 - \lambda^2 \frac{r_{1i}^2}{r_i^2} + \mu^2 r_{1e}^2 - \nu^2 \frac{r_{1e}^2}{r_e^2}\right)$	$0 \leqslant \frac{\lambda^2}{\kappa^2} \leqslant b_i^2$ $0 \leqslant \frac{\nu^2}{\mu^2} \leqslant b_e^2$	$\lambda^2 k$	$\nu^2 k$	$\kappa^2 k$	$\mu^2 k$	$-\lambda^2 k$	$-\nu^2 k$	$-\kappa^2 k$	$-\mu^2 k$

The quantity $2\bar{A}_0 + 2\bar{\bar{A}}_0$ may be found from the condition

$$(7.4)\quad 2\bar{A}_0 + 2\bar{\bar{A}}_0 = \bar{B}_0\left(1 - \frac{h_i^2}{a_i^2} - 2\log a_i\right) + \bar{\bar{B}}_0\left(1 - \frac{h_e^2}{a_e^2} - 2\log a_e\right) +$$

$$+ \bar{D}_0(a_i^2 - h_i^2 - 2h_i^2 \log a_i) + \bar{\bar{D}}_0(a_e^2 - h_e^2 - 2h_e^2 \log a_e) - p =$$

$$= \bar{B}_0\left(1 - \frac{h_i^2}{b_i^2} - 2\log b_i\right) + \bar{\bar{B}}_0\left(1 - \frac{h_e^2}{b_e^2} - 2\log b_e\right) +$$

$$+ \bar{D}_0(b_i^2 - h_i^2 - 2h_i^2 \log b_i) + \bar{\bar{D}}_0(b_e^2 - h_e^2 - 2h_e^2 \log b_e) - q;$$

the corresponding values of the functions $\bar{B}_0$, $\bar{\bar{B}}_0$, $\bar{D}_0$, $\bar{\bar{D}}_0$ are given in Table I as functions of the pressures p and q.

The pressure difference $\Pi(t)$ necessary for maintaining the material of the ring in a plastic state takes the form

$$(7.5)\quad \Pi(t) = p(t) - q(t) =$$

$$= \bar{B}_0\left[\frac{a_i^2 - b_i^2}{a_i^2 b_i^2} h_i^2 - 2\log\frac{a_i}{b_i}\right] + \bar{\bar{B}}_0\left[\frac{a_e^2 - b_e^2}{a_e^2 b_e^2} h_e^2 - 2\log\frac{a_e}{b_e}\right] +$$

$$+ \bar{D}_0\left[(a_i^2 - b_i^2) - 2h_i^2 \log\frac{a_i}{b_i}\right] + \bar{\bar{D}}_0\left[(a_e^2 - b_e^2) - 2h_e^2 \log\frac{a_e}{b_e}\right].$$

The resultant normal force in the cut and its moment in relation to the pole I_2 are determined in a manner analogous to that described above.

§ 8. Concluding remarks

By judicious choice of the functions $\kappa^2(t)$, $\lambda^2(t)$, $\mu^2(t)$, $\nu^2(t)$ we can attenuate or strengthen the influence of each particular type of non-homogeneity.

Setting three of the four functions equal to zero we obtain, of course, one of the four types of non-homogeneity discussed above.

The manner of choosing the functions $\kappa^2(t)$, $\lambda^2(t)$, $\mu^2(t)$, $\nu^2(t)$ and $k(t)$ in order to approach in the best possible way the actual field of mechanical properties of the material, will be discussed separately. This concerns both the case when the material flowing between eccentric surfaces is homogeneous and the case when it is non-homogeneous with a specified type of non-homogeneity.

These properties (i.e., the homogeneity or a determined type of non-homogeneity) may be either independent of time or else functions of the time, given in an appropriate form for a non-stationary or quasi-stationary process of plastic flow of the material in question.

Lower and upper bounds may also be obtained, as will be shown separately.

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ON THE PROPERTIES OF A SINGULAR INTEGRAL IN SPACE AND THEIR APPLICATION TO A SYSTEM OF SINGULAR INTEGRAL EQUATIONS

W. POGORZELSKI
Warsaw, Poland

§ 1. Introduction

In [1] we gave the definition and studied the principal properties of a class of discontinuous functions in space, analogous to the class $\mathsf{H}_\alpha^\hbar$ of discontinuous functions defined on the union of open arcs in the complex plane which we investigated in [2] and [3]. We recall briefly the definition and principal properties of this class, as studied in [1].

Let there be given in the n-dimensional Euclidean space E_n a finite set of $p+1$ closed $(n-1)$-dimensional Liapunov surfaces $S_0, S_1, \cdots, S_p$ $(p \geq 0)$. We assume that these surfaces do not have common points and that the surfaces $S_1, \cdots, S_p$ lie in an arbitrary manner inside a finite region Ω_0 bounded by the surface S_0.

We denote by Ω the set of all points of the region Ω_0 which do not lie on the surfaces $S_1, S_2, \cdots, S_p$. This set is the sum $\Omega = \Omega_1 + \Omega_2 + \cdots + \Omega_q$ of separate, simply or multiply connected regions $\Omega_1, \cdots, \Omega_q$. In the case of absence of the surfaces $S_1, \cdots, S_p$ we have $\Omega = \Omega_0$.

DEFINITION 1. We call the class $\mathsf{H}_\alpha^\hbar$ the union of all complex or real functions $f(x)$ defined on the set of points $x(\xi_1, \cdots, \xi_n) \in \Omega_0 + S_0$ (or only on the set Ω) which satisfy the inequality

$$|f(x)| < \frac{M_f}{|x - x_S|^\alpha}, \quad (x \in \Omega) \tag{1.1}$$

and a generalized Hölder condition

$$|f(x) - f(y)| < \frac{k_f |x - y|^\hbar}{|x - x_S|^{\alpha+\hbar}}. \tag{1.2}$$

Here $|x - y|$ is the Euclidean distance between the arbitrary points $x(\xi_1, \cdots, \xi_n)$ and $y(\eta_1, \cdots, \eta_n)$, which lie simultaneously inside one of the regions $\Omega_1, \cdots, \Omega_q$, x_S is a point on the surfaces of discontinuity $S_0, S_1, \cdots, S_p$ at which the distance $|x - x_S|$ from an internal point $x \in \Omega$ attains its lower bound. We assume that $|x - x_S| \leq |y - y_S|$;

the constant parameters α and h of the class satisfy the inequalities

$$0 \leq \alpha < 1,\ 0 < h < 1,\ \alpha + h < 1, \tag{1.3}$$

the positive constants M_f and k_f are arbitrary.

THEOREM 1. *The class* $\mathsf{H}_{\alpha_1}^{h_1}$ *is a part of the class* H_{α}^{h}, *if the parameters of these classes satisfy the inequalities*

$$\alpha_1 \leq \alpha,\ h_1 \geq h.$$

The defined functions of the class H_{α}^{h} are closely linked to the concept of the singular integral in the Euclidean space E_n. These integrals were studied in the works of F. Tricomi [4], G. Giraud [5], S. G. Mikhlin [6] and recently in the paper [7] by A. Zygmund. We adopt the following form of singular integral, introduced in [7]:

$$\int_{\Omega} N(x-y)f(y)dy = \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} N(x-y)f(y)dy. \tag{1.4}$$

Here Ω_ϵ denotes the set of all points y of the region Ω, the distance $|x-y|$ of which from an internal point x is larger than a positive number ε, $f(y)$ is a complex integrable function defined in the region Ω and $N(x)$ is a function of the form

$$N(x) = \frac{K(x')}{|x|^n}, \tag{1.5}$$

where x' denotes the central projection of the point x of the space E_n on the surface ω of the sphere with centre at the point $(0, \cdots, 0)$ and unit radius, so that $x = x'|x|$.

We assume that the complex function $K(x')$, defined on the surface ω, satisfies the fundamental condition

$$\int_{\omega} K(x')dx' = 0, \tag{1.6}$$

where dx' is the element of the surface ω. Then we assume that the function $K(x')$ satisfies on the surface ω a simple Hölder condition

$$|K(x') - K(y')| < k_\omega |x' - y'|^{h_\omega},\ (0 < h_\omega \leq 1). \tag{1.7}$$

A fundamental property of the functions of the class H_{α}^{h} in conformity with the singular integral is presented by the following theorem.

THEOREM 2. *If the complex function $f(y)$, defined on the set Ω, belongs to the class H_α^h, $(\alpha > 0)$, the function $\varphi(x)$, defined by the singular integral*

$$\varphi(x) = \int_\Omega N(x - y)f(y)dy \tag{1.8}$$

at each point $x \in \Omega$, belongs to the class $H_\alpha^{h'}$, and, in fact, it satisfies the inequalities $(h \neq h_\omega)$

$$\begin{aligned} |\varphi(x)| &< \frac{C_1 M_f + C_2 k_f}{|x - x_S|^\alpha}, \\ |\varphi(x) - \varphi(z)| &< \frac{(C_1' M_f + C_2' k_f)|x - z|^{h'}}{|x - x_S|^{\alpha + h'}}, \end{aligned} \tag{1.9}$$

where $h' = \min(h, h_\omega)$, M_f and k_f are positive constants, linked to the function $f(x)$ by the inequalities (1.1) *and* (1.2), *C_1, C_2, C_1', C_2' are positive constants which are independent of the function $f(x)$; in addition, it is assumed that $|x - x_S| \leq |z - z_S|$.*

This theorem was proved in our paper [1]. In the case when $h < h_\omega$, the transformed function [8] belongs to the same class H_α^h as the given function $f(y)$; this property has a basic significance for the study of singular equations.

DEFINITION 2. We call $H_\alpha^h(S_{k_1}, \cdots, S_{k_r})$ the union of all functions of the class H_α^h which are bounded or almost bounded in the neighborhood of the surfaces of discontinuity $S_{k_1}, \cdots, S_{k_r}$ $(r \leq p)$, i.e., which satisfy the inequalities

$$\begin{aligned} |f(x)| &< \frac{M_f}{|x - x_S|^{\theta\alpha}}, \\ |f(x) - f(y)| &< \frac{k_f |x - y|^h}{|x - x_S|^{\theta\alpha + h}}, \end{aligned} \tag{1.10}$$

where θ is zero or an arbitrarily small positive number, if $x_S \in S_{k_1}, \cdots, S_{k_r}$ and otherwise $\theta = 1$.

THEOREM 3. *If the function $f(y)$ belongs to the class $H_\alpha^h(S_{k_1}, \cdots, S_{k_r})$, $(\alpha > 0)$, the function* (1.8) *belongs to the class $H_\alpha^{h'}(S_{k_1}, \cdots, S_{k_r})$.*

In the present paper we will show one new property of the singular integral and then we will apply this property to the solution of a certain system of singular integral equations in space by the method of Schauder.

§ 2. Study of a singular integral with respect to a parameter

We will study a certain property of the function of the two variables x and t, defined by the singular integral

$$\Phi(x,t)=\int_{\Omega} N(x-y)g(y,t)dy. \tag{2.1}$$

We will prove the following theorem.

THEOREM 4. *If the complex function $g(y,t)$ of two variable points, defined on the set*

$$y\in\Omega,\ t\in\Omega^*,$$

where Ω^* is an arbitrary region in the space E_n, satisfies the inequality

$$|g(y,t)|<\frac{M_g}{|y-y_S|^{\alpha}},\ (\alpha>0) \tag{2.2}$$

and a generalized Hölder condition

$$|g(y,t)|-g(\tilde{y},\tau)|<\frac{k_g(|y-\tilde{y}|^{h}+|t-\tau|^{h_1})}{|y-y_S|^{\alpha+h}}, \tag{2.3}$$

$$|y-y_S|\leq|\tilde{y}-\tilde{y}_S|,\ 0<h'<h_1\leq 1,$$

then the function (2.1) *is defined on the set $x\in\Omega$, $t\in\Omega^*$ and satisfies the inequalities*

$$|\Phi(x,t)|<\frac{C_1M_g+C_2k_g}{|x-x_S|^{\alpha}}, \tag{2.4}$$

$$|\Phi(x,t)-\Phi(z,\tau)|<\frac{(C_1'M_g+C_2'k_g)(|x-z|^{h'}+|t-\tau|^{h'})}{|x-x_S|^{\alpha+h'}}, \tag{2.5}$$

$$|x-x_S|\leq|z-z_S|,$$

where τ, t are arbitrary points of the region Ω^*, and the positive constants C_1' and C_2' do not depend on the function f.

PROOF. The truth of the property (2.4) and the property (2.5) with respect to the variables x and z follows from the Theorem 2 which has been proved. Therefore it is sufficient to prove the inequality (2.5) for $x=z$.

With this objective in mind we study the singular integral

$$J(x,t)=\int_{\Omega_r} N(x-y)g(y,t)dy, \tag{2.6}$$

taken over the region Ω_ν selected arbitrarily from the regions $\Omega_1, \cdots, \Omega_m$. We assume first that the point x lies inside the region of integration Ω_ν. We note it is sufficient to investigate the case

$$|t - \tau| \leq \tfrac{1}{2}|x - x_S|, \tag{2.7}$$

since otherwise the inequality (2.5) follows directly from the inequality (2.4). Let there be given two spheres, Π with center x and radius $|x - x_S|$ and Π_1 with center x and radius $|t - \tau|$. We subdivide the integral (2.6) into the sum of the integrals

$$J(x, t) = J^{\Pi_1}(x, t) + J^{\Pi - \Pi_1}(x, t) + J^{\Omega_\nu - \Pi}(x, t), \tag{2.8}$$

taken over the regions Π_1, $\Pi - \Pi_1$, $\Omega_\nu - \Pi$. Taking into consideration the assumption (1.6) one may write

$$J^{\Pi_1}(x, t) = \int_{\Pi_1} N(x, y)[g(y, t) - g(x, t)]dy,$$

whence follows, in view of (2.3) and the obvious inequality

$$|y - y_S| \geq |x - x_S| - |x - y|,$$

that

$$|J^{\Pi_1}(x, t)| < \int_{\Pi_1} \frac{\text{const. } k_g dy}{|x - y|^{n-h}|y - y_S|^{\alpha+h}} < \tag{2.9}$$

$$< \text{const. } k_g \int_0^{|t-\tau|} \frac{\rho^{n-1}d\rho}{\rho^{n-h}(|x - x_S| - \rho)^{\alpha+h}} <$$

$$< \frac{\text{const. } k_g}{|x - x_S|^\alpha} \int_0^{|t-\tau|/|x-x_S|} \frac{ds}{s^{1-h}(1 - s)^{\alpha+h}} < \frac{\text{const. } k_g|t - \tau|^h}{|x - x_S|^{\alpha+h}}.$$

We obtain an analogous inequality for the integral $J^{\Pi_1}(x, \tau)$. Further, by the formula (1.5) and the inequality (2.3), we may write

$$|J^{\Pi - \Pi_1}(x, t) - J^{\Pi - \Pi_1}(x, \tau)| < \tag{2.10}$$

$$< \int_{\Pi - \Pi_1} \frac{\text{const. } k_g|t-\tau|^{h_1}}{|x-y|^n|y-y_S|^{\alpha+h}} dy < \text{const. } k_g|t-\tau|^{h_1} \int_{|t-\tau|}^{|x-x_S|} \frac{\rho^{n-1}d\rho}{\rho^n(|x-x_S|-\rho)^{\alpha+h}} <$$

$$< \frac{\text{const. } k_g|t-\tau|^{h_1}}{|x-x_S|^{\alpha+h}} \int_{|t-\tau|/|x-x_S|}^1 \frac{ds}{s(1-s)^{\alpha+h}} < \frac{\text{const. } k_g|t-\tau|^{\theta h_1}}{|x-x_S|^{\alpha+h}},$$

where θ is an arbitrary, positive constant which is smaller than unity.

For the difference of the integrals, taken over the outer region $\Omega_\nu - \Pi$, we obtain the inequality

$$(2.11)\quad |J^{\Omega_\nu - \Pi}(x,t) - J^{\Omega_\nu - \Pi}(x,\tau)| < \text{const. } k_g |t-\tau|^{h_1} \int\limits_{\Omega_\nu - \Pi} \frac{dy}{|x-y|^n |y - y_S|^{\alpha+h}}.$$

We have studied the last integral in [1, 8], where the following estimate has been deduced:

$$(2.12)\quad \int\limits_{\Omega_\nu - \Pi} \frac{dy}{|x-y|^n |y-y_S|^{\alpha+h}} < \frac{\text{const.}}{|x-x_S|^{\alpha+h}}, \quad (x \in \Omega_\nu).$$

Hence follows the estimate

$$(2.13)\quad |J^{\Omega_\nu - \Pi}(x,t) - J^{\Omega_\nu - \Pi}(x,\tau)| < \frac{\text{const. } k_g |t-\tau|^h}{|x-x_S|^{\alpha+h}}.$$

Combining the results (2.9), (2.10) and (2.13) we obtain the unknown inequality

$$(2.14)\quad |J(x,t) - J(x,\tau)| < \frac{\text{const. } k_g |t-\tau|^{h'}}{|x-x_S|^{\alpha+h'}}, \quad (x \in \Omega_\nu).$$

There still remains to consider the case when the point x lies outside the region Ω_ν sufficiently close to one of the surfaces S_{ν_1} bounding the region Ω_ν. In this case we write down directly the estimate

$$(2.15)\quad |J(x,t) - J(x,\tau)| < \text{const. } k_g |t-\tau|^{h_1} \int\limits_{\Omega_\nu} \frac{dy}{|x-y|^n |y-y_S|^{\alpha+h}}.$$

We study the last integral in the same manner as in [8]. Let there be given a cylinder W with center at the point x_S on the surface S_{ν_1}, whose axis is normal at the point x_S, whose height is $2r_W$ and radius r_W is so small that for the part S_W of the surface S_{ν_1}, lying inside W, Liapunov's known conditions are fulfilled. We denote by $W' = W \times \Omega_\nu$ the part of the cylinder W which lies inside the region Ω_ν. We now divide the region W' into layers of the surfaces, obtained by parallel displacements of the surface S_W in the direction of the normal at the point x_S. We denote by $S_W{}^\zeta$ the surface obtained by the parallel displacement of the surface S_W in the direction of the internal normal at the point x_S by the distance $|x_S - x_S{}^\zeta| = \zeta \geq 0$, where we have denoted

by x_S^ζ the point of intersection of the surface S_W^ζ with the normal at x_S. We then assume that the radius r_W is sufficiently small, so that the relation $|y - \tilde{y}_0|/|y - y_S|$ has an upper bound for the points $y \in W'$; we denote by $\tilde{y}_\zeta$ the point at which the parallel to the normal at the point x_S, passing through the point y, intersects the surface S_W^ζ. It is likewise known that the relation $|x - \tilde{y}_\zeta|/|x - y|$ has an upper bound for $y \in W'$. We may now write down for the part of the integral (2.15) taken over the region W' the estimate

$$\int_{W'} \frac{dy}{|x-y|^n |y-y_S|^{\alpha+h}} < \text{const.} \int_{W'} \frac{dy}{|x-y|^n |y-\tilde{y}_0|^{\alpha+h}} <$$

$$< \text{const.} \int_0^H \frac{d\zeta}{\zeta^{\alpha+h}} \int_0^{r_W} \frac{\rho^{n-2} d\rho}{[\rho^2 + (|x-x_S| + \zeta)^2]^{n/2}},$$

where $H > 0$ is chosen in order that the region $\tilde{W}' \supset W'$ will be formed by the surface S_W^ζ for $0 \leq \zeta \leq H$. Substituting the values $\rho = (|x - x_S| + \zeta)s$ and $\zeta = |x - x_S|u$, we obtain

$$\text{(2.16)} \quad \int_{W'} \frac{dy}{|x-y|^n |y-y_S|^{\alpha+h}} <$$

$$< \frac{\text{const.}}{|x-x_S|^{\alpha+h}} \int_0^\infty \frac{du}{u^{\alpha+h}(u+1)} \int_0^\infty \frac{s^{n-2} ds}{(1+s^2)^{n/2}}.$$

Starting from this estimate we obtain for the difference (2.15) the inequality

$$\text{(2.17)} \qquad |J(x,t) - J(x,\tau)| < \frac{\text{const.}\ k_g (t-\tau)^{h_1}}{|x-x_S|^{\alpha+h}}.$$

Taking into consideration the results (2.15) and (2.17), we draw the conclusion that the assumption (2.5) of Theorem 4 has been proven.

§ 3. Investigation of a system of singular integral equations in space

Let there be given the system of m singular integral equations of the form

$$\text{(3.1)} \quad \varphi_\nu(x) = f_\nu(x) + \int_\Omega N_\nu(x-y) F_\nu[x, y, \varphi_1(y), \cdots, \varphi_m(y)] dy,$$

$$(\nu = 1, 2, \cdots, m)$$

with the m unknown functions

$$\varphi_1(x), \cdots, \varphi_m(x),$$

defined on the set $x \in \Omega$ (cf. § 1).

We make the following assumptions.

I. The functions $N_\nu(x)$ are defined at each point $x \neq (0, \cdots, 0)$ by the formula

$$N_\nu(x) = \frac{K_\nu(x')}{|x|^n}, \quad (x = x'|x|), \tag{3.2}$$

where the complex functions $K_\nu(x')$, defined on the surface ω of the unit sphere, satisfy a Hölder condition

$$|K_\nu(x') - K_\nu(y')| < k_\omega |x' - y'|^{h_\omega}, \quad (0 < h_\omega \leq 1) \tag{3.3}$$

and, in addition,

$$\int_\omega K_\nu(x')dx' = 0, \quad (\nu = 1, 2, \cdots, m). \tag{3.4}$$

II. The complex functions $F_\nu(x, y, u_1, \cdots, u_m)$, defined on the set

$$x \in \Omega, \ y \in \Omega, \ u_\gamma \in \Pi, \ (\gamma = 1, 2, \cdots, m), \tag{3.5}$$

where Π is the entire plane of the complex variable, satisfy the inequalities

$$|F_\nu(x, y, u_1, \cdots, u_m)| < \frac{M_F}{|y - y_S|^\alpha} + M'_F \sum_{\gamma=1}^{m} |u_\gamma|, \tag{3.6}$$

$$|F_\nu(x, y, u_1, \cdots, u_m) - F_\nu(\tilde{x}, \tilde{y}, \tilde{u}_1, \cdots, \tilde{u}_m)| < \tag{3.6'}$$

$$< \frac{k_F(|x - \tilde{x}|^{h_1} + |y - \tilde{y}|^h)}{|y - y_S|^{\alpha+h}} + M'_F \sum_{\gamma=1}^{m} |u_\gamma - \tilde{u}_\gamma|,$$

in which the constant exponents h, h_1, α satisfy the inequalities

$$0 < h < h_1 \leq 1, \ h < h_\omega, \ \alpha > 0, \ \alpha + h < 1, \tag{3.7}$$

and M_F, M_F', k_F are given positive constants.

III. The given functions $f_\nu(x)$, defined on the set Ω, belong to the class H_α^h, i.e., they satisfy the inequalities

$$\begin{aligned} |f_\nu(x)| &< \frac{M_f}{|x - x_S|^\alpha}, \\ |f_\nu(x) - f_\nu(y)| &< \frac{k_f|x - y|^h}{|x - x_S|^{\alpha+h}}. \end{aligned} \tag{3.8}$$

In view of the very great generality of these assumptions the system (3.1) cannot be solved by the classical method of successive approximations. We will study it by a topological method, based on the following known theorem of J. Schauder [9]. "If in Banach space a continuous transformation turns a closed and convex set into its compact subset of itself, then there exists in this set at least one fixed point." Application of this theorem [1] to the system of singular equations (3.1) requires more delicate arguments about the behavior of the selected surface of discontinuity of the functions entering into the system (3.1).

Let there be given the functional space A, consisting of all systems

$$[\varphi_1(x), \varphi_2(x), \cdots, \varphi_m(x)]$$

of m complex functions, defined and continuous on the open set Ω and satisfying the inequalities

$$|x - x_S|^{\alpha+h}|\varphi_\nu(x)| < \infty, \quad (\nu = 1, \cdots, m), \tag{3.9}$$

where the positive constants α and h characterize the given functions F_ν and f_ν in the inequalities (3.6), (3.6') and (3.8).

We employ the known definitions of linear operations

$$\begin{gathered}[\varphi_1, \cdots, \varphi_m] + [\tilde{\varphi}_1, \cdots, \tilde{\varphi}_m] = [\varphi_1 + \tilde{\varphi}_1, \cdots, \varphi_m + \tilde{\varphi}_m], \\ k[\varphi_1, \cdots, \varphi_m] = [k\varphi_1, \cdots, k\varphi_m],\end{gathered} \tag{3.10}$$

and take the following definitions of the norm of a point $U = [\varphi_1, \cdots, \varphi_m]$ and distance of two points $\delta(U, V)$:

$$\begin{gathered}\|U\| = \max_{1 \le \nu \le m} \sup_{x \in \Omega} [|x - x_S|^{\alpha+h}|\varphi_\nu(x)|], \\ \delta(U, V) = \|U - V\|.\end{gathered} \tag{3.11}$$

As a consequence of these definitions the space A is a linear, normalized, metric space. In addition, it may be shown that the Cauchy condition is a necessary and sufficient condition for the convergence of an arbitrary sequence of points in the space A, in the sense of the norm (3.11). Consequently, the space A is complete and is a Banach space.

We consider now in the space A the set E of points $[\varphi_1, \cdots, \varphi_m]$, satisfying the inequalities

$$|x - x_S|^{\alpha}|\varphi_\nu(x)| \le \rho_\varphi, \quad (\nu = 1, 2, \cdots, m), \tag{3.12}$$

$$|x - x_S|^{\alpha+h}|\varphi_\nu(x) - \varphi_\nu(z)| \le \kappa_\varphi |x - z|^h, \tag{3.12'}$$

where the points x, z lie simultaneously in one of the regions $\Omega_1, \cdots, \Omega_q$ and $|x - x_S| \le |z - z_S|$, ρ_φ and κ_φ are arbitrarily fixed positive constants.

[1] Schauder's topological theorem was first applied to the study of singular integral equations by A. I. Guseinov [10].

By Definition 1, the functions $\varphi_\nu(x)$ connected with the set E belong to the class $\mathsf{H}_\alpha{}^h$. The set E is obviously a closed and convex set.

Taking into consideration the integral equations (3.1) we transform the set E with the help of the relations

$$\psi_\nu(x) = f_\nu(x) + \int_\Omega N_\nu(x-y)F_\nu[x, y, \varphi_1(y), \cdots, \varphi_m(y)]dy, \quad (\nu = 1, 2, \cdots, m), \tag{3.13}$$

which set up a correspondence between each point $[\varphi_1, \cdots, \varphi_m]$ of the set E and the defined system of functions $[\psi_1, \cdots, \psi_m]$. We determine the conditions for which the system $[\psi_1, \cdots, \psi_m]$ would be points belonging to the set E. As a consequence of the assumptions (3.6) and the inequalities (3.12) we may write down the inequalities

$$|F[x, y, \varphi_1(y), \cdots, \varphi_m(y)]| < \frac{M_F + M_F{}'m\rho_\varphi}{|y - y_S|^\alpha}, \tag{3.14}$$

$$|F[x, y, \varphi_1(y), \cdots, \varphi_m(y)] - F[\tilde{x}, \tilde{y}, \varphi_1(\tilde{y}), \cdots, \varphi_m(\tilde{y})]| < \frac{(k_F + k_F{}'m\kappa_\varphi)(|x-\tilde{x}|^{h_1} + |y-\tilde{y}|^h)}{|y - y_S|^{\alpha+h}}, \tag{3.14'}$$

where $x, \tilde{x} \in \Omega$, y and $\tilde{y}$ are two arbitrary points, lying at the same time inside one of the regions $\Omega_1, \cdots, \Omega_q$ and $|y - y_S| \leq |\tilde{y} - \tilde{y}_S|$. In view of the inequalities (3.8), (3.14) and relying on the basic Theorem 4, where one must set $t = x$, $\tau = \tilde{x}$, we conclude that the functions (3.13) belong to the class $\mathsf{H}_\alpha{}^h$, as do also the functions $\varphi_\nu(x)$. In fact, these functions satisfy the inequalities

$$|\psi_\nu(x)| < \frac{C_1(M_F + M_F{}'m\rho_\varphi) + C_2(k_F + M_F{}'m\kappa_\varphi) + M_f}{|x - x_S|^\alpha}, \tag{3.15}$$

$$|\psi_\nu(x) - \psi_\nu(z)| < \frac{2C_1{}'(M_F + M_F{}'m\rho_\varphi) + 2C_2{}'(k_F + M_F{}'m\kappa_\varphi) + k_f}{|x - x_S|^{\alpha+h}} |x - z|^h, \tag{3.15'}$$

where $|x - x_S| \leq |z - z_S|$ and the positive constants C_1, C_2, $C_1{}'$, $C_2{}'$ do not depend on the functions f_ν, F_ν, φ_ν.

In view of the inequalities (3.12), (3.12'), (3.15) and (3.15') we draw the conclusion that the inequalities

$$\begin{aligned} &C_1(M_F + M_F{}'m\rho_\varphi) + C_2(k_F + M_F{}'m\kappa_\varphi) + M_f \leq \rho_\varphi, \\ &2C_1{}'(M_F + M_F{}'m\rho_\varphi) + 2C_2{}'(k_F + M_F{}'m\kappa_\varphi) + k_f \leq \kappa_\varphi \end{aligned} \tag{3.16}$$

are sufficient conditions for the set E' of all points $V = [\psi_1, \cdots, \psi_m]$, corresponding to the points $U = [\varphi_1, \cdots, \varphi_m]$ of the set E, to be a subset of this set. We see that the condition (3.16) can be always fulfilled, if the constant of the problem M_F' is sufficiently small. Since the positive constants ρ_φ and κ_φ are arbitrary, we use this fact and choose their values in such a manner that the possible interval for M_F' by the condition (3.16) will be a maximum. It is not difficult to show that this interval is

$$0 < M_F' < \frac{1}{mC_1 + 2mC_2'}. \tag{3.17}$$

We prove now the following two Lemmas.

LEMMA 1. *The transformation of the points of the set E, defined by the relations* (3.13), *is continuous in the space Λ.*

In accordance with the definition of the continuity of the transformation it is necessary and sufficient to show that, if $\{U^{(j)}\}$ is an arbitrary sequence of the points $U^{(j)} = [\varphi_1^{(j)}, \cdots, \varphi_m^{(j)}]$ of the set E, converging to the point $U = [\varphi_1, \cdots, \varphi_m]$ of this set in the sense of the norm (3.11), then the sequence of the corresponding points $V^{(j)} = [\psi_1^{(j)}, \cdots, \psi_m^{(j)}]$ of the set E' converges to the point $V = [\psi_1, \cdots, \psi_m]$ which corresponds to the limiting point U by the relations (3.13). Thus, in view of the definition of the norm (3.11), we have

$$\lim_{j\to\infty} \max_{\nu} \sup_{j\in\Omega} \left(|x - x_S|^{\alpha+h}|\varphi_\nu^{(j)}(x) - \varphi_\nu(x)|\right) = 0. \tag{3.18}$$

Relying on this property, we consider the behavior of the product

$$|x - x_S|^{\alpha+h}|\psi_\nu^{(j)}(x) - \psi_\nu(x)|$$

for $j \to \infty$. By (3.13) we may write

$$\psi_\nu^{(j)}(x) - \psi_\nu(x) = \int_\Omega N_\nu(x - y)\{F_\nu[x, y, \varphi_1^{(j)}(y), \cdots, \varphi_m^{(j)}(y)] - \tag{3.19}$$

$$- F_\nu[x, y, \varphi_1(y), \cdots, \varphi_m(y)]\}dy, \quad (\nu = 1, 2, \cdots, m).$$

Now let there be given a sphere $K(x, R_\epsilon)$ with center at the point x and radius R_ϵ, independent of x, but depending on an arbitrarily fixed number $\varepsilon > 0$. We decompose the integral (3.19) into the sum of two integrals

$$\psi_\nu^{(j)}(x) - \psi_\nu(x) = J_\nu^{\tilde{K}}(x) + J_\nu^{\Omega-\tilde{K}}(x), \tag{3.20}$$

taken over the region $\tilde{K} = \Omega \times K(x, R_\epsilon)$, representing the common

part of the region Ω and the sphere $K(x, R_\epsilon)$, and over the outer set $\Omega - \tilde{K}$. For the first part one may write down the following expression, containing two integrals with weak singularities:

$$
\begin{aligned}
(3.21)\quad J_\nu^{\tilde{K}}(x) = &\int_{\tilde{K}} N_\nu(x-y)\{F_\nu[x, y, \varphi_1^{(j)}(y), \cdots, \varphi_m^{(j)}(y)] - \\
&- F_\nu[x, x, \varphi_1^{(j)}(x), \cdots, \varphi_m^{(j)}(x)]\}dy - \\
&- \int_{\tilde{K}} N_\nu(x-y)\{F_\nu[x, y, \varphi_1(y), \cdots, \varphi_m(y)] - \\
&- F_\nu[x, x, \varphi_1(x), \cdots, \varphi_m(x)]\}dy + \{F_\nu[x, x, \varphi_1^{(j)}(x), \cdots, \varphi_m^{(j)}(x)] - \\
&- F[x, x, \varphi_1(x), \cdots, \varphi_m(x)]\} \int_{\tilde{K}} N_\nu(x-y)dy.
\end{aligned}
$$

Taking into consideration the inequalities (3.12′) and (3.14′), we draw the conclusion that the first of the integrals on the right hand side of (3.21), $I_1(x)$, satisfies for every j the inequality

$$
\begin{aligned}
(3.22)\quad |I_1(x)| &< \text{const.} \int_{\tilde{K}} \frac{dy}{|x-y|^{n-h}|y-y_S|^{\alpha+h}} < \\
&< \text{const.} \int_0^{R_\epsilon} \frac{\rho^{n-1}d\rho}{\rho^{n-h}(|x-x_S|-\rho)^{\alpha+h}} < \frac{\text{const.}}{|x-x_S|^\alpha} \int_0^{R_\epsilon/|x-x_S|} \frac{dt}{t^{1-h}(1-t)^{\alpha+h}} < \\
&< \frac{\text{const.}}{|x-x_S|^{\alpha+h}} R_\epsilon^h, \quad R_\epsilon \leq |x-x_S|.
\end{aligned}
$$

In the case when $R_\epsilon > |x - x_S|$, one may study the integral (3.22) by a method stated in [1, 8] and obtain an estimate of the same form.

The second of the integrals (3.21), $I_2(x)$, likewise satisfies the inequality (3.22), since the limiting point $[\varphi_1(x), \cdots, \varphi_m(x)]$ belongs to the set E. Thus, we may choose for arbitrary $\varepsilon > 0$ an R_ϵ such that for every point $x \in \Omega$ there are fulfilled the inequalities

$$
\begin{aligned}
(3.23)\quad |x-x_S|^{\alpha+h}|I_1(x)| &< \frac{\varepsilon}{6}, \\
|x-x_S|^{\alpha+h}|I_2(x)| &< \frac{\varepsilon}{6}.
\end{aligned}
$$

In order to obtain an estimate for the third term, $I_3(x)$, of the sum (3.21), we note that by the inequality (3.14) it satisfies the inequality

$$|I_3(x)| < \frac{2M_F + 2M_F' m\rho_\varphi}{|x - x_S|^\alpha} \int_{\widetilde{K}} N(x - y)dy. \tag{3.24}$$

The last integral is zero by (3.4), if $|x - x_S| \geq R_\epsilon$. However, if $|x - x_S| < R_\epsilon$, we have

$$\left| \int_{\widetilde{K}} N(x - y)dy \right| < \int_{|x-x_S|}^{R_\epsilon} \text{const.} \frac{\rho^{n-1}d\rho}{\rho^n} = \text{const.} \log \frac{R_\epsilon}{|x - x_S|}, \tag{3.25}$$

and, consequently, for every point $x \in \Omega$ for sufficiently small R_ϵ

$$|x - x_S|^{\alpha+h}|I_3(x)| < \text{const.}\, R_\epsilon^h(|\log R_\epsilon| + 1) < \frac{\varepsilon}{6}. \tag{3.26}$$

As a result, we find for the difference (3.21) the estimate

$$|x - x_S|^{\alpha+h}|J_\nu^{\widetilde{K}}(x)| < \frac{\varepsilon}{2}, \quad (x \in \Omega) \tag{3.27}$$

for a suitable choice of the radius R_ϵ of the sphere $K(x, R_\epsilon)$. Having fixed the radius R_ϵ of the sphere, we consider now the second term $J^{\Omega-\widetilde{K}}(x)$ of the difference (3.20). Taking into consideration the property (3.6'), we write

$$|J^{\Omega-\widetilde{K}}(x)| < \tag{3.28}$$
$$< \max_\nu \sup_{y\in\Omega} [m|y - y_S|^{\alpha+h}|\varphi_\nu^{(j)}(y) - \varphi_\nu(y)|] \int_{\Omega-\widetilde{K}} \frac{\text{const.}\, dy}{|x - y|^n|y - y_S|^{\alpha+h}}.$$

Following the method used in [1, 8] and employing the estimate (2.16), it may be proved that the product

$$|\dot{x} - x_S|^{\alpha+h} \int_{\Omega-\widetilde{K}} \frac{\text{const.}\, dy}{|x - y|^n|y - y_S|^{\alpha+h}}$$

is bounded for $|x - x_S| \to 0$. Hence we draw the conclusion, by the strength of (3.18) and (3.28), that one may choose for $\varepsilon > 0$ an integer $N(\varepsilon)$ such that for $j > N(\varepsilon)$,

$$\sup_{x\in\Omega} [|x - x_S|^{\alpha+h}|J_\nu^{\Omega-\widetilde{K}}(x)|] < \frac{\varepsilon}{2} \quad (\nu = 1, 2, \cdots, m). \tag{3.29}$$

Adding the estimates (3.27) and (3.29), we conclude that

$$\lim_{j\leftarrow\infty}\sup_{x\in\Omega}[|x-x_S|^{\alpha+\hbar}|\psi_\nu^{(j)}(x)-\psi_\nu(x)|]=0. \tag{3.30}$$

Consequently, Lemma 1 has been proved.

LEMMA 2. *The transformed set E' of the set E is compact.*

By definition, it is necessary and sufficient to prove that one may select from an arbitrary sequence $\{V^{(j)}\}$ of points $V^j=[\psi_1^{(j)},\cdots,\psi_m^{(j)}]$ of the set E' a sequence $\{V^{(k_j)}\}$, converging in the sense of the norm (3.11).

Let there be given the union of closed surfaces S_0^ε, $S_1'^\varepsilon$, $S_1''^\varepsilon$, $\cdots$, $S_p'^\varepsilon$, $S_p''^\varepsilon$, lying inside Ω on one side of the surface S_0 and on both sides of the surfaces $S_1, S_2, \cdots, S_p$ and so far from these surfaces that at each point x, lying in the union Σ^ε of layers, bounded by the surfaces (S_0, S_0^ε), $\cdots$, $(S_p'^\varepsilon, S_p''^\varepsilon)$, and not lying on the surfaces of discontinuity $S_0, S_1, \cdots, S_p$, the components $\psi_\nu^{(j)}(x)$ of all points V^j of an arbitrarily selected sequence $\{V^{(j)}\}$ of points of the set E' satisfy the inequality

$$|x-x_S|^{\alpha+\hbar}|\psi_\nu^{(j)}(x)|<\frac{\varepsilon}{2}\quad(\nu=1,2,\cdots,m), \tag{3.31}$$

where $\varepsilon>0$ is an arbitrary number.

We note that the existence of the set Σ^ε of points x at which the inequality (3.31) will be fulfilled is obvious by the strength of the inequalities (3.15) which are satisfied by all the components $\psi_\nu^{(j)}(x)$ of all points of the sequence $\{V^{(j)}\}$.

Fixing the set Σ^ε, chosen for an arbitrary $\varepsilon>0$, we note by (3.15) and (3.15′) that on the set Ω^ε of points of the set Ω, not belonging to the set Σ^ε, all functions $\psi_\nu^{(j)}(x)$ satisfy the same inequalities

$$\begin{gathered}|\psi_\nu^{(j)}|\le C_\epsilon,\\ |\psi_\nu^{(j)}(x)-\psi_\nu^{(j)}(z)|\le C_\epsilon'|x-z|^\hbar,\end{gathered} \tag{3.32}$$

in which the positive constants C_ϵ and C_ϵ' depend on ε, but not on the indices ν and j. Consequently, on the set Ω^ε all the functions $\psi_\nu^{(j)}(x)$ are bounded by one number and they are uniformly continuous. Relying on this property, we arrive by a known theorem of Arzela at a conclusion regarding the existence of the sequence of points $V^{(k_j)}$ the components $\psi^{(k_j)}(x)$ of which converge uniformly in the ordinary sense on the set Ω^ε. Consequently, there exists for arbitrary $\varepsilon>0$ an integer $N(\varepsilon)$ which for $j, s>N(\varepsilon)$ at each point $x\in\Omega^\varepsilon$ satisfies the Cauchy condition

$$\delta^{\alpha+\hbar}|\psi_\nu^{(k_j)}(x)-\psi_\nu^{(k_s)}(x)|<\varepsilon; \tag{3.33}$$

$\delta > 0$ denotes here the upper bound of the distance $|x - x_S|$ for $x \in \Omega^\varepsilon$. It follows from the inequalities (3.31) and (3.33) that for j, $s > N(\varepsilon)$

$$\|V^{(k_j)} - V^{(k_s)}\| < \varepsilon. \tag{3.34}$$

But the space Λ is a complete space, and, consequently, the property (3.34) is a sufficient condition for the convergence of the sequence $\{V^{(k_j)}\}$ in the sense of the norm (3.11), and the space E' is *compact.*

The results obtained permit one to arrive, on the basis of the cited theorem of J. Schauder, at a conclusion regarding the existence of at least one point $V^* = [\varphi_1^*(x), \cdots, \varphi_m^*(x)]$ on the set E which is fixed with respect to the transformation (3.13), and this means also with regard to the existence of a system of functions $[\varphi_1^*(x), \cdots, \varphi_m^*(x)]$ of the class H_α^h, satisfying the system of integral equations (3.1) at each point $x \in \Omega$.

We may now formulate the following theorem.

THEOREM 5. *If the given functions* $f_\nu(x)$, $N_\nu(x)$, $F_\nu(x, y, u_1, \cdots, u_m)$ *satisfy the conditions I, II, III and if the constant* $M_{F'}$ *satisfies the condition* (3.17), *then the system of singular integral equations* (3.1) *has at least one solution* $[\varphi_1^*(x), \cdots, \varphi_m^*(x)]$ *consisting of functions belonging to the class* H_α^h.

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ON THE PLASTIC ANALYSIS OF SANDWICH STRUCTURES

WILLIAM PRAGER
Providence, U.S.A.

§ 1. Introduction

The historical development of the mathematical theory of plasticity begins with the concept of a stress threshold, called *yield limit*, below which macroscopic plastic deformation cannot occur. The second basic ingredient of the theory of perfectly plastic solids is the *flow rule*, which specifies the plastic flow produced by a given state of stress at the yield limit.

When the first theory of plasticity was developed by Tresca, Saint Venant, and Lévy, the mathematical theory of elasticity was already well established. The flow rule plays a similar role in plasticity as Hooke's law does in elasticity, but the yield limit has no counterpart in elasticity. It is therefore not surprising that for decades the yield limit was the sole object of experimental investigations in plasticity, whereas the flow rule that Saint Venant and Lévy proposed in close analogy to Hooke's law remained unchallenged. This emphasis on the yield limit was further accentuated by the discovery of the statically determinate problems of plasticity, which can be treated without reference to the flow rule. In recent years, however, the development of limit analysis brought a deeper insight into the dual roles played by the statical and the kinematical approach, by which the load carrying capacity can be respectively bounded from below and from above.

The duality between stress and rate of strain in the theory of perfectly plastic solids precisely corresponds to the well-known geometric duality between the vectors specifying the position of a point or the orientation of a plane. In the same way, in which a convex surface can be characterized by its distance function or by its supporting function [1], a perfectly plastic continuum can be characterized by its *yield function* or by its *dissipation function*. The first depends on the stress components and is smaller than, equal to, or greater than 1, according to whether the considered state of stress is below, at, or above the yield limit. The dissipation function depends on the components of the strain rate and

[1] See, for instance, T. Bonnesen and W. Fenchel, Theorie der konvexen Körper, Berlin, 1934, (reprinted, New York, 1948).

represents the rate at which mechanical energy is dissipated in the unit of volume during plastic flow with the considered strain rate.

Because the energy dissipated in a composite structure is the sum of the energies dissipated in its elements, the dissipation function proves to be a particularly useful tool in the plastic analysis of composite structures. To illustrate this remark, some sandwich structures are discussed in the following.

§ 2. Sandwich layers in uniaxial stress

Consider a rigid, perfectly plastic material that has the tensile yield stress $\sigma_t > 0$ and the compressive yield stress $\sigma_c < 0$ (Fig. 1). For the uniaxial state of stress considered here, the stress intensity σ is then restricted to the interval $\sigma_c \leq \sigma \leq \sigma_t$, the endpoints of which mark the yield limit. Denote the half-length and the midpoint of this interval by

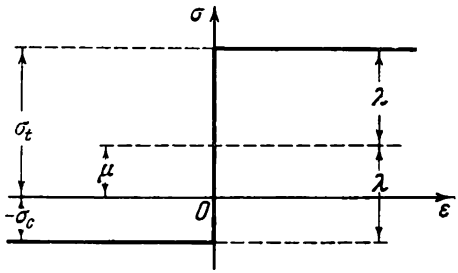

Fig. 1.

$$\lambda = \tfrac{1}{2}(\sigma_t - \sigma_c) \tag{2.1}$$

and

$$\mu = \tfrac{1}{2}(\sigma_t + \sigma_c), \tag{2.2}$$

respectively. Under the action of the uniaxial stress σ, let this material experience the strain rate ε in the direction of the axis of the state of stress. The power of dissipation per unit volume then is given by

$$D = \mu\varepsilon + \lambda|\varepsilon|. \tag{2.3}$$

Indeed, this relation furnishes $D = (\mu + \lambda)\varepsilon$ for $\varepsilon > 0$ and $D = (\mu - \lambda)\varepsilon$ for $\varepsilon < 0$. Since the power of dissipation (2.3) can also be written as $\sigma\varepsilon$, it follows that $\sigma = \mu + \lambda$ for $\varepsilon > 0$ and $\sigma = \mu - \lambda$ for $\varepsilon < 0$. Substitution of (2.1) and (2.2) into these expressions gives the correct yield stresses: $\sigma = \sigma_t$ for plastic extension and $\sigma = \sigma_c$ for plastic compression.

Let us now discuss a sandwich beam that is subjected to the axial force N and the bending moment M. The beam is to have a core with rectangular cross-section of the breadth b and the height $2h$, that is covered by two thin sheets of the material considered above; the common thickness of these sheets is to be denoted by t (see the inset in Fig. 2). As usual, it will be assumed that all direct stresses are carried by the cover sheets. If ε is the rate of axial extension and κ the rate of curvature

of the beam, the cover sheets experience the rates of extension $\varepsilon \pm h\kappa$. According to (2.3), the power of dissipation per unit length of the beam is

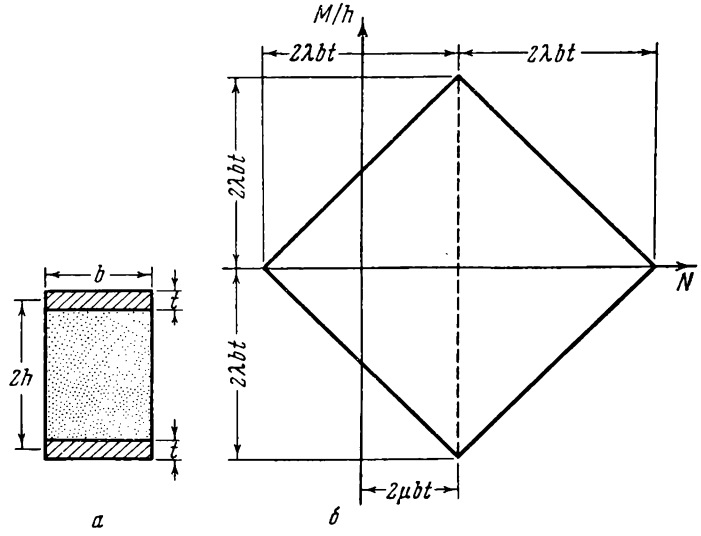

Fig. 2.

therefore given by

$$(2.4)\quad D = bt\{\mu(\varepsilon + h\kappa) + \lambda|\varepsilon + h\kappa| + \mu(\varepsilon - h\kappa) + \lambda|\varepsilon - h\kappa|\} = \\ = bt\{2\mu\varepsilon + \lambda|\varepsilon + h\kappa| + \lambda|\varepsilon - h\kappa|\}.$$

It will be convenient to treat ε and $h\kappa$ as *generalized strain rates*, and N and M/h as the corresponding *generalized stresses*. The expression (2.4) for the *dissipation function* of the sandwich beam is homogeneous and piecewise linear in the generalized strain rates. By comparing it with the alternative expression $N\varepsilon + M\kappa$ for the power of dissipation per unit length of the beam, we can determine the states of stress that are capable of producing given strain rates.

In making this comparison, it proves expedient to focus attention on strain rates for which all but one pair of absolute value signs can be removed in the expression for the specific power of dissipation. Assume, for instance, that ε is negative and $h\kappa$ positive, and that these strain rates have nearly the same absolute value, so that (2.4) reduces to

$$(2.5)\qquad D = bt\{2\mu\varepsilon - \lambda(\varepsilon - h\kappa) + \lambda|\varepsilon + h\kappa|\}.$$

According to whether $\varepsilon + h\kappa$ is positive or negative, comparison of (2.5) with the alternative expression $N\varepsilon + M\kappa$ furnishes

$$(2.6)\qquad N = 2\mu bt, \quad M/h = 2\lambda bt$$

or

$$N = 2(\mu - \lambda)bt, \quad M/h = 0. \tag{2.7}$$

These relations specify the endpoints of a side of the *yield polygon* in the *stress plane* with the rectangular Cartesian coordinates N, M/h. Note that the specifications of this side may be obtained as follows. Firstly, dropping the absolute value from (2.5) compare the remaining terms with the expression $N\varepsilon + M\kappa$; this comparison furnishes

$$N = (2\mu - \lambda)bt, \quad M/h = \lambda bt, \tag{2.8}$$

i.e., the coordinates of the midpoint of the side. Secondly, retaining only the term with the absolute value in (2.5), drop the absolute value sign and compare with $N\varepsilon + M\kappa$; this comparison furnishes

$$N = \lambda bt, \quad M/h = \lambda bt, \tag{2.9}$$

i.e., the differences between corresponding coordinates of one corner and the midpoint of the side.

Assuming again that ε and $h\kappa$ have nearly equal absolute values, but that ε is positive and $h\kappa$ negative, we find another side of the yield polygon. The two sides determined so far are related to each other by symmetry with respect to the center $N = 2\mu bt$, $M/h = 0$, the coordinates

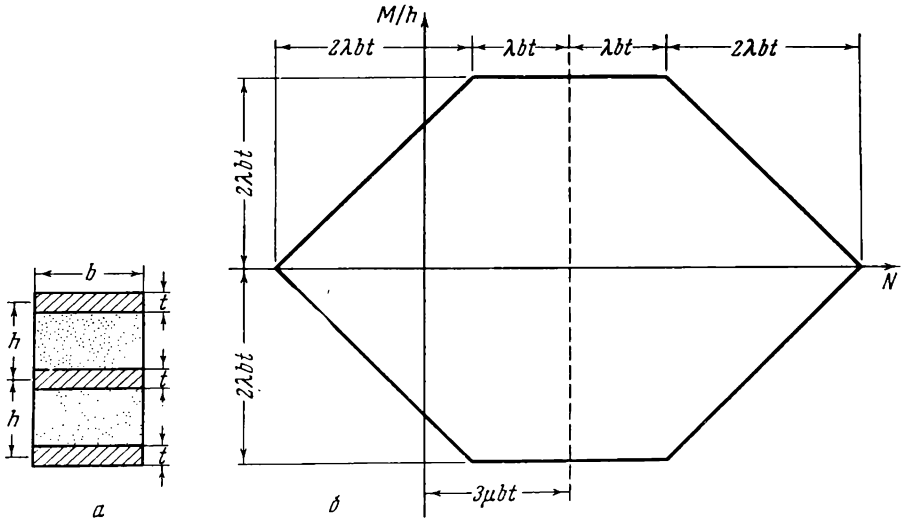

Fig. 3.

of which are obtained from (2.4) by deleting all absolute values in (2.4) before the comparison with $N\varepsilon + M\kappa$ is made. The remaining pair of sides

can be discussed in a similar manner. The quadratic yield locus is shown in Fig. 2.

The dissipation function of the sandwich beam shown in the inset of Fig. 3 is obtained by adding the power of dissipation per unit length of the central sheet, i.e. $bt(\mu\varepsilon + \lambda|\varepsilon|)$, to the right-hand side of (2.4). Thus,

$$D = bt\{3\mu\varepsilon + \lambda|\varepsilon| + \lambda|\varepsilon + h\kappa| + \lambda|\varepsilon - h\kappa|\}. \tag{2.10}$$

As is indicated by the appearance of three absolute values in (2.10), the yield polygon is now a hexagon; it is shown in Fig. 3, which is drawn to the same scale as Fig. 2.

§ 3. Sandwich layers in plane stress

Let the rigid, perfectly plastic material considered in Section 2 be subjected to a plane state of stress with the principal components σ_1 and σ_2, and denote the corresponding principal strain rates by ε_1 and ε_2. It will be assumed that the material begins to yield under the proportionally increasing stresses σ_1, σ_2, when at least one of the following events takes place: the greater of the stresses σ_1, σ_2 reaches the tensile yield limit σ_t, or the smaller one reaches the compressive yield limit σ_c, or the absolute

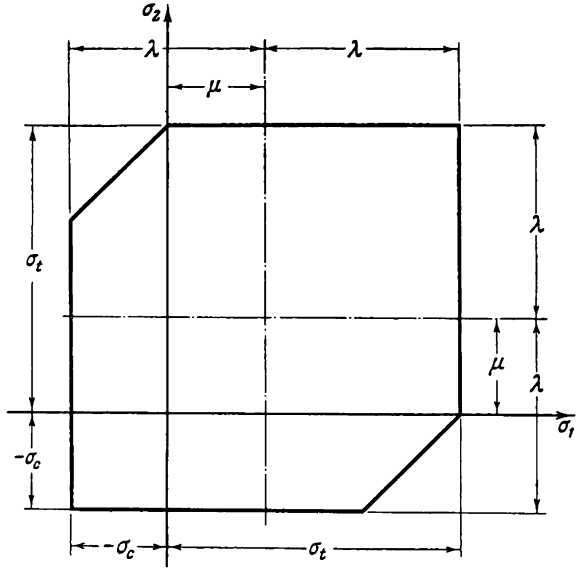

Fig. 4.

value of their difference reaches the value σ_t. The corresponding yield locus is shown in Fig. 4, and the dissipation function is seen to have the

form

$$(3.1)\quad D = \mu(\varepsilon_1 + \varepsilon_2) + \tfrac{1}{2}(\lambda + \mu)(|\varepsilon_1| + |\varepsilon_2|) + \tfrac{1}{2}(\lambda - \mu)|\varepsilon_1 + \varepsilon_2|,$$

where λ and μ are defined by Eqs. (2.1) and (2.2) .Indeed, comparing (3.1) with the alternative expression $\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2$ in the manner discussed in § 2, we readily obtain the sides of this yield polygon.

Let us now discuss a cylindrical sandwich shell under rotationally symmetric loading. As for the beam in Fig. 2, the core thickness will be denoted by $2h$ and the thickness of the covering sheets by t. By symmetry, the principal directions of the stress and the strain rate in these sheets are the axial and circumferential directions.

Let ε_z and ε_θ denote the axial and circumferential rates of extension in the median surface of the shell, and κ_z its axial rate of curvature. Since the ratio of the thickness of the shell to its radius will be treated as small in comparison to 1 and the deformation is rotationally symmetric, the effects of the circumferential rate of curvature can be neglected. The cover sheets therefore have the axial rates of extension $\varepsilon_1 = \varepsilon_z \pm h\kappa_z$ and the common circumferential rate of extension $\varepsilon_2 = \varepsilon_\theta$. According to (3.1) ,the power of dissipation per unit area of the median surface of the shell equals

$$(3.2)\quad D = t\{2\mu(\varepsilon_z + \varepsilon_\theta) + \tfrac{1}{2}(\lambda + \mu)(|\varepsilon_z + h\kappa_z| + |\varepsilon_z - h\kappa_z| + 2|\varepsilon_\theta|) + \\ + \tfrac{1}{2}(\lambda - \mu)(|\varepsilon_z + h\kappa_z + \varepsilon_\theta| + |\varepsilon_z - h\kappa_z + \varepsilon_\theta|)\}.$$

If the axial and circumferential membrane forces are denoted by N_z and N_θ, and the axial bending moment by M_z, the yield polyhedron in the stress space with the rectangular Cartesian coordinates N_z, N_θ, M_z/h can be obtained by comparing the right-hand side of (3.2) with the alternative expression $N_z\varepsilon_z + N_\theta\varepsilon_z + M_z\kappa_z$.

In making these comparisons for all possible combinations of the signs of the quantities whose absolute values appear in (3.2), we temporarily omit the term $2\mu t(\varepsilon_z + \varepsilon_\theta)$. We then obtain a polyhedron that is centered at the origin. To coincide with the yield polyhedron, this preliminary polyhedron must be given the translation with the components $2\mu t$, $2\mu t$, 0 corresponding to the terms that were temporarily omitted.

Table I shows a convenient way of determining the vertices of the preliminary polyhedron. In the first two lines, for instance, $\varepsilon_z + h\kappa_z$ is assumed to be arbitrarily close to zero but of unspecified sign, whereas $\varepsilon_z - h\kappa_z$ and ε_θ are assumed to be positive. It follows that $\varepsilon_z + h\kappa_z + \varepsilon_\theta$ and $\varepsilon_z - h\kappa_z + \varepsilon_\theta$ are positive. If now $\varepsilon_z + h\kappa_z$ is assumed to be positive, the comparison of the coefficients of the strain rate components in the

Signs of					Vertices of preliminary polyhedron			
$\varepsilon_z + h\kappa_z$	$\varepsilon_z - h\kappa_z$	$2\varepsilon_\theta$	$\varepsilon_z + h\kappa_z + \varepsilon_\theta$	$\varepsilon_z - h\kappa_z + \varepsilon_\theta$	N_z/t	N_θ/t	M_z/t	Vertex designation
± 0	$+$	$+$	$+$	$+$	2λ $\lambda - \mu$	2λ 2λ	0 $-\lambda - \mu$	1 2
± 0	$+$	$-$	$-$	$+$	$\lambda + \mu$ 0	$-\lambda - \mu$ $-\lambda - \mu$	$-\lambda + \mu$ -2λ	3 4
± 0	$+$	$-$	$-$	$-$	2μ $-\lambda + \mu$	-2λ -2λ	0 $-\lambda - \mu$	5 6
$+$	± 0	$+$	$+$	$+$	2λ $\lambda - \mu$	2λ 2λ	0 $\lambda + \mu$	1 6*
$+$	± 0	$-$	$+$	$-$	$\lambda + \mu$ 0	$-\lambda - \mu$ $-\lambda - \mu$	$\lambda - \mu$ 2λ	7 8
$+$	± 0	$-$	$-$	$-$	2μ $-\lambda + \mu$	-2λ -2λ	0 $\lambda + \mu$	5 2*
$+$	$+$	± 0	$+$	$+$	2λ 2λ	2λ -2μ	0 0	1 9
$+$	$-$	± 0	$+$	$-$	0 0	$\lambda + \mu$ $-\lambda - \mu$	2λ 2λ	4* 8
$+$	$+$	$-$	± 0	$+$	2λ $\lambda + \mu$	-2μ $-\lambda - \mu$	0 $-\lambda + \mu$	9 3
$+$	$+$	$-$	± 0	$-$	$\lambda + \mu$ 2μ	$-\lambda - \mu$ -2λ	$\lambda - \mu$ 0	7 5
$+$	$-$	$-$	± 0	$-$	0 $-\lambda + \mu$	$-\lambda - \mu$ -2λ	2λ $\lambda + \mu$	8 2*
$+$	$+$	$-$	$+$	± 0	2λ $\lambda + \mu$	-2μ $-\lambda - \mu$	0 $\lambda - \mu$	9 7
$+$	$+$	$-$	$-$	± 0	$\lambda + \mu$ 2μ	$-\lambda - \mu$ -2λ	$-\lambda + \mu$ 0	3 5
$+$	$-$	$+$	$+$	± 0	$\lambda - \mu$ 0	2λ $\lambda + \mu$	$\lambda + \mu$ 2λ	6* 4*

two expressions for the specific power of dissipation furnishes the vertex with the coordinates appearing in the first line. If, on the other hand, $\varepsilon_z + h\kappa_z$ is assumed to be negative, the vertex in the second line is obtained.

In all of the first block (lines 1 to 6), $\varepsilon_z - h\kappa_z$ is taken as positive. Any case in which $\varepsilon_z + h\kappa_z$ is arbitrarily close to zero while $\varepsilon_z - h\kappa_z$ is negative, can be obtained from one of the cases in the first block by the reversal of all signs. Instead of the vertex 1, for instance, its image 1* by symmetry with respect to the origin would be obtained.

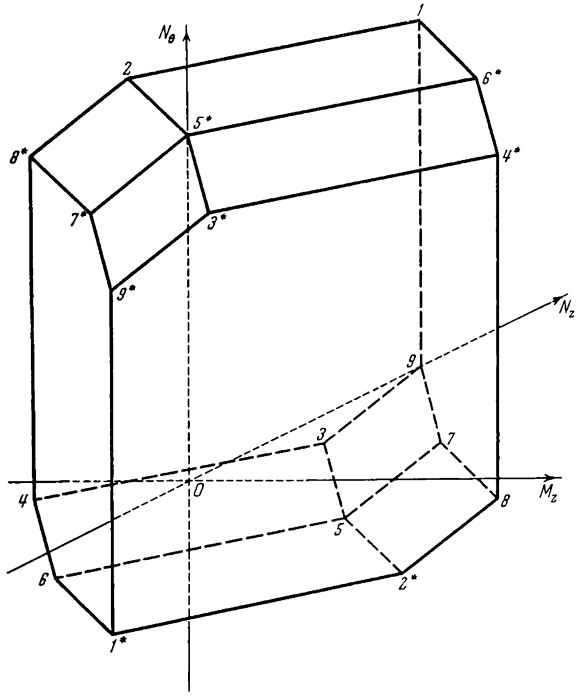

Fig. 5.

Vertices that correspond to the two choices of sign for one of the terms marked ± 0 are connected by an edge.

Figure 5 shows the yield polyhedron obtained by the appropriate translation of the preliminary polyhedron in Table I. As the preceding figures, Fig. 5 corresponds to $\lambda = 2\mu$.

ON TWO REPRESENTATIONS OF FUNCTIONS, ANALYTIC IN THE UPPER HALF-PLANE

G. N. PYKHTEEV
Novosibirsk, U.S.S.R.

§ 1. In [1], functions have been introduced which were given on the segment $[-1, 1]$ in the form of series, composed of Chebyshev polynomials of the first and second kind

$$(1.1)\quad p_1^{(s)}(x) = \sum_{n=2}^{\infty} \frac{1}{(2n-1)^s} T_{2n-1}(x), \quad p_2^{(s)}(x) = \sum_{n=2}^{\infty} \frac{1}{(2n)^s} T_{2n}(x),$$

$$(s = 1, 2, \cdots; -1 \leq x \leq 1),$$

$$(1.2)\quad q_1^{(s)}(x) = \sum_{n=2}^{\infty} \frac{1}{(2n-1)^s} U_{2n-1}(x), \quad q_2^{(s)}(x) = \sum_{n=2}^{\infty} \frac{1}{(2n)^s} U_{2n}(x),$$

$$(s = 1, 2, \cdots; -1 \leq x \leq 1),$$

$$(1.3)\qquad T_n(x) = \cos n \arccos x, \quad U_n(x) = \sin n \arccos x,$$

In another reference [2], functions have been studied which are given on the segment $[0, 1]$ in the form of the power series

$$(1.4)\quad M_1^{(s)}(x) = \sum_{n=2}^{\infty} \frac{1}{(2n-1)^s} x^{2n-1}, \quad M_2^{(s)}(x) = \sum_{n=2}^{\infty} \frac{1}{(2n)^s} x^{2n},$$

$$(s = 1, 2, \cdots; 0 \leq x \leq 1),$$

$$(1.5)\quad N_1^{(s)}(x) = \sum_{n=2}^{\infty} \frac{(-1)^n}{(2n-1)^s} x^{2n-1}, \quad N_2^{(s)}(x) = \sum_{n=2}^{\infty} \frac{(-1)^n}{(2n)^s} x^{2n}.$$

Certain properties of the functions, defined by the formulae (1.1), (1.2) (1.4) and (1.5) were stated in [1, 2]. They also contain tables of these functions evaluated with an accuracy of 4 places.

We will consider functions given in the upper half-plane of the complex variable $z = x + iy$ in the form of the series

$$(1.6)\quad \phi_1^{(s)}(z) = \sum_{n=2}^{\infty} \frac{1}{(2n-1)^s} \frac{1}{(z + \sqrt{z^2-1})^{2n-1}},$$

$$\phi_2^{(s)}(z) = \sum_{n=2}^{\infty} \frac{1}{(2n)^s} \frac{1}{(z + \sqrt{z^2-1})^{2n}}, \quad (s = 1, 2, \cdots),$$

$$\psi_1^{(s)}(z) = \sum_{n=2}^{\infty} \frac{(-1)^n}{(2n-1)^s} \frac{1}{(z+\sqrt{z^2-1})^{2n-1}},$$

(1.7)

$$\psi_2^{(s)}(z) = \sum_{n=2}^{\infty} \frac{(-1)^n}{(2n)^s} \frac{1}{(z+\sqrt{z^2-1})^{2n}},$$

where we will choose for $(z^2-1)^{\frac{1}{2}}$ the branch which takes real positive values on the segment $[1, \infty]$ of the Ox axis. All these functions are for $s > 1$ analytic functions of the complex variable z. The functions $\varphi_1^{(1)}(z)$, $\varphi_2^{(1)}(z)$ have logarithmic singularities at the points $z = \pm 1$:

$$\phi_1^{(1)}(z) = \operatorname{artanh}(z - \sqrt{z^2-1}) - z + \sqrt{z^2-1},$$

$$\phi_2^{(1)}(z) = -\tfrac{1}{2}\log[1 - (z-\sqrt{z^2-1})^2] - \tfrac{1}{2}(z-\sqrt{z^2-1})^2.$$

The functions $\psi_1^{(1)}(z)$ and $\psi_2^{(1)}(z)$ will be represented in the form

$$\psi_1^{(1)}(z) = z - \sqrt{z^2-1} - \operatorname{arc\,tan}(z-\sqrt{z^2-1}),$$

$$\psi_2^{(1)}(z) = \tfrac{1}{2}(z-\sqrt{z^2-1})^2 - \tfrac{1}{2}\log[1 + (z-\sqrt{z^2-1})^2].$$

We will introduce variables x^* and x^{**}, related to x by the equalities

$$x^* = \sqrt{1-x^2}, \qquad (0 \le x^* \le 1,\ -1 \le x \le 1),$$

(1.8)

$$x^{**} = |x| - \sqrt{x^2-1}, \quad (0 \le x^{**} \le 1,\ 1 \le |x| \le \infty),$$

and will note that on the segment $[-1, 1]$ of the Ox axis one has the relations

$$T_{2n-1}(x^*) = (-1)^{n-1}U_{2n-1}(x), \quad T_{2n}(x^*) = (-1)^n T_{2n}(x),$$

(1.9)

$$(\operatorname{sign} x)U_{2n-1}(x^*) = (-1)^{n-1}T_{2n-1}(x),$$

$$(\operatorname{sign} x)U_{2n}(x^*) = (-1)^{n-1}U_{2n}(x).$$

We will also note that at any point of the upper half-plane of the complex variable z one has the integral relations

$$\frac{\sqrt{z^2-1}}{\pi i}\int_{-1}^{1} \frac{T_n(t)}{t-z}\frac{dt}{\sqrt{1-t^2}} = -\frac{1}{i}\frac{1}{(z+\sqrt{z^2-1})^n},$$

(1.10)

$$\frac{1}{\pi i}\int_{-1}^{1} \frac{U_n(t)}{t-z}\,dt = -\frac{1}{i}\frac{1}{(z+\sqrt{z^2-1})^n}.$$

If one goes to the limit $z \to x$ in the equalities (1.6), (1.7), taking into consideration the relations (1.8), it is readily seen that the functions

under consideration assume on the real axis Ox the values

$$\phi_1^{(s)}(x) = \begin{cases} p_1^{(s)}(x) - iq_1^{(s)}(x) & \text{for } -1 \le x \le 1, \\ (\operatorname{sign} x) M_1^{(s)}(x^{**}) & \text{for } 1 \le |x| \le \infty, \end{cases}$$

$$\phi_2^{(s)}(x) = \begin{cases} p_2^{(s)}(x) - iq_2^{(s)}(x) & \text{for } -1 \le x \le 1, \\ M_2^{(s)}(x^{**}) & \text{for } 1 \le |x| \le \infty, \end{cases} \tag{1.11}$$

$$\psi_1^{(s)}(x) = \begin{cases} -(\operatorname{sign} x) q_1^{(s)}(x^*) + ip_1^{(s)}(x^*) & \text{for } -1 \le x \le 1, \\ (\operatorname{sign} x) N_1^{(s)}(x^{**}) & \text{for } 1 \le |x| \le \infty, \end{cases}$$

$$\psi_2^{(s)}(x) = \begin{cases} p_2^{(s)}(x^*) + i(\operatorname{sign} x) q_2^{(s)}(x^*) & \text{for } -1 \le x \le 1, \\ N_2^{(s)}(x^{**}) & \text{for } 1 \le |x| \le \infty. \end{cases} \tag{1.12}$$

It is not difficult to show with the help of (1.9), (1.10) that the functions considered satisfy at any point z of the upper half-plane the integral relations

$$\frac{1}{\pi}\int_{-1}^{1} \frac{p_1^{(s)}(t^*)}{t-z}\,dt = -i\psi_1^{(s)}(z), \quad \frac{1}{\pi}\int_{-1}^{1} \frac{q_1^{(s)}(t)}{t-z}\,dt = i\phi_1^{(s)}(z),$$

$$\frac{1}{\pi}\int_{-1}^{1} \frac{q_2^{(s)}(t)}{t-z}\,dt = i\phi_2^{(s)}(z), \quad \frac{1}{\pi}\int_{-1}^{1} \frac{(\operatorname{sign} t) q_2^{(s)}(t^*)}{t-z}\,dt = -i\psi_2^{(s)}(z), \tag{1.13}$$

$$\frac{\sqrt{z^2-1}}{\pi i}\int_{-1}^{1} \frac{p_1^{(s)}(t)}{t-z}\,\frac{dt}{\sqrt{1-t^2}} = i\phi_1^{(s)}(z),$$

$$\frac{\sqrt{z^2-1}}{\pi i}\int_{-1}^{1} \frac{(\operatorname{sign} t) q_1^{(s)}(t^*)}{t-z}\,\frac{dt}{\sqrt{1-t^2}} = -i\psi^{(s)}(z),$$

$$\frac{\sqrt{z^2-1}}{\pi i}\int_{-1}^{1} \frac{p_2^{(s)}(t)}{t-z}\,\frac{dt}{\sqrt{1-t^2}} = i\phi_2^{(s)}(z), \tag{1.14}$$

$$\frac{\sqrt{z^2-1}}{\pi i}\int_{-1}^{1} \frac{p_2^{(s)}(t^*)}{t-z}\,\frac{dt}{\sqrt{1-t^2}} = i\psi_2^{(s)}(z).$$

§ 2. We will define a function $f^{(s)}(x)$ by the equality

$$f^{(s)}(x) = \left(\frac{d^s}{d\theta^s} f(\cos\theta)\right)_{\theta=\arccos x}, \quad (s = 0, 1, \cdots)$$

and call it the trigonometric derivative of s-th order of the function $f(x)$.

We will denote by $W_\gamma^{(2k)}(M_{2k}; -1, 1)$ the class of functions which satisfy the conditions,

1) any function $f(x)$, belonging to this class, has continuous trigonometric derivatives $f^{(s)}(x)$, $(s = 0, 1, \cdots)$, up to order $(2k-1)$ inclusively on the segment $[-1, 1]$, with the exclusion of the point $x = 0$,

2) the $2k$-th trigonometric derivative of this function satisfies the inequality

$$|f^{(2k)}(x)| \leq M_{2k},$$

3) at the point $x = 0$, the function $f(x)$ and its trigonometric derivatives $f^{(s)}(x)$ may have first order discontinuities

$$2\gamma^{(s)} = f^{(s)}(+0) - f^{(s)}(-0), \quad (s = 0, 1, 2, \cdots, 2k).$$

THEOREM. *Let the function $\phi(z)$, analytic in the upper half-plane of the complex variable z and having at the point at infinity the value $\phi(\infty)$, assume on the real axis the values*

$$\begin{aligned} &\operatorname{Re}\phi(x) = 0 && \text{for } -\infty \leq x \leq -1, \\ &\operatorname{Re}\phi(x) = 0 && \text{for } 1 \leq x \leq \infty, \\ &\phi(x) = f_1(x) + if_2(x) && \text{for } -1 \leq x \leq 1, \end{aligned} \tag{2.1}$$

and let

$$f_1(x) \in W^{(2k)}(M_{2k}; -1, 1), \quad f_2(x) \in W^{(2k)}(M_{2k}, -1, 1);$$

then one has the following two representations of the function $\phi(z)$:

$$\begin{aligned} \phi(z) = \frac{4}{\pi} i \sum_{s=1}^{k} (-1)^{s-1}[&\gamma_1{}^{2(s-1)}\phi_1{}^{(2s-1)}(z) + \gamma_2{}^{2(s-1)}\phi_2{}^{(2s-1)}(z) - \\ &- \gamma^{(2s-1)}\psi_1{}^{(2s)}(z) - \gamma^{2(s-1)}\psi_2{}^{(2s-1)}(z)] + \sum_{m=1}^{n} \frac{b_m{}^*}{(z + \sqrt{z^2-1})^m} + \\ &+ \phi(\infty) + R_n{}^{(1)}(z), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} &2\gamma_1{}^{2(s-1)} = f_1{}^{2(s-1)}(1) + f_1{}^{2(s-1)}(-1), \\ &2\gamma_2{}^{2(s-1)} = f_1{}^{2(s-1)}(1) - f_1{}^{2(s-1)}(-1), \quad (s = 1, 2, \cdots, k), \\ &2\gamma^{(s)} = f_1{}^{(s)}(+0) - f_1{}^{(s)}(-0), \quad (s = 0, 1, \cdots, 2k-1), \end{aligned} \tag{2.2'}$$

$$b_1^* = b_1, \quad b_2^* = b_2,$$

$$b_{2m}^* = b_{2m} - \frac{4}{\pi}\sum_{s=1}^{k}\frac{(-1)^s}{(2m)^{2s-1}}\left((-1)^m\gamma^{2(s-1)} - \gamma_2{}^{2(s-1)}\right),$$

$$b_{2m-1}^* = b_{2m-1} - \frac{4}{\pi}\sum_{s=1}^{k}\frac{(-1)^s}{(2m-1)^{2s}}\left((-1)^m\gamma^{(2s-1)} - (2m-1)\gamma_1{}^{2(s-1)}\right), \tag{2.2''}$$

$$b_m = \frac{2}{\pi}\int_{-1}^{1} f(x)U_m(x)\frac{dx}{\sqrt{1-x^2}},$$

$$\phi(z) = \frac{4}{\pi}i\sum_{s=1}^{k}(-1)^s[\gamma_1{}^{(2s-1)}\phi_1{}^{(2s)}(z) + \gamma_2{}^{(2s-1)}\phi_2{}^{(2s)}(z) + \tag{2.3}$$

$$+ \gamma^{2(s-1)}\psi_1{}^{(2s-1)}(z) - \gamma^{(2s-1)}\psi_2{}^{(2s)}(z)] + i\frac{a_0}{2} +$$

$$+ i\sum_{m=1}^{n}\frac{a_m^*}{(z+\sqrt{z^2-1})^m} + \phi(\infty) - iC + R_n{}^{(2)}(z),$$

where

$$2\gamma_1{}^{(2s-1)} = f_2{}^{(2s-1)}(1) + f_2{}^{(2s-1)}(-1),$$

$$2\gamma_2{}^{(2s-1)} = f_2{}^{(2s-1)}(1) - f_2{}^{(2s-1)}(-1), \quad (s = 1, 2, \cdots k) \tag{2.3'}$$

$$2\gamma^{(s)} = f_2{}^{(s)}(+0) - f_2{}^{(s)}(-0), \quad (s = 0, 1, \cdots, 2k-1),$$

$$a_1^* = a_1, \quad a_2^* = a_2,$$

$$a_{2m}^* = a_{2m} + \frac{4}{\pi}\sum_{s=1}^{k}(-1)^s\left((-1)^m\gamma^{(2s-1)} - \gamma_2{}^{(2s-1)}\right),$$

$$a_{2m-1}^* = a_{2m-1} - \frac{4}{\pi}\sum_{s=1}^{k}(-1)^s\left((-1)^m(2m-1)\gamma^{2(s-1)} + \gamma_1{}^{(2s-1)}\right), \tag{2.3''}$$

$$a_m = \frac{2}{\pi}\int_{-1}^{1} f_2(x)T_m(x)\frac{dx}{\sqrt{1-x^2}},$$

$$C = \frac{1}{\pi}\int_{-1}^{1} f_2(x)\frac{dx}{\sqrt{1-x^2}}. \tag{2.3'''}$$

In this connection

$$|R_n{}^{(1)}(z)| < 2M_{2k}N(n, k), \quad |R_n{}^{(2)}(z)| < 2M_{2k}N(n, k), \tag{2.4}$$

where

$$N(n, k) = 4(n+1)^{-2k}\left(1 + \log\frac{\pi}{2} + \frac{1}{2k} + \frac{1}{2n} + \log n\right). \tag{2.5}$$

In other words, one has the approximate equalities

$$\phi(z) \approx \phi_n^{(k)}(z) = \frac{4}{\pi} i \sum_{s=1}^{k} (-1)^{s-1}[\gamma_1^{2(s-1)}\phi_1^{(2s-1)}(z) + \tag{2.6}$$

$$+ \gamma_2^{2(s-1)}\phi_2^{(2s-1)}(z) - \gamma^{(2s-1)}\psi_1^{(2s)}(z) - \gamma^{2(s-1)}\psi_2^{(2s-1)}(z)] +$$

$$+ i \sum_{m=1}^{n} \frac{b_m^*}{(z + \sqrt{z^2-1})^m} + \phi(\infty),$$

or

$$\phi(z) \approx \psi_n^{(k)}(z) = \tag{2.7}$$

$$= \frac{4}{\pi} i \sum_{s=1}^{k} (-1)^{s}[\gamma_1^{(2s-1)}\phi_1^{(2s)}(z) + \gamma_2^{(2s-1)}\phi_2^{(2s)}(z) -$$

$$- \gamma^{2(s-1)}\psi_1^{(2s-1)}(z) - \gamma^{(2s-1)}\psi_2^{(2s)}(z)] + i\frac{a_0}{2} +$$

$$+ i \sum_{m=1}^{n} \frac{a_m^*}{(z + \sqrt{z^2-1})^m} + \phi(\infty) - iC,$$

where

$$|\phi(z) - \phi_n^{(k)}(z)| < 2M_{2k}N(n, k), \tag{2.8}$$
$$|\phi(z) - \psi_n^{(k)}(z)| < 2M_{2k}N(n, k).$$

PROOF. We will apply Schwarz's formula to the function $\phi(z)$ under consideration, assuming that its real part $f_1(x)$ is known on the segment $[-1, 1]$ of the Ox axis; we obtain

$$\phi(z) = \frac{1}{\pi i}\int_{-1}^{1} \frac{f_1(t)}{t - z}\, dt + \phi(\infty). \tag{2.9}$$

Then, assuming the imaginary part $f_2(x)$ of the function $\phi(z)$ to be known, we apply to the last the formula giving the solution of the mixed problem for the upper half-plane [3, 4]; as the result we obtain

$$\phi(z) = \frac{\sqrt{z^2-1}}{\pi i}\int_{-1}^{1} \frac{f_2(t)}{t - z}\, \frac{dt}{\sqrt{1-t^2}} - iC + \phi(\infty), \tag{2.10}$$

where C is given by the equality (2.3‴). We note that for a given value of $\phi(\infty)$ the formula (2.10) holds only in the case when the function $f_2(x)$ satisfies the condition $iC = \phi(\infty)$.

We turn now to the functions $f_1(x)$ and $f_2(x)$. Since they belong to the class $W^{(2k)}(M_{2k}; -1, 1)$, the representations (3.8) and (3.9) of [1] are applicable to them. Applying the first of these representations to the function $f_2(x)$ and the second to the function $f_1(x)$, we find

$$(2.11)\quad f_1(x) = -\frac{4}{\pi}\sum_{s=1}^{k}(-1)^s[\gamma^{(2s-1)}p_1^{(2s)}(x^*) + \gamma_1^{2(s-1)}q_1^{(2s-1)}(x) + \gamma_2^{2(s-1)}q_2^{(2s-1)}(x) + (\operatorname{sign} x)q_2^{(2s-1)}(x^*)] + \sum_{m=1}^{n} b_m^* U_m(x) + r_n^{(2)}(x),$$

$$(2.12)\quad f_2(x) = \frac{4}{\pi}\sum_{s=1}^{k}(-1)^s[\gamma_1^{(2s-1)}p_1^{(2s)}(x) - (\operatorname{sign} x)\gamma^{2(s-1)}q_1^{(2s-1)}(x^*) + \gamma_2^{(2s-1)}p_2^{(2s)}(x) - \gamma^{(2s-1)}p_2^{(2s)}(x^*)] + \frac{a_0}{2} + \sum_{m=1}^{n} a_m^* T_m(x) + r_n^{(1)}(x).$$

The constants in the formula (2.11) are determined by the equalities (2.2′), (2.2″), and the constants in (2.12) by (2.3′), (2.3″). Substituting (2.11) and (2.12) and using the relations (1.10), (1.13) and (1.14), we obtain

$$\phi(z) = \phi_n^{(k)}(z) + R_n^{(1)}(z), \quad \phi(z) = \psi_n^{(k)}(z) + R_n^{(2)}(z),$$

where

$$R_n^{(1)}(z) = \frac{1}{\pi i}\int_{-1}^{1}\frac{r_n^{(2)}(t)}{t - z}\,dt,$$

$$R_n^{(2)}(z) = \frac{\sqrt{z^2 - 1}}{\pi i}\int_{-1}^{1}\frac{r_n^{(1)}(t)}{t - z}\,\frac{dt}{\sqrt{1 - t^2}}.$$

There remains to prove that $R_n^{(1)}(z)$ and $R_n^{(2)}(z)$ satisfy the inequalities (2.4). Since they are analytic functions in the upper half-plane of the variable z and, consequently, have a maximum on the Ox axis, one has

$$(2.13)\qquad |R_n^{(s)}(z)| < \max_{-\infty \le x \le \infty} |(R^{(s)}(z))_{z=x}|, \quad (s = 1, 2).$$

Introducing the notation

$$R_n^{(-1)}(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{r_n^{(2)}(t)}{t-x}\,dt,$$

$$R_n^{(-2)}(x) = \frac{\sqrt{x^2-1}}{\pi}\int_{-1}^{1} \frac{r_n^{(1)}(t)}{t-x}\,\frac{dt}{\sqrt{1-t^2}}, \quad (-\infty \le x \le -1),$$

$$r_n^{(3)}(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{r_n^{(2)}(t)}{t-x}\,dx,$$

$$r_n^{(4)}(x) = \frac{\sqrt{1-x^2}}{\pi}\int_{-1}^{1} \frac{r_n^{(1)}(t)}{t-x}\,\frac{dx}{\sqrt{1-t^2}}, \quad (-1 \le x \le 1),$$

$$R_n^{(1)}(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{r_n^{(2)}(t)}{t-x}\,dt,$$

$$R_n^{(2)}(x) = \frac{\sqrt{x^2-1}}{\pi}\int_{-1}^{1} \frac{r_n^{(1)}(t)}{t-x}\,\frac{dt}{\sqrt{1-t^2}}, \quad (1 \le x \le \infty),$$

we have

$$|(R_n^{(s)}(z))_{z=x}| = |R_n^{(-s)}(x)| \text{ for } -\infty \le x \le -1,$$

$$|(R_n^{(s)}(z))_{z=x}| = |R_n^{(s)}(x)| \text{ for } 1 \le x \le \infty, \quad s = 1, 2,$$

$$|(R_n^{(1)}(z))_{z=x}| = \sqrt{|r_n^{(2)}(x)|^2 + |r_n^{(3)}(x)|^2} \text{ for } -1 \le x \le 1,$$

$$|(R_n^{(2)}(z))_{z=x}| = \sqrt{|r_n^{(1)}(x)|^2 + |r_n^{(4)}(x)|^2} \text{ for } -1 \le x \le 1.$$

The estimates obtained in [1] for $r_n^{(s)}(x)$, $(s = 1, 2, 3, 4)$ and in [2] for $R_n^{(s)}(x)$, $(s = 1, 2)$ give

$$|r_n^{(s)}(x)| < M_{2k}N(n, k), \quad (s = 1, 2, 3, 4),$$

$$|R_n^{(s)}(x)| < 2M_{2k}N(n, k), \quad (s = 1, 2).$$

Replacing in the integrals $R_n^{(-1)}(x)$ and $R_n^{(-2)}(x)$ the arguments x by $-x$, t by $-t$ and using results of [2], it is not difficult to show that

$$|R_n^{(-s)}(x)| < 2M_{2k}N(n, k), \quad (s = 1, 2).$$

Thus

$$|(R_n^{(s)}(z))_{z=x}| < 2M_{2k}N(n, k),$$

whence we obtain by the strength of (2.13)

$$|R_n^{(s)}(z)| < 2M_{2k}N(n, k), \ (s = 1, 2),$$

as was to be proved.

The representations (2.2), (2.3) may be used for the solution of some plane problems of hydrodynamics, aerodynamics and the theory of elasticity.

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PLANE PROBLEMS OF LINEAR VISCOELASTICITY

J. R. M. RADOK
New York, U.S.A.

After reduction of the problems of the plane theory of elasticity to boundary value problems of the theory of functions of a complex variable [1, 2], the scene had been set for a revival of research into effective methods of solution in this classical field of mathematics. As a consequence, there occurred the development of the theory of singular integral equations with Cauchy type kernels and of the associated boundary value problems which subsequently extended its rule into many branches of classical and modern applied mathematics [3, 4].

The plane theory of linear viscoelasticity is probably the latest and one of the most obvious additions to the sphere of influence of this work. Its basic formulae and problems will be formulated in relation to those of elasticity and it will be shown that the solutions of many of the corresponding problems of the two theories are interconnected through well-defined ordinary differential equations in time. Some problems of viscoelasticity will be presented for which this rule does not apply and the solution of which may be considered a major challenge of the mathematical theory of viscoelasticity.

§ 1. Basic elastic and viscoelastic formulae

The basic equations of the static theory of plane elastic strain are written most conveniently in terms of the complex variables

$$z = x + iy, \quad \bar{z} = x - iy \tag{1.1}$$

in the form [1, 5]

$$\frac{\partial}{\partial \bar{z}} (Y_y - X_x + 2iX_y) = \frac{\partial}{\partial z} (X_x + Y_y), \tag{1.2}$$

$$Y_y - X_x + 2iX_y = -4\mu \frac{\partial}{\partial z} (u - iv), \tag{1.3}$$

$$X_x + Y_y = \frac{2\mu}{1 - 2\sigma} \left[\frac{\partial}{\partial z} (u + iv) + \frac{\partial}{\partial \bar{z}} (u - iv) \right], \tag{1.4}$$

where μ, σ are the shear modulus and Poisson's ratio respectively. In the

case of generalized plane stress, one has to replace in (1.4)

$$\frac{2\mu}{1-2\sigma} \text{ by } \frac{1+\sigma}{1-\sigma}.$$

When the material under consideration is incompressible, Poisson's ratio must be given the value $\frac{1}{2}$ and for plane strain the place of (1.4) is taken by the condition of incompressibility

$$\frac{\partial}{\partial z}(u+iv)+\frac{\partial}{\partial \bar{z}}(u-iv)=0 \tag{1.4'}$$

The formulae of Kolosov-Muskhelishvili follow directly from these equations by quadrature:

$$X_x+Y_y=X'_{x'}+Y'_{y'}=2[\varphi'(z)+\overline{\varphi'(z)}], \tag{1.5}$$

$$Y_y-X_x+2iX_y=(Y'_{y'}-X'_{x'}+2iX'_{y'})e^{-2i\alpha}=2[\bar{z}\varphi''(z)+\psi'(z)], \tag{1.6}$$

$$2\mu(u+iv)=2\mu(u'+iv')e^{i\alpha}=\kappa\varphi(z)-z\,\overline{\varphi'(z)}-\overline{\psi(z)}, \tag{1.7}$$

where

$$\kappa=3-4\sigma \text{ for plane strain,}$$

$$\kappa=\frac{3-\sigma}{1+\sigma} \text{ for generalized plane stress.} \tag{1.8}$$

For the sake of definiteness, the following considerations will be confined to the case of plane strain.

The quasi-static, viscoelastic theory, corresponding to the above elastic theory of plane strain, will be based on the modified stress-displacement relations (1.3), (1.4)

$$P^p\{Y_y-X_x+2iX_y\}=-2Q^q\left\{\frac{\partial}{\partial z}(u-iv)\right\}, \tag{1.9}$$

$$R^r\{X_x+Y_y\}=S^s\left\{\frac{\partial}{\partial z}(u+iv)+\frac{\partial}{\partial \bar{z}}(u-iv)\right\}, \tag{1.10}$$

where a typical operator

$$M^m\{\ \}=\sum_{k=0}^{m} m_k\frac{\partial^k}{\partial t^k}\{\ \} \tag{1.11}$$

involves $(m+1)$ parameters m_k which will be assumed to be known for a given material. If a material is incompressible, the relation (1.4') takes the place of (1.10). It should also be noted that extension of this theory to

include time dependence of the parameters m_k presents no principal difficulties [6].

The formal similarity of the stress displacement relations (1.3), (1.4) and (1.9), (1.10) suggests that modified Kolosov-Muskhelishvili formulae must apply for such a viscoelastic theory. In fact, executing step by step the manipulations leading from (1.2)–(1.4) to (1.5)–(1.7) and taking into consideration that P, Q, R, S are linear differential operators with respect to the time t, one finds

$$(Q^qR^r + P^pS^s)\{X_x + Y_y\} = 2[\varphi'(z) + \overline{\varphi'(z)}], \tag{1.12}$$

$$(Q^qR^r + P^pS^s)\{Y_y - X_x + 2iX_y\} = 2[\bar{z}\varphi''(z) + \psi'(z)], \tag{1.13}$$

$$Q^qS^s(Q^qR^r + P^pS^s)\{u + iv\} = (2Q^qR^r + P^pS^s)\{\varphi(z)\} - P^pS^s\{z\,\overline{\varphi'(z)} + \overline{\psi(z)}\}, \tag{1.14}$$

where the functions φ, ψ have the same properties as in the elastic theory [2]. In particular, these functions are holomorphic in the region occupied by the body, so that, with certain possible restrictions, they possess at any point a of their region of definition expansions of the form

$$\sum_{j=0}^{\infty} A_j(z - a)^j.$$

The above formulae must be interpreted in the sense that the operators apply to the coefficients A_j in these expansions which may become time dependent. It will be assumed throughout that the operators do not change the character of the convergence of these expansions. It is readily verified that in many physical problems the quantities which may exhibit time dependence occupy the positions of scale factors of these functions, so that their differentiation or integration indeed does not affect the analytic properties of the functions φ, ψ.

If the material is incompressible, the above formulae assume the form

$$P^p\{X_x + Y_y\} = 2[\varphi'(z) + \overline{\varphi'(z)}], \tag{1.12'}$$

$$P^p\{Y_y - X_x + 2iX_y\} = 2[\bar{z}\varphi''(z) + \psi'(z)], \tag{1.13'}$$

$$Q^q\{u + iv\} = \varphi(z) - z\,\overline{\varphi'(z)} - \overline{\psi(z)}. \tag{1.14'}$$

It may now be noted that, apart from the operators

$$(Q^qR^r + P^pS^s) \text{ and } P^p \tag{1.15}$$

in the relations (1.12), (1.13) and (1'12'), (1.13') respectively, the Kolosov-Muskhelishvili formulae for viscoelasticity (1.12)–(1.14), (1.12')–(1.14')

could have been obtained formally from (1.5)–(1.7) by the substitutions

$$2\mu = \frac{Q^q}{P^p}, \quad \frac{2\mu}{1-2\sigma} = \frac{S^s}{R^r}.$$

The additional operators (1.15) do not appear as a result of such a formal process of deduction, since most developments of the theory of plane strain are based on the equation of compatibility

$$(\lambda + 2\mu)\Delta(X_x + Y_y) = -2(\lambda + \mu)\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right).$$

They assume that the body forces X, Y vanish identically and that therefore the elastic constants can be removed by division.

§ 2. Formulation of fundamental problems

The fundamental boundary value problems of the viscoelastic theory will now be formulated in correspondence with the elastic theory. They will differ from those of the elastic theory only in the manner in which the time variable may enter into initial and boundary conditions as well as into the specification of the region occupied by the viscoelastic material.

First fundamental problem (*Problem I*): For a given initial state of stress and given, time dependent stresses applied to the boundary L of the region S, find the state of viscoelastic equilibrium at all subsequent times.

Second fundamental problem (*Problem II*): For a given initial state of deformation and given, time dependent displacements of points of the boundary L of the region S, find the state of viscoelastic equilibrium at all subsequent times.

A detailed discussion of the terms region S and boundary L will not be necessary here, since these quantities do not differ from those occurring in the elastic theory [2]. However, in the viscoelastic theory, boundaries may change with time without altering their general geometric and analytic character; for example, in the problem of immersion in a fluid of a viscoelastic cylinder, the rate of immersion will play a significant role, since it specifies the dividing line between different loading conditions.

The fundamental mixed problem involves combinations of the above two problems, applying to different parts of a body. The initial state of the complete body will be assumed to be a possible instantaneous elastic state of deformation, while the boundary between the parts of S which are subject to the regimes of Problem I and Problem II may change with time.

It is not proposed to consider here other types of problems of the viscoelastic theory which are readily obtained, for example, by modification of the initial conditions. The questions of uniqueness and existence of solutions of the posed problems also go beyond the scope of the present treatment of the viscoelastic theory. However, in view of the following deductions, it will be suggested that the available knowledge of elastic theory combined with known results on solutions of ordinary linear differential equations may well suffice to dispose of these questions in a relatively straight forward manner.

§ 3. Reduction of the fundamental viscoelastic problems to fundamental elastic problems

It will be convenient to introduce here the generalized stress and displacement rates

$$(3.1) \qquad (Q^q R^r + P^p S^s)\{X_x, Y_y, X_y, u, v\{ = (\underline{X}_x, \underline{Y}_y, \underline{X}_y, \underline{u}, \underline{v})$$

in terms of which the formulae (1.12)–(1.14) may be rewritten

$$(3.2) \qquad \underline{X}_x + \underline{Y}_y = 2[\varphi'(z) + \overline{\varphi'(z)}].$$

$$(3.3) \qquad \underline{Y}_y - \underline{X}_x + 2i\underline{X}_y = 2[\bar{z}\varphi''(z) + \psi'(z)],$$

$$(3.4) \quad Q^q S^s\{(\underline{u} + i\underline{v})\} = (2Q^q R^r + P^p S^s)\{\varphi(z)\} - P^p S^s\{z\,\overline{\varphi'(z)} + \overline{\psi(z)}\}.$$

Thus, if the given external stresses have time derivatives of order $\max(q + r, p + s)$, one may evaluate the boundary values of $\underline{X}_x$, $\underline{Y}_y$, $\underline{X}_y$. Hence, using the transformation formulae (1.5), (1.6), one arrives at the first form of Problem I of the elastic theory [2]:

$$(3.5) \quad \varphi'(z) + \overline{\varphi'(z)} - [\bar{z}\varphi''(z) + \psi'(z)]e^{2i\alpha} = \\ = \underline{X}_x + \underline{Y}_y - [\underline{Y}_y - \underline{X}_x + 2i\underline{X}_y]e^{2i\alpha}$$

for z on L, where α is the angle between the outward normal n to L and the axis Ox.

An alternative form of the boundary condition of Problem I is obtained by multiplying (3.5) by $e^{-i\alpha}$ and by integrating along L [2]:

$$(3.5)' \qquad \varphi(z) + z\,\overline{\varphi'(z)} + \overline{\psi(z)} = i\int_0^s (\underline{X}_n + i\underline{Y}_n)\,ds$$

for z on L, where $\underline{X}_n$, $\underline{Y}_n$ are the Cartesian components of the external generalized stress rates.

Thus it is seen that, as far as determination of the generalized stress rates is concerned, Problem I of the viscoelastic theory has been reduced to a corresponding Problem I of the elastic theory.

In the case of Problem I for incompressible materials, analogous reasoning may be applied with regard to the generalized stress rates

$$P^p\{X_x, Y_y, X_y\} = (\underline{X}_x, \underline{Y}_y, \underline{X}_y). \tag{3.1'}$$

In contrast, the incompressibility of a material plays an important role in Problem II. In fact, it will always ensure reducibility of the viscoelastic Problem II to corresponding elastic problems.

Introducing the generalized deformation rates

$$Q^q\{u, v\} = (\underline{u}, \underline{v}), \tag{3.6}$$

the boundary condition of the viscoelastic Problem II for incompressible material becomes

$$\underline{u} + i\underline{v} = \varphi(z) - z\,\overline{\varphi'(z)} - \overline{\psi(z)}, \tag{3.7}$$

for z on L. Hence the above statement is proved, since the functions φ, ψ solving the elastic problem (3.7) will determine the generalized deformation rates in the region S bounded by L, provided the given displacements of the boundary have derivatives of order q with respect to t.

For compressible materials, the formula for the generalized displacement rates (3.4) shows that reducibility of a viscoelastic Problem II to an elastic Problem II will, in general, not be possible. However, if the functions φ, ψ have the earlier mentioned property by which the differential operators in (3.4) do not change their analytic character, then one may use, for example, the method of Cauchy integrals or reduction to a problem of linear relationship (Hilbert Problem) for the solution of the viscoelastic Problem II [2]. In this case a formal interpretation as an elastic problem will not be possible.

As will by now be expected, mixed problems for incompressible materials are readily reducible to elastic problems and all the methods of Refs. [2, 3] are immediately applicable. For compressible material, no general statement is possible at the present time. In particular, it will be recalled that in certain mixed elastic problems, the constant κ plays a decisive role in the solution of the associated problems of linear relationship (Hilbert Problem) [2, 3]. In other problems, the geometry of a given region and the boundary conditions permit manipulations by which κ assumes effectively the role of a scale factor of φ, ψ. Since in the viscoelastic theory the elastic constants are replaced by differential operators,

the first group of problems cannot be solved by the same methods as the corresponding elastic problems, because these methods involve analytic operations which lose their meaning when applied to operators. In general, great care must also be observed, when dealing with multiply connected regions and conditions of single-valuedness of displacements.

Summing up the results of this section, it may be concluded that in the plane theory of viscoelasticity formal reduction to elastic problems is possible in the case of Problems I for all materials, of Problems II and Mixed Problems for incompressible materials and of certain Problems II and Mixed Problems for compressible materials. Most of the corresponding elastic problems involve as boundary conditions generalized stress and displacement rates and their solutions determine these rates throughout the regions of their definition. The problem of the evaluation of the actual viscoelastic stresses and displacements will be discussed next.

§ 4. Viscoelastic stresses and displacements

For the present, consideration will be restricted to problems of viscoelasticity which are reducible to elastic problems in the sense of the preceding section and it will be assumed that the functions φ, ψ solving the elastic problems have been determined. Under these circumstances, the relations (1.12)–(1.14) for compressible materials and (1.12′)–(1.14′) for incompressible materials render ordinary differential equations in the time variable t for the determination of the viscoelastic stresses and displacements. The order of these differential equations varies from $\max(q + r, p + s)$ to p for the stresses and from $\max(2q + r+s, 2s+q+p$ to q for the displacements. Given the corresponding numbers of initial conditions, these equations may be solved by any preferred method for the solution of ordinary differential equations for any selected point in the body. The so-called method of Laplace Transforms and the associated elastic problems [7] are thus seen to have little physical significance; in fact, they only fulfill the purpose of dealing with the mathematical problem of the solution of these differential equations. The same conclusion was reached at an earlier stage [8] by more formal and heuristic means which did not elucidate the role played by the generalized stress and displacement rates.

The present subdivision of the process of solution of viscoelastic problems into two stages, concerned with the determination of the stress and displacement rates and the stresses and displacements respectively, is, of course, only possible due to the quasi-static nature of the theory by which time effects are confined to the stress-displacements relations.

§ 5. Applications

As a first example of a Problem I the stretching of an infinite body with an elliptical cylindrical cavity will be considered. In the case of biaxial tension $p(t)$ at infinity, the instantaneous elastic solution is given by [2]

$$\varphi(\zeta) = \frac{p(t)R}{2}\left(\zeta - \frac{m}{\zeta}\right), \quad \psi(\zeta) = -p(t)\frac{R(1+m^2)\zeta}{\zeta^2 - m}, \tag{5.1}$$

where

$$z = \omega(\zeta) = R\left(\zeta + \frac{m}{\zeta}\right), \quad 0 \leq m \leq 1 \tag{5.2}$$

maps the unit circle $\zeta = 1$ onto the ellipse with semi-axes

$$a = R(1+m), \quad b = R(1-m) \tag{5.3}$$

in the physical plane. For the corresponding viscoelastic problem, the functions (5.1) determine the generalized stress and displacement rates (3.1), when the generalized stress rate at infinity is

$$\underline{p}(t) = (Q^q R^r + P^p S^s)p(t). \tag{5.4}$$

In particular, at the edge of the cavity, one thus finds

$$\underline{\widehat{\vartheta\vartheta}}(1, \vartheta, t) = (Q^q R^r + P^p S^s)\{\widehat{\vartheta\vartheta}(1, \vartheta, t)\} = \tag{5.5}$$

$$= 2\underline{p}(t)\frac{1 - m^2}{1 - 2m\cos 2\vartheta + m^2}$$

which determines the time history of the circumferential stress $\widehat{\vartheta\vartheta}$.

As a second example, Problem II for the circular cylinder will be studied. Let it be assumed that the material of the cylinder is incompressible. The solution of the corresponding elastic problem

$$\varphi(\sigma) - \sigma\overline{\varphi'(\sigma)} - \overline{\psi(\sigma)} = g_1 + ig_2 = g \tag{5.6}$$

for $\sigma = e^{i\vartheta}$ a point of the boundary γ is [2]

$$\varphi(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{g\,d\sigma}{\sigma - \zeta} + \bar{a}_1\zeta + 2\bar{a}_2 + \bar{a},$$

$$\psi(\zeta) = -\frac{1}{2\pi i}\int_\gamma \frac{\bar{g}\,d\sigma}{\sigma - \zeta} - \frac{1}{\zeta}\varphi'(\zeta) + \frac{a_1}{\zeta}, \tag{5.7}$$

where

$$2\bar{a}_2 + \bar{a} = -\frac{1}{2\pi i}\int_\gamma g\frac{d\sigma}{\sigma},$$

$$\int_\gamma \frac{g d\sigma}{\sigma^2} = -\int_\gamma \bar{g} d\sigma, \tag{5.8}$$

$$\varphi(\zeta) = \sum_1^\infty a_j\zeta^j, \quad \psi(0) = a, \quad z = \omega(\zeta) = R\zeta.$$

The viscoelastic solution for the stress and displacement rates is obtained from (5.7), (5.8) by replacing g by

$$g = Q^q\{g(t)\} = Q^q\{u(\sigma, t) + iv(\sigma, t)\}, \tag{5.9}$$

where $u(\sigma, t)$, $v(\sigma, t)$ are the prescribed displacements of the boundary. The ordinary differential equations for the stresses and displacements are now given by (1.12′)–(1.14′).

The application of the problem of linear relationship will be illustrated by reference to the Mixed Problem of pressure of rigid stamps in the absence of friction on a viscoelastic half-space. The corresponding elastic problem is [2]

$$\varphi'^+(s) + \varphi'^-(s) = \frac{4\mu i f'(s)}{\kappa + 1} \text{ on } L', \tag{5.10}$$

where L' is the region of contact between the stamp and the real axis and $f'(s) = df/ds$. In order to arrive at the boundary condition for the viscoelastic problem, one has to give due consideration to the physical boundary conditions

$$\begin{aligned} X_y^- &= 0 \text{ for all } s, \\ Y_y^- &= 0 \text{ for } s \text{ outside } L', \\ v^- &= f(s) \text{ on } L'. \end{aligned} \tag{5.11}$$

The first of these, after introduction of the analytic continuation of $\varphi'(z)$ from the lower into the upper half-plane [2]

$$\varphi'(z) = -\bar{\varphi}'(z) - z\bar{\varphi}''(z) - \bar{\psi}'(z), \tag{5.12}$$

gives

$$\bar{\psi}'(z) = -\varphi'(z). \tag{5.13}$$

As a consequence (1.14) may be rewritten

$$Q^qS^s(Q^qR^r + P^pS^s)\{u' + iv'\} = (2Q^qR^r + P^pS^s)\{\varphi'(z)\} + P^pS^s\{\varphi'(\bar{z})\},$$

or

$$Q^qS^s(Q^qR^r + P^pS^s)\{u' - iv'\} = -(2Q^qR^r + P^pS^s)\{\varphi'(\bar{z})\} - P^pS^s\{\varphi'(z)\},$$

whence follows

$$iQ^qS^s(Q^qR^r + P^pS^s)\{v'\} = (Q^qR^r + P^pS^s)\{\varphi'(z) + \varphi'(\bar{z})\}. \tag{5.14}$$

The boundary condition for the viscoelastic problem now becomes

$$i\underline{v}'^- = \underline{\varphi}'^+(s) + \underline{\varphi}'^-(s) = i\underline{f}'(s), \tag{5.15}$$

where

$$\begin{aligned} \underline{v} &= Q^qS^s(Q^qR^r + P^pS^s)\{v\}, \\ \underline{\varphi} &= (Q^qR^r + P^pS^s)\{\varphi\}. \end{aligned} \tag{5.16}$$

When the region of contact L' consists of a single segment (a, b) of the real axis, the solution of the elastic problem (5.10) with $2\mu = 1$, $\kappa = 1$ is [2]

$$\varphi'(z) = \frac{1}{2\pi\sqrt{(z-a)(z-b)}} \int_a^b \frac{\sqrt{(s-a)(s-b)}f'(s)\,ds}{s-z} + \frac{D}{\sqrt{(z-a)(z-b)}} \tag{5.17}$$

and the corresponding viscoelastic solution follows directly from this result [9].

§ 6. Other problems

An example of a simple Mixed Problem which is not reducible to an elastic problem is obtained, if in the last problem of the preceding section the boundary conditions (5.11) are replaced by

$$\begin{aligned} Y_y^- = X^-{}_y &= 0 \text{ for } s \text{ outside } L', \\ u^- + iv^- &= g(s) \text{ for } s \text{ inside } L'. \end{aligned} \tag{6.1}$$

The first of these, with the analytic continuation (5.12) of $\varphi'(z)$, gives

$$\varphi'^+(s) - \varphi'^-(s) = 0 \text{ outside } L', \tag{6.2}$$

so that $\varphi'(z)$ must be holomorphic in the entire plane cut along L'. The second gives

$$Q^qS^s(Q^qR^r + P^pS^s)\{g'(s)\} = (2Q^qR^r + P^pS^s)\{\varphi'^+(s)\} + P^pS^s\{\varphi'^-(s)\}, \tag{6.3}$$

and it is seen that this boundary condition cannot be reformulated as an elastic problem for displacement rates. The same observation applies to the Mixed Problem of the last section when friction occurs between the punch and the indented half-space. On the other hand, the punch need not be rigid in order that reducibility to an elastic problem will be possible, since the plane Hertz Problem can be solved by the methods of plane elastic theory [9].

It will be interesting to note here another aspect of the Hertz Problem which can only arise in the case of viscoelastic theory. As long as the indenting bodies are pressed together, the boundary conditions, in the absence of friction, will have the form (5.11). However, when the contacting bodies are disengaged, one is no longer dealing with a Mixed Problem, but with a Problem I, and hence there occurs a change in the type of the elastic problem to which the problem under consideration is reducible. Nevertheless, the proposed method of solution is very suitable for problems of this type, because the initial state of the contacting bodies at the instant of disengagement can be described in terms of the functions φ, ψ for the determination of the stress rates. If at disengagement the two bodies are moved with respect to each other by a small distance only reengagement may occur at a later stage, a phenomenon which may be of considerable interest in the study of impact problems.

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ON SOME VARIATIONAL THEOREMS IN ELASTICITY

ERIC REISSNER
Cambridge, U.S.A.

§ 1. Introduction

The present note is concerned with further considerations of a general variational theorem in elasticity which has been studied by the author in some previous publications [1, 2, 3, 4]. The theorem is for stresses *and* displacements, and is such that equilibrium equations as well as stress displacement relations are Euler (differential) equations and all boundary conditions are natural (Euler) boundary conditions. We have previously shown in which way the classical theorems of minimum potential energy (minimum principle for displacements) and of minimum complementary energy (maximum principle for stresses) may be considered as *consequences* of the general theorem [4] and also in which way the general theorem may be used for the approximate solution of specific boundary value problems, in Ritz fashion [2].

In what follows, we formulate the general variational theorem for a somewhat larger class of problems than has previously been done. We also derive the corresponding generalization of the classical extremum principles for displacements and for stresses. For these latter principles, and with the restriction of linearity, we obtain a concise and explicit sufficient condition for the existence of an extremum, again for a larger class of boundary conditions and body force distributions than have previously been considered.

Finally, we use the general variational theorem as the starting point for a derivation of a third specialized theorem, involving surface integrals only, for which all boundary conditions and none of the differential equations appear as Euler equations.

It may be added that within the framework of the calculus of variations our general variational theorem is nothing but the canonical form of the problem [5, 6]. Our contribution, from this point of view, is a complete formulation of the theorem, treating all differential equations and all boundary conditions as Euler equations. From the point of view of the theory of elasticity our contribution is the use of the canonical form as the common starting point for the derivation of generalizations of the classical extremum principles, and also to show the possibility of using the direct methods of the calculus of variations in conjunction with the general variational theorem.

§ 2. The boundary value problem

We take as differential equations of the theory of elasticity six stress strain relations of the form

$$\frac{\partial u}{\partial x} = \frac{\partial W}{\partial \sigma_x}, \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial W}{\partial \tau_{xy}}, \text{ etc.} \tag{2.1}$$

and three equilibrium equations of the form

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = \frac{\partial U}{\partial u}, \text{ etc.} \tag{2.2}$$

The nine equations (2.1) and (2.2) are to be solved for six components of stress σ_x, τ_{xy}, etc., and three components of displacement u, v, w, in the interior of a given surface S. The functions W and U are assumed arbitrarily given as

$$W = W(\sigma_x, \tau_{xy}, \cdots), \quad U = U(u, \cdots). \tag{2.3}$$

The system (2.1) and (2.2) is complemented by boundary conditions of the form

$$p + \frac{\partial \psi_s}{\partial u} = 0, \quad q + \frac{\partial \psi_s}{\partial v} = 0, \cdots \text{ on } S_s \tag{2.4}$$

and

$$u + \frac{\partial \psi_d}{\partial p} = 0, \quad v + \frac{\partial \psi_d}{\partial q} = 0, \cdots \text{ on } S_d, \tag{2.5}$$

where S_s and S_d together, without overlapping, represent the surface S, where the functions

$$\psi_s = \psi_s(u, \cdots), \quad \psi_d = \psi_d(p, \cdots) \tag{2.6}$$

are assumed arbitrarily given and where p, q, r stand for the x, y, z-components of the surface tractions,

$$p = \sigma_x \cos(n, x) + \tau_{xy} \cos(n, y) + \tau_{xz} \cos(n, z), \text{ etc.} \tag{2.7}$$

We note the boundary value problem is linear provided $U = U_1 + U_2$, $W = W_1 + W_2$, $\psi_s = \psi_{s1} + \psi_{s2}$, $\psi_d = \psi_{d1} + \psi_{d2}$, where the subscripts 1 and 2 indicate that the functions are homogeneous of the first and second degree, respectively. Ordinary stress and displacement boundary conditions are given provided $\psi_s = \psi_{s1} = -(\bar{p}u + \bar{q}v + \bar{r}w)$ and $\psi_d = \psi_{d1} = = -(\bar{u}p + \bar{v}q + \bar{w}r)$, where $\bar{p}$, $\bar{q}$, $\bar{r}$, $\bar{u}$, $\bar{v}$, $\bar{w}$ are given functions of position on S_s and S_d, respectively.

§ 3. The general variational theorem

Our general variational theorem consists in the statement that the variational equation

$$\delta I = 0, \tag{3.1}$$

where

$$I = \iiint \left[\frac{\partial u}{\partial x}\sigma_x + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\tau_{xy} + \cdots + U - W\right] dV - \iint_d [pu + \cdots + \psi_d]\, dS + \iint_s [\psi_s]\, dS \tag{3.2}$$

and where δu, δv, $\delta\sigma_x$, $\delta\tau_{xy}$, etc. are arbitrary variations has as Euler (differential) equations the system (2.1) and (2.2) and as natural (Euler) boundary conditions the system (2.4) and (2.5). In the proof we make use of the relations $\delta(\partial u/\partial x) = \partial\, \delta u/\partial x$, etc. and also of the integration by parts formula

$$\iiint \left[\frac{\partial \delta u}{\partial x}\sigma_x + \left(\frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x}\right)\tau_{xy} + \cdots\right] dV = -\iiint \left[\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}\right)\delta u + \cdots\right] dV + \iint [p\delta u + q\delta v + r\delta w]\, dS \tag{3.3}$$

which reduce (3.1) to the form

$$0 = \iiint \left[\left(\frac{\partial u}{\partial x} - \frac{\partial W}{\partial \sigma_x}\right)\delta\sigma_x + \cdots - \left(\frac{\partial \sigma_x}{\partial x} + \cdots - \frac{\partial U}{\partial u}\right)\delta u - \cdots\right] dV + \iint_s \left[\left(p + \frac{\partial \psi_s}{\partial u}\right)\delta u + \cdots\right] dS - \iint_d \left[\left(u + \frac{\partial \psi_d}{\partial p}\right)\delta p + \cdots\right] dS. \tag{3.4}$$

We note, for subsequent reference, an alternate form of I, which is ob-

tained by introducing (3.3), without the δ, into equation (3.2),

$$(3.5)\quad I = -\iiint \left[\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}\right) u + \cdots + W - U\right] dV + \\ + \iint_s [pu + \cdots + \psi_s]\, dS + \iint_d [\psi_d]\, dS.$$

§ 4. Comparison states

We now consider, in addition to the *solution state* u, σ_x, etc. which satisfies all differential equations and boundary conditions, *comparison* states $\tilde{u}$, $\tilde{\sigma}_x$, etc. in the form

$$(4.1)\qquad \tilde{u} = u + \delta u, \cdots, \quad \tilde{\sigma}_x = \sigma_x + \delta \sigma_x, \cdots.$$

We then have a quantity $\tilde{I}$ which is obtained from I as given by (3.2) or (3.5) by changing u into $\tilde{u}$, etc. Taylor expansion in powers of all the independent variations leads to the formula

$$(4.2)\qquad \tilde{I} = I + \delta I + \tfrac{1}{2}\delta^2 I + \cdots,$$

where I and δI are, as previously stated, and where the second variation $\delta^2 I$ is homogeneous of the second degree in δu, $\delta \sigma_x$, etc. The linear term in (4.2) vanishes, in accordance with (3.1). Whether or not I has an extremum property in relation to the values of the corresponding $\tilde{I}$ will depend on whether or not $\delta^2 I$ is of one sign only, regardless of the possible values of the variations which occur in it.

From (3.2) follows as expression for $\delta^2 I$,

$$(4.3)\quad \delta^2 I = \iiint \left[2\frac{\partial \delta u}{\partial x}\delta \sigma_x + \cdots + \right. \\ + \frac{\partial}{\partial u}\left(\frac{\partial U}{\partial u}\delta u + \frac{\partial U}{\partial v}\delta v + \cdots\right)\delta u + \cdots - \\ \left. - \frac{\partial}{\partial \sigma_x}\left(\frac{\partial W}{\partial \sigma_x}\delta \sigma_x + \frac{\partial W}{\partial \tau_{xy}}\delta \tau_{xy} + \cdots\right)\delta \sigma_x - \cdots\right] dV - \\ - \iint_d \left[2\delta p \delta u + \cdots + \right. \\ \left. + \frac{\partial}{\partial p}\left(\frac{\partial \psi_d}{\partial p}\delta p + \frac{\partial \psi_d}{\partial q}\delta q + \cdots\right)\delta p + \cdots\right] dS + \\ + \iint_s \left[\frac{\partial}{\partial u}\left(\frac{\partial \psi_s}{\partial u}\delta u + \frac{\partial \psi_s}{\partial v}\delta v + \cdots\right)\delta u + \cdots\right] dS$$

and a corresponding expression follows from (3.5) in the form

$$\delta^2 I = -\iiint \Big[2\Big(\frac{\partial \delta\sigma_x}{\partial x} + \frac{\partial \delta\tau_{xy}}{\partial y} + \cdots\Big)\delta u + \cdots - \tag{4.4}$$

$$- \frac{\partial}{\partial u}\Big(\frac{\partial U}{\partial u}\delta u + \cdots\Big)\delta u - \cdots +$$

$$+ \frac{\partial}{\partial \sigma_x}\Big(\frac{\partial W}{\partial \sigma_x}\delta\sigma_x + \cdots\Big)\delta\sigma_x + \cdots\Big] dV -$$

$$- \iint_d \Big[\frac{\partial}{\partial p}\Big(\frac{\partial \psi_d}{\partial p}\delta p + \cdots\Big)\delta p + \cdots\Big] dS +$$

$$+ \iint_s \Big[2\delta p \delta u + \cdots + \frac{\partial}{\partial u}\Big(\frac{\partial \psi_s}{\partial u}\delta u + \cdots\Big)\delta u + \cdots\Big] dS.$$

§ 5. Minimum principle for displacements

We now assume that all comparison states $\tilde{u}$, $\tilde{\sigma}_x$, etc. satisfy the stress strain relations corresponding to (2.1) and the boundary conditions on S_d corresponding to (2.5). This means that the variations δu, $\delta\sigma_x$, etc. are restricted by six relations

$$\frac{\partial \delta u}{\partial x} = \frac{\partial^2 W}{\partial {\sigma_x}^2}\delta\sigma_x + \frac{\partial^2 W}{\partial\sigma_x \partial\tau_{xy}}\delta\tau_{xy} + \cdots, \text{ etc.} \tag{5.1}$$

in the interior of S and by three relations

$$\delta u + \frac{\partial^2 \psi_d}{\partial p^2}\delta p + \frac{\partial^2 \psi_d}{\partial p \partial q}\delta q + \cdots = 0, \text{ etc.} \tag{5.2}$$

on the portion S_d of the surface S.

Use of these restrictions for $\tilde{u}$, $\tilde{\sigma}_x$, etc. allows a transformation of $\tilde{I}$ and I into a form $\tilde{I}_d$ and I_d for which *only* equilibrium differential equations and boundary conditions on S_d appear as Euler equations. This transformation consists in assuming that the six stress strain relations $\partial\tilde{u}/\partial x = \partial W/\partial\tilde{\sigma}_x$, etc. are solved in the form $\tilde{\sigma}_x = \tilde{\sigma}_x(\partial\tilde{u}/\partial x, \cdots)$, etc. and the result used to write

$$\frac{\partial \tilde{u}}{\partial x}\tilde{\sigma}_x + \cdots - W(\tilde{\sigma}_x, \cdots) = W^*\Big(\frac{\partial\tilde{u}}{\partial x}, \cdots\Big). \tag{5.3}$$

Furthermore, the boundary conditions $\tilde{u} + \partial\psi_d/\partial\tilde{p} = 0$, etc. are con-

sidered solved in the form $\tilde{p} = \tilde{p}(\tilde{u}, \cdots)$, etc. and the result used to write, on S_d,

$$\tilde{p}\tilde{u} + \cdots + \psi_d(\tilde{p}, \cdots) = -\psi_d^*(\tilde{u}, \cdots). \tag{5.4}$$

We then have

$$\tilde{I} = \tilde{I}_d = \iiint \left[U(\tilde{u}, \cdots) + W^*\left(\frac{\partial \tilde{u}}{\partial x}, \cdots\right)\right] dV + \tag{5.5}$$

$$+ \iint_d [\psi_d^*(\tilde{u}, \cdots)] dS + \iint_s [\psi_s(\tilde{u}, \cdots)]\, dS$$

with a corresponding expression for $I = I_d$.

If we now observe that in view of (5.3) and (5.4)

$$\sigma_x = \frac{\partial W^*}{\partial(\partial u/\partial x)}, \quad \tau_{xy} = \frac{\partial W^*}{\partial(\partial u/\partial y + \partial v/\partial x)}, \text{ etc.} \tag{5.6}$$

in the interior of S and

$$p = -\frac{\partial \psi_d^*}{\partial u}, \text{ etc.} \tag{5.7}$$

on S_d, then it is readily seen that the variational equation

$$\delta I_d = 0 \tag{5.8}$$

has as Euler equations the three equilibrium equations (2.2) and the three boundary conditions (2.4) on S_s.

In order to see whether I_d represents an extremum value of $\tilde{I}_d$ we consider the form of $\delta^2 I$ when the admissible variations are restricted in accordance with (5.1) and (5.2). We find from (4.3), upon elimination of $\partial \delta u/\partial x$, etc.

$$\delta^2 I_d = \iiint \left[\frac{\partial}{\partial \sigma_x}\left(\frac{\partial W}{\partial \sigma_x}\delta\sigma_x + \cdots\right)\delta\sigma_x + \cdots + \right. \tag{5.9}$$

$$\left. + \frac{\partial}{\partial u}\left(\frac{\partial U}{\partial u}\delta u + \cdots\right)\delta u + \cdots\right] dV +$$

$$+ \iint_d \left[\frac{\partial}{\partial p}\left(\frac{\partial \psi_d}{\partial p}\delta p + \cdots\right)\delta p + \cdots\right] dS +$$

$$+ \iint_s \left[\frac{\partial}{\partial u}\left(\frac{\partial \psi_s}{\partial u}\delta u + \cdots\right)\delta u + \cdots\right] dS.$$

While we know that for a physically realistic material the contribution due to W in (5.9) is positive definite, this same definiteness is not necessarily given for the contributions of the other three functions and to the extent that it is not we do not have an extremum property for I_d.

Further consideration of $\delta^2 I_d$ will be limited to *linear* boundary value problems for which $U = U_1 + U_2$, etc. We then have

$$\frac{\partial}{\partial \sigma_x}\left(\frac{\partial W}{\partial \sigma_x}\delta\sigma_x + \frac{\partial W}{\partial \tau_{xy}}\delta\tau_{xy} + \cdots\right)\delta\sigma_x + \cdots = 2W_2(\delta\sigma_x, \cdots) \tag{5.10}$$

with corresponding expressions for U, ψ_d and ψ_s. Therewith $\delta^2 I_d$ assumes the form

$$\tfrac{1}{2}\delta^2 I_d = \iiint [W_2(\delta\sigma_x, \cdots) + U_2(\delta u, \cdots)]\, dV + \tag{5.11}$$

$$+ \iint_d [\psi_{d2}(\delta p, \cdots)]\, dS + \iint_s [\psi_{s2}(\delta u, \cdots)]\, dS$$

and we have the result that I_d *is a relative minimum of* $\tilde{I}_d$ *provided the four functions* W_2, U_2, ψ_{d2}, ψ_{s2} *are all positive definite.*

§ 6. Maximum principle for stresses

We now assume that all comparison states satisfy the equilibrium conditions corresponding to (2.2) and the boundary conditions on S_s corresponding to (2.4). This means that the variations δu, $\delta\sigma_x$, etc. are restricted by three relations

$$\frac{\partial \delta\sigma_x}{\partial x} + \cdots = \frac{\partial^2 U}{\partial u^2}\delta u + \frac{\partial^2 U}{\partial u \partial v}\delta v + \cdots, \text{ etc.} \tag{6.1}$$

in the interior of S and by three relations

$$\delta p + \frac{\partial \psi^2{}_s}{\partial u^2}\delta u + \frac{\partial^2 \psi_s}{\partial u \partial v}\delta v + \cdots = 0, \text{ etc.} \tag{6.2}$$

on S_s.

We may again use the assumed restrictions for a transformation of I and $\tilde{I}$ into forms I_s and $\tilde{I}_s$ for which only stress strain relations and boundary conditions on S_d appear as Euler equations. Taking I in the

form (3.5) we shall have, upon using (2.2) and (2.4),

$$(6.3)\qquad I_s = -\iiint \left[u\frac{\partial U}{\partial u} + \cdots - U + W\right] dV + \\ + \iint_d [\psi_d]\, dS - \iint_s \left[u\frac{\partial \psi_s}{\partial u} + \cdots - \psi_s\right] dS.$$

We note that when $U = U_1$ and $\psi_s = \psi_{s1}$, which corresponds to the usual case of body forces and surface stresses on S_s prescribed as functions of the coordinates, the quantities U and ψ_s disappear from (6.3). When in addition $\psi_d = \psi_{d1}$, then $-I_s$ reduces to a quantity which is customarily designated as complementary energy.

It may readily be verified that the Euler equations obtained from $\delta I_s = 0$, together with the restrictions (6.1) and (6.2) on admissible variations are the stress strain relations (2.1) and the boundary conditions (2.5) on S_d.

In order to see whether I_s represents an extremum value of $\tilde{I}_s$ we consider the form which is assumed by $\delta^2 I$ upon introduction of the restrictions (6.1) and (6.2). Taking $\delta^2 I$ in the form (4.4) we obtain

$$(6.4)\qquad \delta^2 I_s = -\iiint \left[\frac{\partial}{\partial u}\left(\frac{\partial U}{\partial u}\delta u + \cdots\right)\delta u + \cdots + \right. \\ \left. + \frac{\partial}{\partial \sigma_x}\left(\frac{\partial W}{\partial \sigma_x}\delta\sigma_x + \cdots\right)\delta\sigma_x + \cdots\right] dV + \\ - \iint_d \left[\frac{\partial}{\partial p}\left(\frac{\partial \psi_d}{\partial p}\delta p + \cdots\right)\delta p + \cdots\right] dS - \\ - \iint_s \left[\frac{\partial}{\partial u}\left(\frac{\partial \psi_s}{\partial u}\delta u + \cdots\right)\delta u + \cdots\right] dS.$$

A comparison of (6.4) and (6.9) reveals the important equality

$$(6.5)\qquad \delta^2 I_s = -\delta^2 I_d$$

Accordingly, I_s is a maximum of $\tilde{I}_s$ under the precisely same conditions that I_d is a minimum of $\tilde{I}_d$. In particular, the maximum property of I_s is assured for those linear problems for which W_2, U_2, ψ_{s2}, ψ_{d2} are positive definite.

§ 7. Upper and lower bounds for I

For those cases for which $I_s = \max(\check{I}_s)$ and $I_d = \min(\check{I}_d)$ we have, since $I_d = I_s = I$, the system of inequalities

$$\check{I}_s \leqslant I \leqslant \check{I}_d. \tag{7.1}$$

For an interpretation of I in terms of quantities of physical interest it is useful to transform (3.2) insofar as one half of the combination $\sigma_x \partial u/\partial x + \cdots$ is concerned. We obtain

$$\begin{aligned} I = \iiint \Big[\frac{1}{2}\Big(\frac{\partial u}{\partial x}\sigma_x + \cdots\Big) - W + U - \\ - \frac{1}{2}\Big(\frac{\partial \sigma_x}{\partial x} + \cdots\Big)u - \cdots \Big] dV + \\ + \iint_s [\tfrac{1}{2}(pu + \cdots) + \psi_s]\, dS - \iint_d [\tfrac{1}{2}(pu + \cdots) + \psi_d]\, dS. \end{aligned} \tag{7.2}$$

Introduction of (2.1), (2.2), (2.4) and (2.5) changes (7.2) into

$$\begin{aligned} I = \iiint \Big[\frac{1}{2}\Big(\frac{\partial W}{\partial \sigma_x}\sigma_x + \cdots\Big) - W \Big] dV - \\ - \iiint \Big[\frac{1}{2}\Big(\frac{\partial U}{\partial u}u + \cdots\Big) - U \Big] dV - \\ - \iint_s \Big[\frac{1}{2}\Big(\frac{\partial \psi_s}{\partial u}u + \cdots\Big) - \psi_s \Big] dS - \\ - \iint_d \Big[\frac{1}{2}\Big(\frac{\partial \psi_d}{\partial p}p + \cdots\Big) - \psi_d \Big] dS. \end{aligned} \tag{7.3}$$

In order to reduce (7.3) further it is necessary to have information concerning the form of W, U, ψ_s and ψ_d. We limit ourselves here again to the case of linear boundary value problems for which $U = U_1 + U_2$, etc. For these assumptions we have that I reduces to the form

$$I = \tfrac{1}{2}\iiint [U_1 - W_1]\, dV + \tfrac{1}{2}\iint_s \psi_{s1}\, dS - \tfrac{1}{2}\iint_d \psi_{d1}\, dS. \tag{7.4}$$

Various special cases of this may be used to obtain bounds on influence coefficients for surface deformations due to surface tractions, or on average displacements due to uniform body force distributions, or on the space average of the mean normal stress produced by a uniform temperature change.

§ 8. A variational theorem for boundary values

We now wish to formulate a variational theorem in which the only Euler equations are the boundary conditions of the problem. This means that we assume that all comparison states $\tilde{u}$, $\tilde{\sigma}_x$, etc. satisfy stress strain relations as well as equilibrium differential equations in the interior of S and that the variations δu, $\delta\sigma_x$, etc. are restricted by the six relations (5.1) as well as by the three relations (6.1). A particularly concise theorem of this nature may be obtained if we assume that all nine differential equations for u, σ_x, etc. are linear and homogeneous and we shall therefore limit ourselves here to this assumption which is equivalent to the statement that U as well as W are homogeneous of degree two and which implies the relations

$$u\frac{\partial U}{\partial u} + \cdots = 2U, \quad \sigma_x\frac{\partial W}{\partial \sigma_x} + \cdots = 2W. \tag{8.1}$$

To take advantage of (8.1) we write I in the form

$$\begin{aligned} I = \iiint \Big[\tfrac{1}{2}\sigma_x\frac{\partial u}{\partial x} + \cdots + U - W - \\ - \tfrac{1}{2}\Big(\frac{\partial \sigma_x}{\partial x} + \cdots\Big)u - \cdots\Big] dV + \\ + \tfrac{1}{2}\iint [pu + \cdots]\, dS - \iint_d [pu + \cdots + \psi_d]\, dS + \iint_s \psi_s\, dS. \end{aligned} \tag{8.2}$$

Introduction of the differential equations (2.1) and (2.2) and observation of (8.1) changes I into an expression I_b of the following form

$$I_b = \iint_s [\tfrac{1}{2}pu + \cdots + \psi_s]\, dS - \iint_d [\tfrac{1}{2}pu + \cdots + \psi_d]\, dS. \tag{8.3}$$

It remains to show that the boundary conditions (2.4) and (2.5) are the Euler equations of the variational equation $\delta I_b = 0$. This does not at first seem to be the case, since

$$\begin{aligned} \delta I_b = \iint_s \Big[\tfrac{1}{2}(p\delta u + u\delta p + \cdots + \frac{\partial \psi_s}{\partial u}\delta u + \cdots\Big] dS - \\ - \iint_d \Big[\tfrac{1}{2}(p\delta u + u\delta p + \cdots + \frac{\partial \psi_d}{\partial p}\delta p + \cdots\Big] dS \end{aligned} \tag{8.4}$$

does not contain the terms $(p + \partial\psi_s/\partial u)\delta u$, $(u + \partial\psi_d/\partial p)\delta p$, etc. which should occur in the developed variational equation. However, we can show that the variations δu, δp, etc. on S are not entirely independent but satisfy the relation

$$\iint [u\delta p + \cdots]\, dS = \iint [p\delta u + \cdots]\, dS. \tag{8.5}$$

Introduction of (8.5) into (8.4) transforms the variational equation $\delta I_b = 0$ into

$$\iint_s \left[\left(p + \frac{\partial\psi_s}{\partial u}\right)\delta u + \cdots\right] dS - \iint_d \left[\left(u + \frac{\partial\psi_d}{\partial p}\right)\delta p + \cdots\right] dS = 0 \tag{8.6}$$

which is in fact of the desired form.

In order to prove (8.5) we apply the fundamental integration-by-parts formula (3.3) to both sides of (8.5) and take again account of (2.1), (2.2) and of the degree of homogeneity of U and W. In this way we obtain

$$\begin{aligned} \iint [u\delta p + \cdots]\, dS = \\ = \iiint \left[\frac{\partial u}{\partial x}\delta\sigma_x + \cdots + \left(\frac{\partial\delta\sigma_x}{\partial x} + \cdots\right)u + \cdots\right] dV = \\ = \iiint \left[\frac{\partial W}{\partial\sigma_x}\delta\sigma_x + \cdots + \left(\frac{\partial^2 U}{\partial u^2}\delta u + \cdots\right)u + \cdots\right] dV = \\ = \iiint \left[\frac{\partial W}{\partial\sigma_x}\delta\sigma_x + \cdots + \frac{\partial U}{\partial u}\delta u + \cdots\right] dV \end{aligned} \tag{8.7}$$

and

$$\begin{aligned} \iint [p\delta u + \cdots]\, dS = \\ = \iiint \left[\sigma_x \frac{\partial\delta u}{\partial x} + \cdots + \left(\frac{\partial\sigma_x}{\partial x} + \cdots\right)\delta u + \cdots\right] dV = \\ = \iiint \left[\sigma_x \left(\frac{\partial^2 W}{\partial\sigma_x{}^2}\delta\sigma_x + \cdots\right) + \cdots + \frac{\partial U}{\partial u}\delta u + \cdots\right] dV = \\ = \iiint \left[\frac{\partial W}{\partial\sigma_x}\delta\sigma_x + \cdots + \frac{\partial U}{\partial u}\delta u + \cdots\right] dV, \end{aligned} \tag{8.8}$$

which proves (8.5).

To illustrate the possible use of the theorem $\delta I_b = 0$ assume that we know a system u_k, σ_{xk}, etc. of particular solutions of the nine homogeneous linear differential equations of the problem and consider the series solutions $\sum c_k u_k$, etc., where the c_k are undetermined coefficients. Introduce $u = \sum c_k u_k$, $p = \sum c_k p_k$, etc. into equation (8.3) for I_b. The variational equation $\delta I_b = 0$ then reduces to n simultaneous equations for the n coefficients c_k, of the form $\partial I_b / \partial c_k = 0$.

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THE STRESS DISTRIBUTION IN A THIN SHELL WITH AN ARBITRARY HOLE

G. N. SAVIN
Kiev, U.S.S.R.

§ 1. Formulation of the problem. The basic equations and the boundary conditions

The problem of the stress distribution around holes in shells has been the subject of a comparatively small number of papers. Among them one has the theoretical studies of A. I. Lur'e [3], Iu. A. Shevliakov [4–8], I. M. Pirogov [9–11], A. V. Burlakov [12], N. G. Vasilenko [13], M. I. Dlugach [16], G. A. Van-Fo-Fy and G. N. Savin [18–19], D. Withum [20] and the experimental investigations of A. A. Flerova [14], S. N. Galkin, V. V. Kabanov and S. S. Leshenko [15].

An overwhelming number of the above theoretical studies are further developments of the known problem of A. I. Lur'e [3] and concern the investigation of the state of stress around circular holes in cylindrical shells.

The case of the spherical shell with a circular opening has been considered in the paper of Iu. A. Shevliakov, and with an elliptical hole in the papers of G. A. Van-Fo-Fy and the author [18–19].

As far as the author knows, there do not exist studies of stress concentrations in shells around holes of other shapes [1]. The present paper considers the state of stress in a thin shell of constant thickness h which has an arbitrary, generally speaking, not very small opening, the contour of which is represented by the space curve Γ without any angular points.

We will refer the middle surface of the shell under consideration to a certain isometric, generally speaking, non-conjugate system of curvilinear coordinates α and β (Fig. 1). The equation of the middle surface of the shell in these coordinates may be written in vector form

$$\mathbf{R}(\alpha, \beta) = \mathbf{i}_x f_1(\alpha, \beta) + \mathbf{i}_y f_2(\alpha, \beta) + \mathbf{i}_z f_3(\alpha, \beta), \tag{1.1}$$

[1] In close connection with this problem one has the interesting studies of I. N. Vekua [21–22], where the main attention is focussed on the formulation, for convex shells with any numbers of holes, of the conditions which will guarantee a membrane state of stress in the shell.

where $f_1(\alpha, \beta) = x$, $f_2(\alpha, \beta) = y$, $f_3(\alpha, \beta) = z$ are given functions of the parameters α and β.

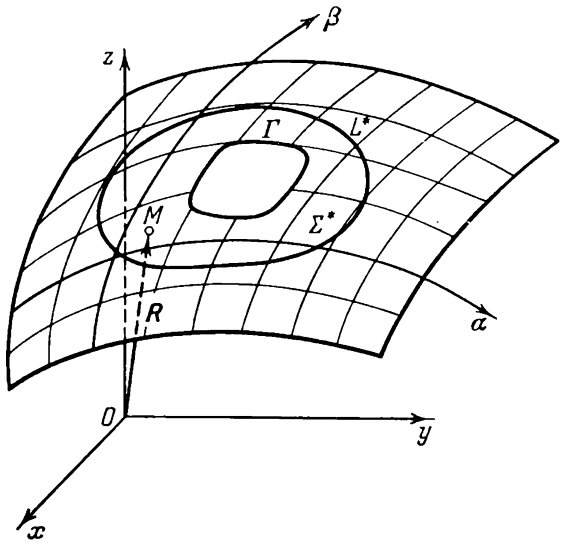

Fig. 1.

In the system of coordinates α and β the first and second quadratic forms are known to have the form

$$I = ds^2 = \lambda^2(\alpha, \beta)(d\alpha^2 + d\beta^2), \tag{1.2}$$

$$II = -(d\mathbf{R}, d\mathbf{n}) = D\, d\alpha^2 + 2D^*\, d\alpha\, d\beta + D^{**}\, d\beta^2. \tag{1.3}$$

The coefficients λ^2, D, D^* and D^{**} in these forms are determined by the given equation of the middle surface of the shell (1.1) in accordance with the formulae

$$\lambda^2(\alpha, \beta) = \left(\frac{\partial \mathbf{R}}{\partial \alpha}, \frac{\partial \mathbf{R}}{\partial \alpha}\right) = \left(\frac{\partial \mathbf{R}}{\partial \beta}, \frac{\partial \mathbf{R}}{\partial \beta}\right), \tag{1.4}$$

$$D = \left(\mathbf{n}, \frac{\partial^2 \mathbf{R}}{\partial \alpha^2}\right), \quad D^* = \left(\mathbf{n}, \frac{\partial^2 \mathbf{R}}{\partial \alpha \partial \beta}\right), \quad D^{**} = \left(\mathbf{n}, \frac{\partial^2 \mathbf{R}}{\partial \beta^2}\right), \tag{1.5}$$

where $\mathbf{n} = \mathbf{n}(\alpha, \beta)$ is the unit normal to the surface (1.1) (cf. Fig. 2).

We will now introduce the notation

$$\begin{aligned} &T_\alpha^*,\ T_\beta^*,\ S_\alpha^*,\ S_\beta^* \ - \text{for the tangential forces,} \\ &Q_\alpha^*,\ Q_\beta^* \ - \text{for the shear forces,} \\ &H_\alpha^*,\ H_\beta^* \ - \text{for the twisting moments,} \\ &G_\alpha^*,\ G_\beta^* \ - \text{for the bending moments,} \end{aligned} \tag{1.6}$$

characterizing the state of stress in the shell with the hole under consideration, subject to the action of a given system of external forces for prescribed boundary conditions.

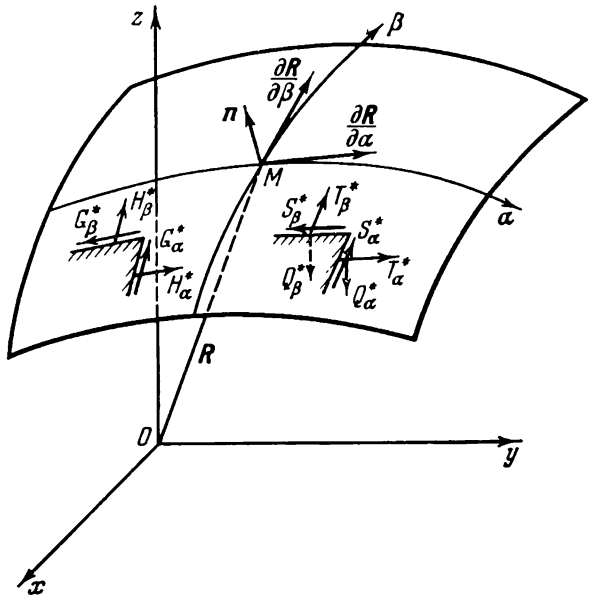

Fig. 2.

The positive directions of the components (1.6) are shown in Fig. 2. The moments H_α^*, H_β^*, G_α^*, G_β^* are shown there by vectors where, if one looks in the positive direction of the vector, the corresponding moment tends to produce a rotation in the clockwise direction.

We will call basic the state of stress of the same, but complete shell (without hole) under the influence of the same external forces and subject to the same boundary conditions and we will refer to it in the following manner:

$$\begin{aligned} &T_\alpha^0,\ T_\beta^0,\ S_\alpha^0,\ S_\beta^0,\ Q_\alpha^0, \\ &Q_\beta^0,\ H_\alpha^0,\ H_\beta^0,\ G_\alpha^0,\ G_\beta^0. \end{aligned} \tag{1.7}$$

We will represent the components $T_\alpha^*, T_\beta^*, \cdots, G_\beta^*$ of (1.6) in the shell with a hole in the form

$$\begin{aligned} &T_\alpha^* = T_\alpha^0 + T_\alpha,\ T_\beta^* = T_\beta^0 + T_\beta,\ S_\alpha^* = S_\alpha^0 + S_\alpha, \\ &S_\beta^* = S_\beta^0 + S_\beta,\ G_\alpha^* = G_\alpha^0 + G_\alpha,\ G_\beta^* = G_\beta^0 + G_\beta, \\ &H_\alpha^* = H_\alpha^0 + H_\alpha,\ H_\beta^* = H_\beta^0 + H_\beta,\ Q_\alpha^* = Q_\alpha^0 + Q_\alpha, \\ &Q_\beta^* = Q_\beta^0 + Q_\beta. \end{aligned} \tag{1.8}$$

It is seen from (1.8) that the components

$$T_\alpha,\ T_\beta,\ S_\alpha,\ S_\beta,\ Q_\alpha,$$
$$Q_\beta,\ H_\alpha,\ H_\beta,\ G_\alpha,\ G_\beta \tag{1.9}$$

represent the additional components of the elastic forces and moments in the shell, caused by the presence of the hole in it and characterizing strictly the "stress concentration" in the shell around the hole under consideration.

Experimental studies of the state of stress in shells around holes with sufficiently smooth contours show that the perturbation in the state of stress around the hole in the shell has a local character, extending over a comparatively small zone Σ^* around the hole, bounded by the contour L^* (Fig. 1). These perturbations rapidly die out with the distance from the hole.

As regards the magnitude of the decay and the zone of the perturbed

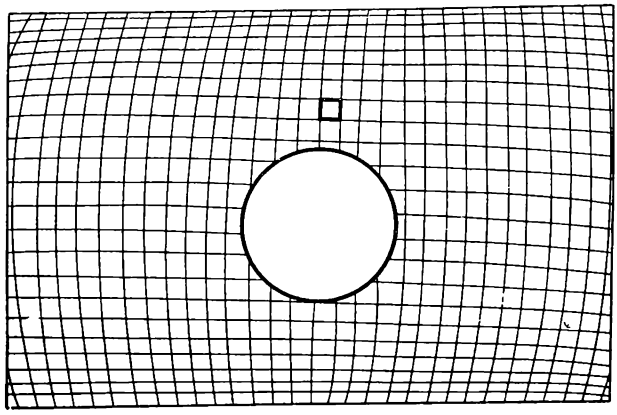

Fig. 3a.

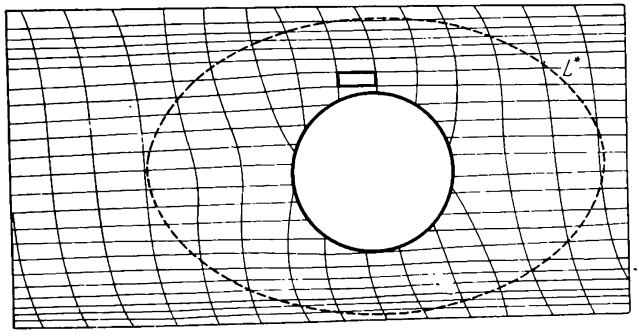

Fig. 3b.

state of stress around a circular disc of thickness $\delta = 3$ mm. and of diameter $d = 15$ mm., glued to a cylindrical shell (of thickness $h = 1$ mm. and of diameter $D = 40$ mm.) and imitating a circular hole in a shell with an elastic ring, subject to the simultaneous action of internal pressure $q_0 = 0.25$ atm. and to an axial extension of the cylindrical shell for $\varepsilon = 0$ and $\varepsilon = 0.5$, this may be judged from the photographs, produced in the Figs. 3a and 3b respectively.

Figure 4 shows photographs of the deformed state of the shell for simultaneous action of internal pressure $q_0 = 0.25$ atm. for extension along the axis of the same cylinder of diameter $D = 40$ mm. with a disc of thickness $\delta = 1$ mm. and of diameter $d = 7$ mm. glued to it for relative extensions $\varepsilon = 0$ (Fig. 4a) and $\varepsilon = 0.5$ (Fig. 4b). For convenient observation these shells were inscribed in their undeformed state with

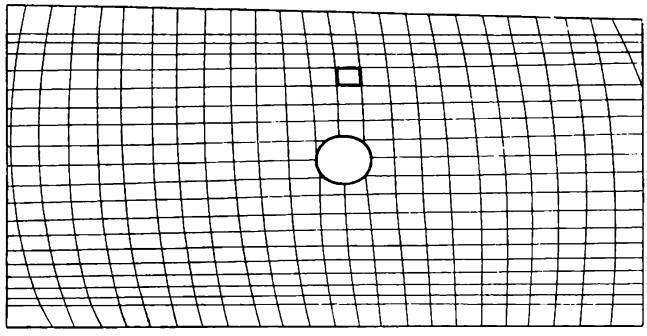

Fig. 4a.

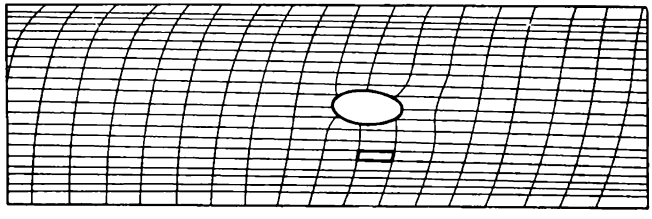

Fig. 4b.

a square net of lines $\alpha = \text{const.}$ and $\beta = \text{const.}$[1] Comparison of the evidence of Figs. 3a and 3b and likewise of Figs. 4a and 4b clearly shows the local character of the perturbation caused by the attached elastic disc, imitating a circular hole with an elastic ring; further, these results

[1] For convenience of comparison figures 3a, b and 4a, b had inscribed a black square which for the deformation of the shell became a rectangle, permitting visual assessment of the degree of deformation of the shell around the hole.

give the possibility of indicating (with some degree of accuracy) the dimension of the zone of perturbation. As is seen from Fig. 3b, the perturbation of the deformation extends from the contour of the disc along the meridian to a distance not exceeding the diameter d of the disc, and along the generators only by a quantity not exceeding $0.5d$. It follows from this that, if one produces on the surface of the considered cylinder around the disc a contour L^* (Fig. 3) having the form of an ellipse with certain semi-axes a and b ($a \approx 1.5d$; $b \approx d$), then the entire perturbed zone will be contained inside the contour L^*.

We obtain the same impression of a local character in the case of a spherical rubber shell of diameter $D = 250$ mm. and of thickness $h = 1$ mm. around a glued on circular rubber disc of thickness $\delta = 3$ mm. and of diameter $d = 25$ mm. for internal pressure $q \approx 0.1$ atm. (Fig. 5). In this case the zone of perturbation Σ^* is bounded by the circle L^* of radius $r = 1.5d$.

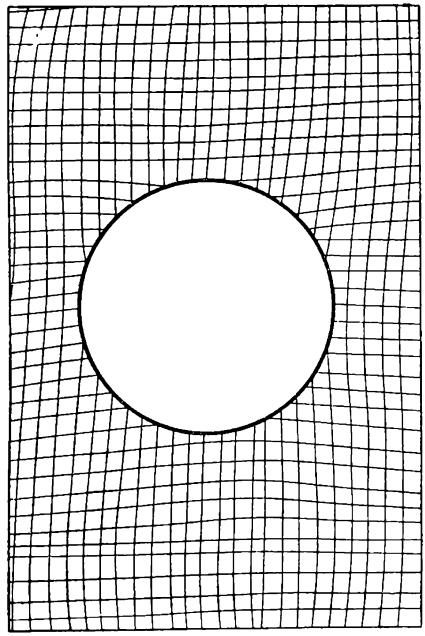

Fig. 5.

The numerous experiments aimed at the study of the state of stress around circular and elliptic holes in spherical and cylindrical shells, performed by V. G. Bessonov in the Laboratory for Model Work of the Institute of Mechanics of the Academy of Sciences of the Ukrainian Soviet Republic, have led to the conclusion that in the general case of a

shell with a hole bounded by a sufficiently smooth contour one may indicate such a zone, having the local character of a perturbed state of stress Σ^*, bounded by a contour L^* (Fig. 1), on which, and the more so outside which, the components $T_\alpha, \cdots, G_\beta$ of (1.9) vanish for all practical purposes and where the components $T_\alpha{}^*, \cdots, G_\beta{}^*$ of (1.6) and $T_\alpha{}^0, \cdots, G_\beta{}^0$ of (1.7) will coincide.[1]

This shows that the components $T_\alpha, \cdots, G_\beta$ of (1.9), characterizing essentially the "stress concentration" in the shell around the hole under consideration, represent rapidly decaying functions of the coordinates and that for their determination one may apply the approximate theory of states of stress with large index of variation which coincides with the theory of shallow shells [1–2].

Before we proceed to the establishment of the basic equations and boundary conditions for $T_\alpha, \cdots, G_\beta$ of (1.9) we note that for shells of revolution and, in particular, for the spherical shells considered by us above, the coefficient $\lambda^2(\alpha, \beta)$ in (1.2) may be represented in the form

$$\lambda^2(\alpha, \beta) = \lambda_0^2[1 + \varepsilon(\alpha, \beta)], \tag{1.10}$$

where $\lambda_0^2 = \text{const}$.

If then in the region of the disturbed state Σ^* the quantity $\varepsilon(\alpha, \beta)$ turns out to be a sufficiently small quantity, namely, such that it may be neglected in comparison with unity, then the expression (1.2) for ds^2 in the region Σ^* may be taken in the form

$$ds^2 = \lambda_0^2(d\alpha^2 + d\beta^2). \tag{1.11}$$

The replacement of ds^2 in the region Σ^* by the expression (1.11), which in the case of cylindrical shells is an exact expression, corresponds to the replacement of the Non-Euclidean metric of the middle surface of the shell by the Euclidean metric of the plane.

The theory of shells in which such a replacement of the form (1.2) by the form (1.11) has been introduced is called the theory of shallow shells ([2], Part III, § 2, pp. 301–315) and it has great value for engineering applications.

In what follows we will assume that in the region Σ^*, bounded by the contour L^*, the quantity $\varepsilon(\alpha, \beta)$ in (1.10) is small and that it may be neglected in comparison with unity; for ds^2 we will take the expression (1.11).

For the sake of simplification of the manipulations and reasoning it will be convenient to change from the old system of coordinates (α, β) to an isometric system (ρ, θ), one of the coordinate lines $\rho = \rho_0$ of

[1] Of course, with a given degree of accuracy.

which is to coincide with the contour Γ of the hole in the shell.

We obtain the vector equation of the contour Γ by replacing α and β in (1.1) by the new variables ρ and θ and by setting $\rho = \rho_0$

$$\mathbf{R}(\rho_0, \theta) = \mathbf{i}_x\varphi_1(\rho_0, \theta) + \mathbf{i}_y\varphi_2(\rho_0, \theta) + \mathbf{i}_z\varphi_3(\rho_0, \theta).$$

In the plane P of the variables α and β the contours Γ and L^* of the surface will correspond to contours γ and l (Fig. 6). We will consider

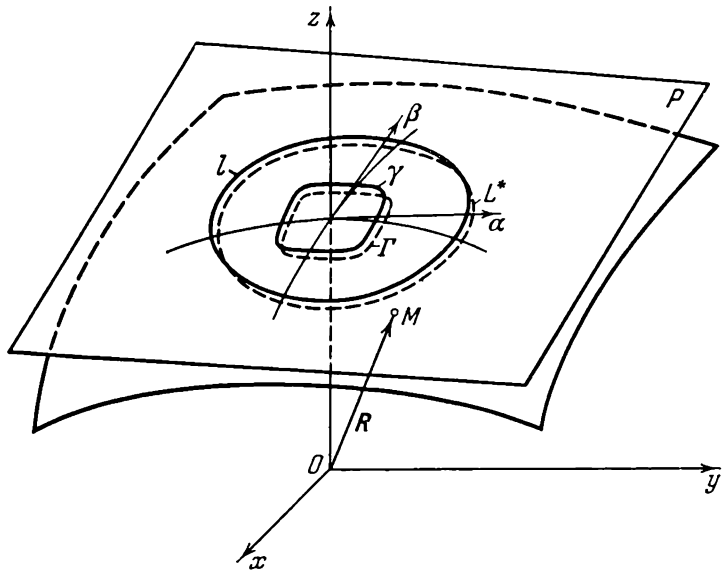

Fig. 6.

shapes of holes in shells bounded by sufficiently smooth contours Γ which in the plane P correspond to contours γ without corners, the parametric equations of which may be given by the analytic functions

$$\alpha + i\beta = \omega(\zeta) = a_1\zeta + \frac{a_{-1}}{\zeta} + \frac{a_{-2}}{\zeta^2} + \cdots + \frac{a_{-n}}{\zeta^n}, \tag{1.12}$$

where $\zeta = \exp(\rho_0 + i\theta)$ and $\rho_0, a_1, a_{-1}, \cdots, a_{-n}$ are constants. In other words, we will assume that the analytic function (1.12) carries out the above coordinate transformation.

In what follows we will classify the hole in the shell by the shape of the curve γ in the plane P.

Separating real and imaginary parts in (1.12), setting $\zeta = \exp(\rho + i\theta)$

and assuming that the coefficients $a_1, a_{-1}, a_{-2}, \cdots, a_{-n}$ are real we obtain the relations

$$\begin{aligned}\alpha &= \left(a_1 e^{\rho} + \frac{a_{-1}}{e^{\rho}}\right)\cos\theta + \frac{a_{-2}}{e^{2\rho}}\cos 2\theta + \cdots + \frac{a_{-n}}{e^{n\rho}}\cos n\theta,\\ \beta &= \left(a_1 e^{\rho} - \frac{a_{-1}}{e^{\rho}}\right)\sin\theta - \frac{a_{-2}}{e^{2\rho}}\sin 2\theta - \cdots - \frac{a_{-n}}{e^{n\rho}}\sin n\theta,\end{aligned} \tag{1.13}$$

relating the coordinates α and β to ρ and θ. We obtain the equation of the contour γ of the hole in the plane of the variables α, β by setting $\rho = \rho_0$ in (1.13) (in particular, $\rho = 0$). It is readily seen from (1.12) that the first quadratic form ds^2 of (1.11) in terms of the new coordinates ρ and θ will have the form

$$ds^2 = \mathcal{H}^2(\rho, \theta)[d\rho^2 + d\theta^2], \tag{1.14}$$

where

$$\mathcal{H}^2(\rho, \theta) = \lambda_0^2 e^{2\rho}[A^2(\rho, \theta) + B^2(\rho, \theta)], \tag{1.15}$$

$$\begin{aligned}A(\rho, \theta) &= a_1 - \frac{a_{-1}}{e^{2\rho}}\cos 2\theta - \frac{2a_{-2}}{e^{3\rho}}\cos 3\theta - \cdots -\\ &\quad - \frac{na_{-n}}{e^{(n+1)\rho}}\cos(n+1)\theta,\\ B(\rho, \theta) &= \frac{a_{-1}}{e^{2\rho}}\sin 2\theta + \frac{2a_{-2}}{e^{3\rho}}\sin 3\theta + \cdots +\\ &\quad + \frac{na_{-n}}{e^{(n+1)\rho}}\sin(n+1)\theta.\end{aligned} \tag{1.16}$$

Thus, depending on the form of the given hole in the shell, one can execute with the help of $\omega(\zeta)$ the transition to the new system of coordinates ρ and θ. The contour l^* in the plane P, corresponding to L^*, may be replaced by the coordinate line $\rho = \rho_1$, lying closest to it.

In order to find the components $T_\alpha, \cdots, G_\beta$ of (1.9) which we will denote in terms of the coordinates ρ and θ by

$$T_\rho, \cdots, G_\theta, \tag{1.17}$$

we will apply the approximate theory of states of stress with large index of variation (cf., [1], equations (20.8)–(20.9), p. 361).

Since each of the states of stress $T_\rho^*, \cdots, G_\theta^*$ of (1.6) and $T_\rho^0, \cdots, G_\theta^0$ of (1.7) satisfy the same equations of the theory of shells (cf. [1], pp. 86–

88) for one and the same external forces and boundary conditions, we obtain for the determination of the components $T_\rho, \cdots, G_\theta$, as is readily seen from (1.8), a homogeneous system of equations which we will present in the form

$$\begin{aligned} &\frac{Eh^3}{12(1-\nu^2)}\nabla^2\nabla^2 w - \nabla_k^2\varphi = 0, \\ &\frac{1}{Eh}\nabla^2\nabla^2\varphi + \nabla_k^2 w = 0, \end{aligned} \tag{1.18}$$

where E is Young's modulus, ν is Poisson's ratio and h is the thickness of the shell. The operators ∇^2 and ∇_k^2 have in the isometric coordinates ρ and θ the form

$$\nabla^2 = \frac{1}{\mathscr{H}^2}\left[\frac{\partial^2}{\partial\rho^2} + \frac{\partial^2}{\partial\theta^2}\right],$$

$$\nabla_k^2 = \frac{1}{\mathscr{H}^2}\left[\frac{\partial}{\partial\rho}\left(\frac{1}{R_\theta}\frac{\partial}{\partial\rho}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{R_{\rho\theta}}\frac{\partial}{\partial\rho}\right) + \right. \tag{1.19}$$

$$\left. + \frac{\partial}{\partial\rho}\left(\frac{1}{R_{\rho\theta}}\frac{\partial}{\partial\theta}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{R_\rho}\frac{\partial}{\partial\theta}\right)\right],$$

where $\mathscr{H}^2(\rho, \theta)$ is the coefficient of the form (1.14), being determined by (1.15).

The quantities

$$\frac{1}{R_\rho} = -\frac{D}{\mathscr{H}^2}, \quad \frac{1}{R_\theta} = -\frac{D^{**}}{\mathscr{H}^2} \tag{1.20}$$

represent the curvatures of normal sections of the middle surface of the shell, cut along the coordinate lines $\rho = \text{const.}$ and $\theta = \text{const.}$ respectively. The quantity

$$\frac{1}{R_{\rho\theta}} = \frac{D^*}{\mathscr{H}^2} \tag{1.21}$$

characterizes the degree of nonconjugacy of the isometric coordinates ρ and θ. Here D, D^* and D^{**} are the coefficients of the form (1.3), expressed in terms of the isometric coordinates ρ and θ. The deflection of the middle surface of the shell $w(\rho, \theta)$ is assumed to be positive for deflection of the shell in the direction of the negative normal. The components of the bending deformation κ_1, κ_2, $\tau^{(1)}$ and $\tau^{(2)}$ are determined

in terms of the deflection $w(\rho, \theta)$ by the formulae

$$\begin{aligned}
\kappa_1 &= \frac{1}{\mathscr{H}} \frac{\partial}{\partial \rho}\left(\frac{1}{\mathscr{H}} \frac{\partial w}{\partial \rho}\right) + \frac{1}{\mathscr{H}^3} \frac{\partial \mathscr{H}}{\partial \theta} \frac{\partial w}{\partial \theta}, \\
\kappa_2 &= \frac{1}{\mathscr{H}} \frac{\partial}{\partial \theta}\left(\frac{1}{\mathscr{H}} \frac{\partial w}{\partial \theta}\right) + \frac{1}{\mathscr{H}^3} \frac{\partial \mathscr{H}}{\partial \rho} \frac{\partial w}{\partial \rho}, \\
\tau^{(1)} &= \frac{1}{\mathscr{H}} \frac{\partial}{\partial \rho}\left(\frac{1}{\mathscr{H}} \frac{\partial w}{\partial \theta}\right) - \frac{1}{\mathscr{H}^3} \frac{\partial \mathscr{H}}{\partial \theta} \frac{\partial w}{\partial \rho}, \\
\tau^{(2)} &= -\frac{1}{\mathscr{H}} \frac{\partial}{\partial \theta}\left(\frac{1}{\mathscr{H}} \frac{\partial w}{\partial \rho}\right) + \frac{1}{\mathscr{H}^3} \frac{\partial \mathscr{H}}{\partial \rho} \frac{\partial w}{\partial \theta},
\end{aligned} \tag{1.22}$$

The moments G_ρ, G_θ, H_ρ and H_θ are determined in terms of the components κ_1, κ_2, $\tau^{(1)}$ and $\tau^{(2)}$ by the formulae

$$\begin{aligned}
G_\rho &= -\frac{Eh^3}{12(1-\nu^2)}(\kappa_1 + \nu\kappa_2), \quad G_\theta = -\frac{Eh^3}{12(1-\nu^2)}(\kappa_2 + \nu\kappa_1) \\
H_\rho &= \frac{Eh^3}{12(1-\nu^2)}(\tau^{(1)} + \nu\tau^{(2)}), \quad H_\theta = \frac{Eh^3}{12(1-\nu^2)}(\tau^{(2)} + \nu\tau^{(1)}).
\end{aligned} \tag{1.23}$$

The components T_ρ, T_θ, S_ρ and S_θ are determined by the stress function $\varphi(\rho, \theta)$ in accordance with the formulae

$$\begin{aligned}
T_\rho &= \frac{1}{\mathscr{H}} \frac{\partial}{\partial \theta}\left(\frac{1}{\mathscr{H}} \frac{\partial \varphi}{\partial \theta}\right) + \frac{1}{\mathscr{H}^3} \frac{\partial \mathscr{H}}{\partial \rho} \frac{\partial \varphi}{\partial \rho}, \\
T_\theta &= \frac{1}{\mathscr{H}} \frac{\partial}{\partial \rho}\left(\frac{1}{\mathscr{H}} \frac{\partial \varphi}{\partial \rho}\right) + \frac{1}{\mathscr{H}^3} \frac{\partial \mathscr{H}}{\partial \theta} \frac{\partial \varphi}{\partial \theta}, \\
S_\rho &= -\frac{1}{\mathscr{H}} \frac{\partial}{\partial \theta}\left(\frac{1}{\mathscr{H}} \frac{\partial \varphi}{\partial \rho}\right) + \frac{1}{\mathscr{H}^3} \frac{\partial \mathscr{H}}{\partial \rho} \frac{\partial \varphi}{\partial \theta}, \\
S_\theta &= \frac{1}{\mathscr{H}} \frac{\partial}{\partial \rho}\left(\frac{1}{\mathscr{H}} \frac{\partial \varphi}{\partial \theta}\right) - \frac{1}{\mathscr{H}^3} \frac{\partial \mathscr{H}}{\partial \theta} \frac{\partial \varphi}{\partial \rho}.
\end{aligned} \tag{1.24}$$

The shear resultants Q_ρ and Q_θ are determined in terms of H_ρ, H_θ, G_ρ and G_θ by the formulae

$$\begin{aligned}
Q_\rho &= \frac{1}{\mathscr{H}^2}\left[\frac{\partial}{\partial \rho}(\mathscr{H} G_\rho) - \frac{\partial \mathscr{H}}{\partial \theta} H_\rho + \frac{\partial}{\partial \theta}(\mathscr{H} H_\theta) - \frac{\partial \mathscr{H}}{\partial \rho} G_\theta\right], \\
Q_\theta &= -\frac{1}{\mathscr{H}^2}\left[\frac{\partial}{\partial \rho}(\mathscr{H} H_\rho) + \frac{\partial \mathscr{H}}{\partial \theta} G_\rho - \frac{\partial}{\partial \theta}(\mathscr{H} G_\theta) - \frac{\partial \mathscr{H}}{\partial \rho} H_\theta\right].
\end{aligned} \tag{1.25}$$

One may reduce the system of equations (1.18) by introduction of the complex function

$$\Phi(\rho, \theta) = \frac{Eh^2}{\sqrt{12(1-\nu^2)}} w(\rho, \theta) + i\varphi(\rho, \theta) \tag{1.26}$$

to the equation

$$\nabla^2\nabla^2\Phi + i\frac{\sqrt{12(1-\nu^2)}}{h}\nabla_k^2\Phi = 0. \tag{1.27}$$

This result is easily verified by direct calculations. The equations (1.18) or the equation (1.27) equivalent to them, as has already been mentioned above, are the basic equations of the theory of shallow shells (cf. [1], p. 368; [2], pp. 303–310). Thus the solution of the problem posed is reduced to finding the deflection function $w(\rho, \theta)$ and the stress function $\varphi(\rho, \theta)$ from the system of equations (1.18) or from the single equation (1.27). In order to eliminate the arbitrariness admitted by the general integrals of the stated equations, we must impose certain conditions on the unknown functions w and φ. In our case these conditions will be boundary conditions on the contour of the hole and so called conditions "at infinity", i.e., conditions in regions which are sufficiently far away from the hole of the shell outside the region Σ^*, bounded by the contour L^*.

On the edge of the hole Γ, i.e., for $\rho = \rho_0$, one may be given different conditions: either external forces, or displacements of points of the contour or, on one part of the edge external forces and on the remaining part of the contour displacements, etc. Of all the possible forms of boundary conditions we will consider only the first case, when there are given at the edge of the hole the external forces $F_1(\rho_0, \theta)$, $F_2(\rho_0, \theta)$, $F_3(\rho_{,0}\, \theta)$ and $F_4(\rho_0, \theta)$. In this case the external forces, applied to the contour of the hole Γ, must be equal to the acting resultants $\tilde{T}$, $\tilde{S}$, $\tilde{Q}$ and $\tilde{G}$ (cf. [1], p. 102), i.e.,

$$\begin{aligned} \tilde{T} &= F_1(\rho_0, \theta), \quad \tilde{S} = F_2(\rho_0, \theta), \\ \tilde{Q} &= F_3(\rho_0, \theta), \quad \tilde{G} = F_4(\rho_0, \theta), \end{aligned} \tag{1.28}$$

where

$$\tilde{T} = T_\rho^* + \frac{\sin 2\chi}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)H_\rho^*, \quad \tilde{G} = G_\rho^*,$$

$$\tilde{S} = S_\rho^* + \frac{H_\rho^*}{R_\Gamma}, \quad \tilde{Q} = Q_\rho^* - \frac{\partial H_\rho^*}{\partial S_\Gamma},$$

χ is the angle between the contour Γ of the hole and the line $1/R_1$ on the

contour of the middle surface, R_Γ is the radius of curvature of the normal section of the middle surface along the contour Γ. The derivative $\partial H_\rho^*/\partial S_\Gamma$ in $\tilde{Q}$ is taken along the arc S_Γ of the contour of the hole Γ.

For an approximate solution of the problem posed the formulae for $\tilde{T}$ and $\tilde{S}$ may be simplified considerably; in fact, there one may neglect terms of order $1/R$ and, by taking into consideration (1.8), represent the boundary conditions (1.28) in the form

$$T_\rho = F_1(\rho_0, \theta) - T_\rho^0, \quad S_\rho = F_2(\rho_0, \theta) - S_\rho^0,$$

$$Q_\rho - \frac{\partial H_\rho}{\partial S_\Gamma} = F_4(\rho_0, \theta) - Q_\rho^0 + \frac{\partial H_\rho^0}{\partial S_\Gamma}, \quad G_\rho = F_3(\rho_0, \theta) - G_\rho^0, \tag{1.29}$$

where the derivative $\partial H_\rho^0/\partial S_\Gamma$ is taken along the arc of the contour Γ of the hole P. Often the basic state of stress (1.7) is a membrane state; then, setting in (1.29) $G_\rho^{(0)} = H_\rho^{(0)} = Q_\rho^0 = 0$, we obtain the boundary conditions for this case in the form

$$T_\rho = F_1(\rho_0, \theta) - T_\rho^0, \quad S_\rho = F_2(\rho_0, \theta) - S_\rho^0,$$

$$Q_\rho - \frac{\partial H_\rho}{\partial S_\Gamma} = F_4(\rho_0, \theta), \quad G_\rho = F_3(\rho_0, \theta). \tag{1.30}$$

If the contour of the hole is free from external forces, one must set in (1.30) $F_1 = F_2 = F_3 = F_4 \equiv 0$, and we obtain

$$T_\rho = -T_\rho^0, \quad S_\rho = -S_\rho^0, \quad G_\rho = 0,$$

$$Q_\rho - \frac{\partial H_\rho}{\partial S_\Gamma} = 0. \tag{1.31}$$

Since the components $T_\rho, \cdots, G_\theta$ of (1.17) decay rapidly with the distance from the edge of the hole Γ, the functions $w(\rho, \theta)$ and $\varphi(\rho, \theta)$ found from the equations (1.18) or the equation (1.27) and the boundary conditions (1.29), (1.30) or (1.31) must satisfy these conditions. This is the condition "at infinity". In order to satisfy these conditions the function w and φ must be chosen in such a way that the components $T_\rho, \cdots, G_\theta$ found from the formulae (1.23) and (1.24) tend to zero for $\rho \to \infty$. We note (cf. [1], p. 336) that the equations (1.18) and the relations leading to them were obtained for the condition that the Gaussian curvature K of the middle surface $\mathbf{R}(\rho, \theta)$ of (1.1) is equal to zero, i.e.,

$$K = \frac{1}{R_1 R_2} = 0. \tag{1.32}$$

The equality (1.32) will be satisfied only for surfaces of zero Gaussian curvature for which the first quadratic form is given by (1.11); however, this equality may be fulfilled approximately and on some part of an arbitrary shell, which contains the hole, with a sufficiently smooth middle surface. For this purpose one must choose a proper system of the curvilinear coordinates α, β of (1.13) with origin inside the hole, in order that for given external forces and boundary conditions the contour L^* does not leave the boundary of the section of the shell under consideration; also, one has to replace the quadratic form (1.2) in this region by the form (1.11). Under these conditions the requirement (1.32) will be fulfilled automatically.

It is clear that such a solution of the problem of stress concentration for shells for which $K \neq 0$ will be the more exact the smaller is the zone Σ^*, i.e., the smaller is the size of the hole in the shell. However, as has been shown by the theoretical and experimental investigations performed under the guidance of V. Z. Vlasov in the Central Scientific Research Institute for Engineering Structures [2], approximate solutions based on the theory of shallow shells give good agreement with experiments for shells for which the ratio of the sag f of the shell to its smallest dimension a in plan form is $f/a < \frac{1}{5}$.

§ 2. Circular cylindrical shell with a hole

We will assume that the middle surface of the cylindrical shell of radius a is referred to the isometric system of coordinates α, β, so that its equation in vector form is

$$\mathbf{R}(\alpha, \beta) = a[\mathbf{i}_x \cos\alpha + \mathbf{i}_y \sin\alpha + \mathbf{i}_z\beta], \tag{2.1}$$

or in component form

$$x = a\cos\alpha,\ y = a\sin\alpha,\ z = a\beta. \tag{2.2}$$

The square of the element of length of the middle surface of the cylinder in the coordinates (2.2) will have the form (1.11), i.e.,

$$ds^2 = a^2(d\alpha^2 + d\beta^2). \tag{2.3}$$

We will imagine the cylinder to have been cut from the hole along a generator and developed into the plane; then we obtain the plane P with the hole, referred to rectilinear Cartesian coordinates α and β.

If in the plane P the hole has the form of a circle, ellipse, etc., then we will speak respectively of a cylindrical shell, having a circular, elliptical, etc., hole.

We now proceed, guided by the form of the hole, by use of the formulae (1.13) to the new system of isometric coordinates (ρ, θ). The operators ∇^2 and ∇_k^2 of the basic equations (1.18) or the equivalent equation (1.27) contain the quantities $1/R_\rho$, $1/R_\theta$ and $1/R_{\rho\theta}$, depending on the coefficient $\mathscr{H}^2$ of the quadratic form (1.14) and on the coefficients D, D^* and D^{**} of the form (1.3), expressed in terms of the variables ρ and θ. These coefficients depend on the shape of the hole, since the form of the function $\omega(\zeta)$ must be chosen in dependence on it by use of (1.12).

We will find the form of the coefficients $\mathscr{H}^2$, D, D^* and D^{**} for any coordinate net $\rho = \text{const.}$, $\theta = \text{const.}$, given by the transition formula (1.13). We find the value of $\mathscr{H}^2(\rho, \theta)$ from (1.15) and (1.16) by setting in (1.15)

$$\lambda_0^2 = a^2. \tag{2.4}$$

Substituting in the vector equation (2.1) the values of α and β from (1.13), we obtain the equation of this surface in the variables ρ, θ in the form

$$\mathbf{R} = \mathbf{R}(\rho, \theta). \tag{2.5}$$

For the unit normal $\mathbf{n}$ to the surface of the circular cylinder we obtain

$$\mathbf{n} = \mathbf{i}_x \cos\alpha + \mathbf{i}_y \sin\alpha, \tag{2.6}$$

where one must substitute for α its value expressed in terms of ρ and θ from (1.13).

After finding from (2.5) $\partial^2\mathbf{R}/\partial\rho^2$, $\partial^2\mathbf{R}/\partial\rho\partial\theta$ and $\partial^2\mathbf{R}/\partial\theta^2$ and substituting their values together with $\mathbf{n}$ into (1.5) we obtain

$$D = -a\left(\frac{\partial\alpha}{\partial\rho}\right)^2, \quad D^* = -a\frac{\partial\alpha}{\partial\theta}\frac{\partial\alpha}{\partial\rho}, \quad D^{**} = -a\left(\frac{\partial\alpha}{\partial\theta}\right)^2. \tag{2.7}$$

Substituting into (1.20)–(1.21) the values of D, D^* and D^{**} from (2.7) and of $\mathscr{H}^2$ from (1.15), one easily finds the quantities $1/R_\rho$, $1/R_\theta$ and $1/R_{\rho\theta}$ entering into the operators of the basic equations (1.18) in terms of the isometric system of coordinates ρ, θ. For example, in the case of a circular hole of radius r_0 in the cylindrical shell the function $\omega(\zeta)$ must take the form

$$\alpha + i\beta = \omega(\zeta) = a_1\zeta = \frac{r_0}{a}e^{\rho+i\theta}. \tag{2.8}$$

Hence

$$\alpha = \frac{r_0}{a} e^{\rho} \cos\theta, \quad \beta = \frac{r_0}{a} e^{\rho} \sin\theta. \tag{2.9}$$

The parameters ρ and θ in (2.9) vary over the intervals

$$0 \leq \rho < \infty, \quad 0 \leq \theta \leq 2\pi. \tag{2.10}$$

We obtain the equation of the contour of the hole Γ, lying in the middle surface of the shell, from $\mathbf{R}$ of (2.1) by substituting in it the values for α and β from (2.9) for $\rho = 0$. We obtain the equation of the edge of the hole γ in the plane P for $\rho = 0$ from (2.9).

Substituting the values of α and β from (2.9) into (2.7) we find readily

$$\begin{aligned} D &= -\frac{r_0^2}{a} e^{2\rho} \cos^2\theta, \\ D^{**} &= -\frac{r_0^2}{a} e^{2\rho} \sin^2\theta, \\ D^{*} &= \frac{r_0^2}{a} e^{2\rho} \sin\theta \cos\theta. \end{aligned} \tag{2.11}$$

We obtain for $\mathscr{H}^2$ in this case the value

$$\mathscr{H}^2 = r_0^2 e^{2\rho}. \tag{2.12}$$

Substituting (2.11)–(2.12) into (1.20)–(1.21) we find

$$\frac{1}{R_\rho} = \frac{\cos^2\theta}{a}, \quad \frac{1}{R_\theta} = \frac{\sin^2\theta}{a}, \quad \frac{1}{R_{\rho\theta}} = \frac{1}{2}\frac{\sin 2\theta}{a}. \tag{2.13}$$

It is seen from (2.13) that for passage along the contour of the circular hole, i.e., for changes of θ in the interval $0 \leq \theta \leq 2\pi$, the quantities $1/R_\rho$ and $1/R_\theta$ do not remain constant, but change from zero to $1/a$, while the quantity $1/R_{\rho\theta}$ changes from $-\frac{1}{2}a$ to $+\frac{1}{2}a$.

Substituting the values of $\mathscr{H}^2$ $1/R_\rho$, $1/R_\theta$ and $1/R_{\rho\theta}$ from (2.12) and (2.13), we find for this case the operators

$$\begin{aligned} \nabla_k^2 = \frac{1}{a r_0^2 e^{2\rho_0}} \Bigg[&\sin^2\theta \frac{\partial^2}{\partial\rho^2} + \sin 2\theta \frac{\partial^2}{\partial\rho\partial\theta} + \cos^2\theta \frac{\partial^2}{\partial\theta^2} + \\ &+ \cos 2\theta \frac{\partial}{\partial\rho} - \sin 2\theta \frac{\partial}{\partial\theta} \Bigg], \end{aligned} \tag{2.14}$$

$$\nabla^2 = \frac{1}{r_0^2 e^{2\rho}} \left[\frac{\partial^2}{\partial\rho^2} + \frac{\partial^2}{\partial\theta^2} \right].$$

Introducing (2.14) into (1.27), we obtain the basic equation of the problem of stress concentration in a circular cylindrical shell with a circular hole of radius r_0.

The equation obtained does not coincide with the known equation of A. I. Lur'e [3] for a problem bearing the same title.[1] This had to be expected, because these equations referred to a different system of curvilinear coordinates. It is seen from (2.14) that the coefficients in the equations (1.27), involving the quantities $1/R_\rho$, $1/R_\theta$ and $1/R_{\rho\theta}$, depend in an essential manner on the shape of the hole, i.e., on the selected system of coordinates ρ and θ. Only in the case of the rectangular hole (and, in particular, for the square) the contour of which in terms of the variables α and β (Fig. 7) is formed by coordinate lines $\alpha = \text{const.}$ and $\beta = \text{const.}$, the quantities $1/R_\alpha$ and $1/R_\beta$ will have along these coordinate lines the constant values $1/R_\alpha = 1/a$, $1/R_\beta = 0$, while the quantity $1/R_{\alpha\beta}$ will be identically zero, since the coordinates α, β are isometric conjugate coordinates.

Adding to the above basic equations the corresponding boundary conditions and imposing on the unknown functions $w(\rho, \theta)$ and $\varphi(\rho, \theta)$

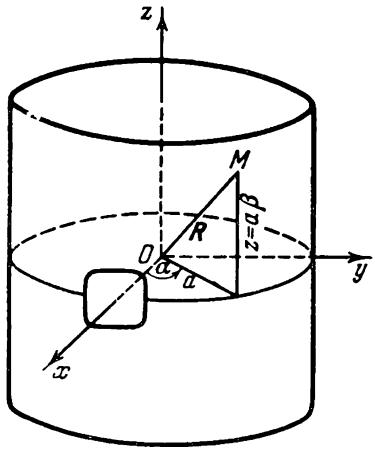

Fig. 7.

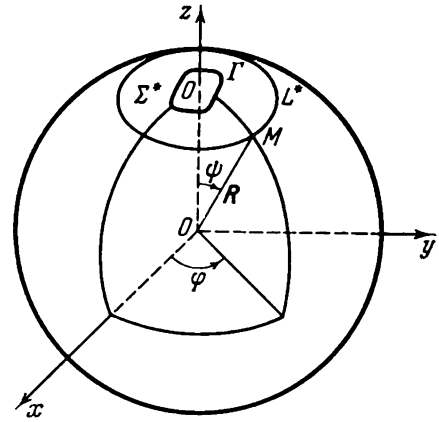

Fig. 8.

the conditions "at infinity", we obtain a sufficient number of equations and conditions for finding the functions w and φ in the case of any hole, the equation of the contour Γ of which we obtain from (2.1), where α and β are given by the formulae (1.13) for $\rho = \rho_0 = \text{const.}$

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[1] This fact was taken into consideration in the thesis of N. G. Vasilenko [13].

§ 3. The spherical shell with a hole

We will consider the state of stress in a spherical shell having a hole with a sufficiently smooth contour Γ. We will assume this hole to lie at the "north pole" of the sphere (Fig. 8).

We will refer the middle surface (the radius of which is equal to a) of the shell to the isometric system of coordinates α, β.

$$\mathbf{R}(\alpha, \beta) = a[\mathbf{i}_x \sin\psi \cos\varphi + \mathbf{i}_y \sin\psi \sin\varphi + \mathbf{i}_z \cos\psi], \tag{3.1}$$

where

$$\varphi = \arctan\frac{\beta}{\alpha}, \quad \psi = 2\arctan\frac{\sqrt{\alpha^2 + \beta^2}}{2a}. \tag{3.2}$$

The coordinates α, β are simultaneously Cartesian coordinates of the stereographic projection of the sphere onto the plane (Fig. 8). The square of the element of length of the middle surface of the sphere (3.1) in terms of the coordinates α and β will have the form

$$ds^2 = \frac{d\alpha^2 + d\beta^2}{\left(1 + \dfrac{\alpha^2 + \beta^2}{4a^2}\right)^2}. \tag{3.3}$$

Under the action of a given system of external forces and boundary conditions a disturbed state of stress arises in the region Σ^* of the shell around the hole under consideration.

We will pass on, in accordance with the formulae (1.13), to new isometric coordinates ρ, θ, choosing them in such a way that one of the coordinate lines, for example, the line $\rho = \rho_0$ will simultaneously be the contour Γ of the hole considered. We obtain the equation of the contour of the hole Γ from (3.1) by expressing the functions φ and ψ in terms of ρ and θ in accordance with (1.13) for $\rho = 0$. The contour L^* may always be replaced by the coordinate line (parallel to it) $\psi = \text{const.}$ which is closest to it (Fig. 8). The closed contours Γ and L^*, bounding the region Σ^* on the sphere of radius a, will correspond in the plane P of the variables α and β to closed curves γ and l^*, bounding a region σ^* (Fig. 9).

Since we assume the region Σ^* to be not large, its corresponding region σ^* in the plane P will be a quantity of the same order. This gives a certain basis for assuming the quantity $\varepsilon(\alpha, \beta) = (\alpha^2 + \beta^2)/4a^2$ in (3.3) to be small compared with unity. Neglecting it we obtain for ds^2 in the region Σ^* the expression

$$ds^2 = d\alpha^2 + d\beta^2, \tag{3.4}$$

where $\lambda_0^2 = 1$. The geometrical meaning of taking ds^2 in accordance with (3.4) is the replacement of the part of the spherical surface, bounded by the contour L^*, by a plane with a net of Cartesian coordinates $\alpha = \text{const.}$ and $\beta = \text{const.}$ By the theory of shallow shells of V. Z. Vlasov [2] this leads to good agreement with experimental evidence for

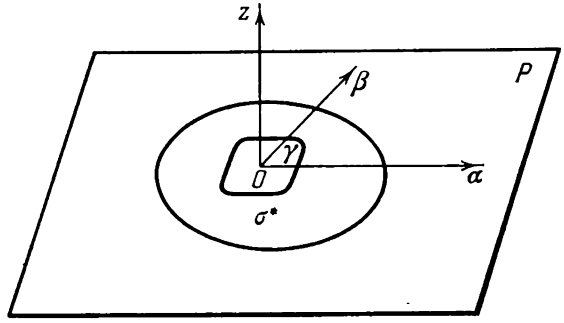

Fig. 9.

shells for which $\frac{1}{2}f/R_0 < \frac{1}{5}$, where f is the displacement of the segment cut from the sphere by the plane which contains the parallel closest to the contour L^*, R_0 is the radius of the circle of this section. If one takes these results, the quantity $(\alpha^2 + \beta^2)/(4a^2)$ in (3.3) may be not too small; for $\frac{1}{2}f/R_0 \leq \frac{1}{5}$,

$$\frac{\sqrt{\alpha^2 + \beta^2}}{2a} \leq 0.4. \tag{3.5}$$

In the future we will assume that for given external forces and conditions of support of the shell, the dimensions of the hole have been chosen in such a manner that the condition (3.5) is automatically satisfied.

The quantities $1/R_\rho$, $1/R_\theta$, $1/R_{\rho\theta}$ in any isometric system of coordinates ρ, θ, i.e., for any hole in the sphere considered by us, are given by

$$\frac{1}{R_\rho} = \frac{1}{R_\theta} = \frac{1}{a}, \quad \frac{1}{R_{\rho\theta}} = 0. \tag{3.6}$$

The general form of the expression for $\mathscr{H}^2$ is given by (1.15)–(1.16).

We will now establish the basic equations of the problem for the case of the circular and the elliptical holes. For other holes the basic equations (1.18) or the equation (1.27) may be found in an analogous manner.

In the case of a circular hole of radius r_0 the function $\omega(\zeta)$ must take the form

$$\alpha + i\beta = \omega(\zeta) = r_0\zeta. \tag{3.7}$$

Hence

$$\alpha = r_0 e^{\rho} \cos\theta, \quad \beta = r_0 e^{\rho} \sin\theta. \tag{3.8}$$

The coefficient $\mathscr{H}^2(\rho, \theta)$ will have the form

$$\mathscr{H}^2 = r_0^2 e^{2\rho}. \tag{3.9}$$

In the case of an elliptical hole with semi-axes a^* and b^* the function $\omega(\zeta)$ will be chosen in the form

$$\alpha + i\beta = \omega(\zeta) = \frac{c}{2}\left(\zeta + \frac{1}{\zeta}\right) = c \cosh(\rho + i\theta), \tag{3.10}$$

where $c = \sqrt{a^{*2} - b^{*2}}$ is the eccentricity of the ellipse. Separating real and imaginary parts in (3.10), we obtain

$$\alpha = c \cosh\rho \cos\theta, \quad \beta = c \sinh\rho \sin\theta. \tag{3.11}$$

We obtain the equation of the contour of the hole by setting $\rho = \rho_0$ in (3.11); we then determine the values of ρ_0 and c for the given semi-axes of the ellipse from the formulae

$$a^* = c \cosh\rho_0, \quad b^* = c \sinh\rho_0. \tag{3.12}$$

The coefficient $\mathscr{H}^2$ in this case takes the form

$$\mathscr{H}^2(\rho, \theta) = \frac{c^2}{2}(\cosh 2\rho - \cos 2\theta). \tag{3.13}$$

We obtain analogous equations in the case of other shapes of holes, given by the formulae (1.13).

Substituting in the operators ∇^2 and ∇_k^2 of (1.19) the values of the quantities $1/R_\rho$, $1/R_\theta$ and $1/R_{\rho\theta}$ in accordance with (3.6) and introducing these quantities into (1.27), we obtain the basic equation of the problem of stress concentration in a spherical shell with an arbitrary hole

$$\nabla^2\nabla^2\Phi + i\kappa^2\nabla^2\Phi = 0, \tag{3.14}$$

where

$$\nabla^2 = \frac{1}{\mathscr{H}^2}\left(\frac{\partial^2}{\partial\rho^2} + \frac{\partial^2}{\partial\theta^2}\right),$$
$$\kappa^2 = \frac{\sqrt{12(1-\nu^2)}}{ah}. \tag{3.15}$$

The form of the equation (3.14) suggests for its general solution the form

$$\Phi(\rho, \theta) = \Phi_1(\rho, \theta) + \Phi_2(\rho, \theta), \tag{3.16}$$

where $\Phi_2(\rho, \theta)$ is a harmonic function satisfying the equation

$$\nabla^2\Phi_2 = 0, \tag{3.17}$$

while Φ_1 is a function which is a solution of the equation

$$\nabla^2\Phi_1 + i\kappa^2\Phi_1 = 0, \tag{3.18}$$

which is conveniently written in the form

$$\frac{\partial^2\Phi_1}{\partial\rho^2} + \frac{\partial^2\Phi_1}{\partial\theta^2} + i\kappa^2\mathscr{H}^2\Phi_1 = 0. \tag{3.19}$$

Substituting in (3.19) the value of $\mathscr{H}^2$ from (3.9) or (3.13) respectively, we find for Φ_1 the basic equation in the form:

for the circular hole

$$\frac{\partial^2\Phi_1}{\partial\rho^2} + \frac{\partial^2\Phi_1}{\partial\theta^2} + i\kappa^2 r_0^2 e^{2\rho}\Phi_1 = 0, \tag{3.20}$$

for the elliptical hole

$$\frac{\partial^2\Phi_1}{\partial\rho^2} + \frac{\partial^2\Phi_1}{\partial\theta^2} + i\frac{\kappa^2 c^2}{2}(\cosh 2\rho - \cos 2\theta)\Phi_1 = 0. \tag{3.21}$$

The boundary conditions (1.30) for $F_1 = F_2 = F_3 = 0$ and the basic membrane state of stress of the spherical shell

$$T_\rho^0 = T_\theta^0 = \frac{aq}{2}, \quad S_\rho^0 = S_\theta^0 = 0, \tag{3.22}$$

caused by internal pressure q will have the form

$$\begin{gathered} T_\rho = -\frac{aq}{2}, \quad S_\rho = 0, \\ G_\rho = 0, \quad Q_\rho = -\frac{1}{\mathscr{H}}\frac{\partial H_\rho}{\partial\theta} = F_4(\theta). \end{gathered} \tag{3.23}$$

In the case of the circular hole $F_4(\theta) = qr_0/2$, one may take the harmonic function $\Phi_2(\rho, \theta)$, satisfying the conditions "at infinity" for the basic state of stress (3.22), in the form

$$\Phi_2(\rho, \theta) = d\log\rho + \sum_{n=0}^{\infty}(a_n + ib_n)e^{-n\rho}\cos n\theta, \tag{3.24}$$

and the function $\Phi_1(\rho, \theta)$ in the form

$$\Phi_1(\rho, \theta) = \sum_{n=0}^{\infty}(A_n + iB_n)H_n^{(1)}(\kappa\sqrt{i}\rho)\cos n\theta, \tag{3.25}$$

where $H_n^{(1)}(\kappa\sqrt{i}\rho)$ is the Hankel function, the asymptotic expansion of which for large values of the argument $\kappa\sqrt{i}\rho$ has the form [17]

$$H_n^{(1)}(\kappa\sqrt{i}\rho) \approx \sqrt{\frac{2}{\pi\kappa\sqrt{i}\rho}}\, e^{i\left(\kappa\sqrt{i}\rho - \frac{n\pi}{2} - \frac{\pi}{4}\right)}. \tag{3.26}$$

Separating in the functions Φ_2 and Φ_1 real and imaginary parts we find the general form of the stress function $\varphi(\rho, \theta)$ and the deflection function $w(\rho, \theta)$. Expressing now in the boundary conditions (3.23), the components T_ρ, H_ρ, S_ρ, G_ρ and Q_ρ in terms of the functions $\varphi(\rho, \theta)$ and $w(\rho, \theta)$ by use of the formulae (1.23)–(1.25), we obtain a system of equations from which we find the constants d, a_n, b_n, A_n and B_n $(n = 0, 1, 2, \cdots)$ entering into the expansions (3.24)–(3.25).

By determining the functions φ and w in one or the other approximation, we find from (1.23)–(1.25) the unknown components characterizing the perturbed state of stress around a circular hole of radius r_0 in a spherical shell.

In the case of an elliptical hole with semi-axes a^* and b^* one may choose the harmonic function Φ_2 for the same basic state of stress (3.22) and the boundary conditions (3.23) where,

$$F_\gamma(\theta) = q\frac{b^*}{a^* + b^*}[a - (a - b)\cos 2\theta],$$

taking into consideration the symmetry of the perturbed state of stress, in the form

$$\Phi_2 = d \log\left\{\frac{c^2}{2}(\cosh 2\rho + \cos 2\theta)\right\} + \sum_{n=0}^{\infty}(a_n + ib_n)e^{-n\rho}\cos n\theta, \tag{3.27}$$

and the function $\Phi_1(\rho, \theta)$, satisfying the equation (3.21), in the form

$$\Phi_1(\rho, \theta) = \sum_{n=0}^{\infty}(A_n + iB_n)\,\mathrm{Me}_n^{(1)}(\rho)\,\mathrm{Ce}_n(\theta), \tag{3.28}$$

where $\mathrm{Me}_n^{(1)}(\rho)$ is the Matthieu-Hankel function the asymptotic expansion of which for large values of the argument is identical with the expansion of the Hankel function $H_n^{(1)}$ of (3.26), and $\mathrm{Ce}_n(\theta)$ is the periodic Matthieu function. A study of the state of stress and deformation around a circular hole of radius $r_0 = 10$ cm. and around an elliptical hole with $a^*/b^* = 1.1$, $E = 7.2 \times 10^5$ kg./cm.^{2}, $b^* = 10$ cm., $q = 1$ kg./cm^2., $h = 0.4$ cm., $a = 200$ cm., $\nu = 0.3$, $T_\rho^0 = T_\theta^0 = 100$ kg./cm., has shown that the largest membrane stress near the above holes arises at their edge. In the first approximation, i.e., taking two terms of the expansions of the functions (3.24)–(3.25) and (3.27)–(3.28) into account, these stresses become

for the circular hole

$$T_\theta^* = 411 \text{ kg./cm.}, \tag{3.29}$$

for the elliptical hole

$$T_\theta^* = 441 + 65 \cos 2\theta \text{ kg./cm.} \tag{3.30}$$

Hence we obtain for the stress concentration factor

$$K = \frac{T_\theta^*}{T_\theta^0} = 4.41 + 0.65 \cos 2\theta. \tag{3.31}$$

The largest value of the coefficient $K_{\max} \approx 5.1$ will occur on the edge of the hole at the end of the major semi-axis of the ellipse.

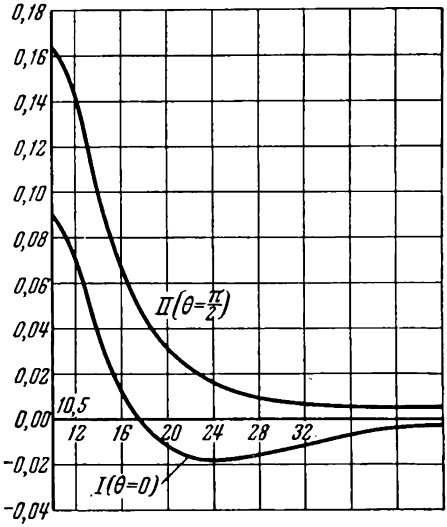

Fig. 10.

Figure 10 gives graphs characterizing the deflection w along the major semi-axis ($\theta = 0$) (curve I) and along the minor semi-axis $\theta = \frac{1}{2}\pi$ (curve II) around the above-mentioned elliptical hole. The abscissa in this graph is ρ in mm., the ordinate w in mm., assumed positive when the direction of the deflection w coincides with the direction of the normal **n** (Fig. 2) to the surface.

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STABILITY OF FINITE DEFORMATION

B. R. SETH
Kharagpur, India

By making use of a linear stress-strain tensor law [1] and finite strain components, the equations of neutral equilibrium have been solved for a solid right circular cylinder of a compressible elastic material under finite extension and torsion. The condition for stability of equilibrium has been determined in terms of the roots of an equation involving the frequency of the small deformation superposed over the finite deformation of the cylinder. We find the frequency equation to be quite similar to that given by Green [2] on the basis of incompressibility and Neo-Hookean-ness of the elastic material. The numerical solution of this equation is reserved for a future paper.

Recently R. Hill has also dealt with the stability of such deformations [3].

§ 1. Introduction

The problem of finite extension and torsion of a solid right circular cylinder is well known and has been attacked by many authors like Seth [1], Rivlin [4], Greenhill [5], Nicolai [6], and Trösch [7], using various methods of approach. The main feature of all such attempts has, of course, been to take certain forms for the displacements and the stress-strain law and then to determine the forces required to maintain this deformation in equilibrium. The stability of equilibrium of the cylinder under the above mentioned deformation has only been discussed recently by Green [2]. The method employed is due to Wilkes [8], but it was later developed by Green, Rivlin and Shield [9]. By superposing a small deformation over the finite extension and torsion of the cylinder, he formulates the equations of neutral equilibrium and gives the criterion for stability in terms of the roots of an equation involving the frequency of the small deformation. He assumes, however, the material to be incompressible and Neo-Hookean, and this assumption, according to him, is necessary to be able to solve the equations of neutral equilibrium. In this paper, it is shown that we can do away with the assumptions of incompressibility and Neo-Hookean-ness to solve the said equations. We use Seth's [1] linear stress-strain law and finite strain components referred to the strained state. The particular case when Poisson's ratio is equal to $1/3$ has been discussed and the condition for stability obtained in terms of the roots of the frequency equation similar to that of Green [2].

§ 2. Nature of the problem

We seek to examine the stability of the equilibrium of a solid right circular cylinder of compressible elastic material subjected to finite extension and torsion. The forces to maintain such a deformation in equilibrium have already been determined by many authors, but here we confine ourselves to Seth's [1] analysis. We superpose a small deformation over the given finite deformation and find out the conditions under which an adjacent equilibrium position can exist to determine the stability of original equilibrium. If we denote the body in original equilibrium by B and after superposition by B', then for an adjacent equilibrium position to exist, with zero surface traction over the curved surface of B' as well as over that of B, it should be possible to solve the equations of neutral equilibrium. The criterion for such a solution has been put in terms of the roots of an equation involving the frequency of the superposed deformation.

§ 3. Equations of neutral equilibrium

The displacement and stresses in cylindrical coordinates in the body B, as given by Seth [1], are as follows:

$$u = r(1 - \beta), \quad v = r\tau z, \quad w = \alpha z, \tag{3.1}$$

and

$$\begin{aligned} T_{rr} &= \lambda\Delta + \mu[1 - (r\beta' + \beta)^2], \\ T_{\theta\theta} &= \lambda\Delta + \mu[1 - \beta^2], \\ T_{zz} &= \lambda\Delta + \mu[2\alpha - \alpha^2 - r^2\tau^2\beta^2], \\ T_{\theta z} &= \mu\tau r\beta^2, \quad T_{r\theta} = T_{rz} = 0, \end{aligned} \tag{3.2}$$

where

$$\Delta = 1 - \beta^2 - \tfrac{1}{2}r^2\beta'^2 - r\beta\beta' + \alpha - \tfrac{1}{2}\alpha^2 - \tfrac{1}{2}r^2\tau^2\beta^2, \quad \beta' = \mathrm{d}\beta/\mathrm{d}r,$$

$$\beta^2 = 1 - \tfrac{1}{4}(1 - c)\tau^2(r^2 - \alpha^2) + \cdots, \quad c = 2\mu/(\lambda + 2\mu),$$

$$\alpha = \tfrac{1}{4}\tau^2 a^2 + \cdots,$$

τ = angular displacement per unit length, and λ and μ are elastic constants.

These stresses in (3.2) are based on the linear law

$$t_j^i = \lambda\Delta\delta_j^i + 2\mu e_j^i.$$

The physical components of displacement after superposition are

$$u' = u + \varepsilon U, \quad v' = v + \varepsilon \frac{V}{r}, \quad w' = w + \varepsilon W, \tag{3.3}$$

where ε is a small quantity, and

$$\begin{aligned} U &= f_1(r)\cos(n\theta - vz + A), \\ V &= f_2(r)\sin(n\theta - vz + A), \\ W &= f_3(r)\sin(n\theta - vz + A), \end{aligned} \tag{3.3'}$$

v being the frequency of the small superposed deformation and A a constant.

The physical components of stress after superposition are $T_{rr} + T_{rr}'$, $T_{\theta\theta} + T_{\theta\theta}'$, $T_{zz} + T_{zz}'$, $T_{r\theta} + T_{r\theta}'$, $T_{\theta z} + T_{\theta z}'$, $T_{rz} + T_{rz}'$, where we have

$$\begin{aligned} T_{rr}' &= \varepsilon\left[\lambda\delta + 2\mu(r\beta' + \beta)\frac{\partial U}{\partial r}\right], \\ T_{\theta\theta}' &= \varepsilon\left[\lambda\delta + 2\mu\beta\left(\frac{U}{r} + \frac{\beta}{r^2}\frac{\partial V}{\partial \theta}\right)\right], \\ T_{zz}' &= \varepsilon\left[\lambda\delta + 2\mu\left\{\frac{\partial W}{\partial z}(1-\alpha) - r\beta^2\frac{\partial V}{\partial z} + Ur\beta\tau^2\right\}\right], \\ T_{r\theta}' &= \mu\varepsilon\left[\frac{1}{r}\frac{\partial U}{\partial \theta}(r\beta' + \beta) + r\left(\frac{V}{r^2}\right)'\beta^2\right], \\ T_{\theta z}' &= \mu\varepsilon\left[(1-\alpha)\frac{\partial W}{r\partial\theta} + \beta^2\left(\frac{1}{r}\frac{\partial V}{\partial z} - \frac{\tau}{r}\frac{\partial V}{\partial \theta}\right) - 2U\beta\tau\right], \\ T_{rz}' &= \mu\varepsilon\left[\frac{\partial U}{\partial z}(r\beta' + \beta) + (1-\alpha)\frac{\partial W}{\partial r} - \tau\beta^2\left(V' - \frac{2V}{r}\right)\right], \end{aligned} \tag{3.4}$$

dashes denoting differentiation with respect to r and

$$\delta \equiv \frac{\partial U}{\partial r}(r\beta' + \beta) + \beta\left(\frac{U}{r} + \frac{1}{r^2}\frac{\partial V}{\partial \theta}\beta\right) + \frac{\partial W}{\partial z}(1-\alpha) - \tau\beta^2\frac{\partial V}{\partial z} + Ur\beta\tau^2.$$

The equations of equilibrium in B are

$$T^{ik}{}_{,i} = 0 \tag{3.5}$$

and in B' are

$$(T'^{ik} + T^{ir}\omega^k{}_{,r})_{,i} = 0, \tag{3.6}$$

where a comma denotes covariant differentiation, T^{ik} are contravariant stress components in B, $T'^{ik} + T^{ik}$ are the contravariant stress components in B' and ω^k are the contravariant components of the superposed displacement.

The contravariant components T^{kh} can easily be found from the physical components $\bar{T}_{ij}$ through the formula

$$\bar{T}_{ij} = \sqrt{g^{ii}g^{jj}}(g_{ki}g_{hj})T^{kh}, \tag{3.7}$$

where g_{ij} is the metric tensor of the deformed state.

Similarly, the contravariant components ω^k can be computed from the physical components $\bar{\omega}^k$ through the following rule:

$$\bar{\omega}^k = \sqrt{g_{kk}}\,\omega^k. \tag{3.8}$$

When surface forces are prescribed at the surface of B', then the boundary conditions are

$$(T^{ik} + T'^{ik})(n_i + \varepsilon n'_i) = P^k + \varepsilon P'^k, \tag{3.9}$$

where $n_i + \varepsilon n'_i$ are the covariant components of the unit normal to the surface of B' and $P^k + \varepsilon P'^k$ are the contravariant components of the surface force vector.

Now we substitute the stresses and the displacements given by (3.1)–(3.4) in (3.6) and get the equations of neutral equilibrium as follows:

$$\begin{aligned}
&\frac{d}{dr}[\lambda\delta_1 + 2\mu(r\beta' + \beta)f_1' + [\lambda\Delta + \mu\{1 - (r\beta' + \beta)^2\}]f_1'] + \\
&+\left[\frac{\mu}{r}\left\{\frac{1}{r}(r\beta' + \beta)(-f_1n^2) + r\beta^2n\left(\frac{f_2}{r^2}\right)'\right\}\right] - \\
&-\frac{1}{r^2}\left(f_1n^2 + \frac{f_2n}{r}\right)\left[\lambda\Delta + \mu(1 - \beta^2)\right] + \mu\tau n\nu f_1\beta^2 + \\
&+\frac{1}{r}\left[2\mu f_1'(r\beta' + \beta) - 2\mu\beta\left(\frac{f_1}{r} + \frac{f_2}{r^2}\beta n\right) + \right. \\
&+ f_1'\{\lambda\Delta + \mu - \mu(r\beta' + \beta)^2\} + \\
&\left.+ (f_1' - \nu f_3)\{\lambda\Delta + \mu(1 - \beta^2)\}\right] + \\
&+\mu\left[(-\nu^2 f_1)(r\beta' + \beta) - (1 - \alpha)f_3'\nu + \nu\tau\beta^2 f_2'\right] + \\
&+\left[\mu\tau\beta^2 f_1 n\nu - f_1\nu^2\{\lambda\Delta + 2\mu(\alpha - \tfrac{1}{2}\alpha^2 - \tfrac{1}{2}r^2\tau^2\beta^2)\}\right] = 0,
\end{aligned} \tag{3.10}$$

$$(3.11)\quad \frac{d}{dr}\left[\frac{\mu}{r}\left\{-f_1\frac{n}{r}(r\beta'+\beta)+r\left(\frac{f_2}{r^2}\right)'\beta^2\right\}+\right.$$

$$\left.+\frac{1}{r^2}\left(f_2'-\frac{f_2}{r}\right)\{\lambda\Delta+\mu-\mu(r\beta'+\beta)^2\}\right]-$$

$$-\frac{1}{r^2}\left[\lambda\delta_1 n+2\mu\beta\frac{n}{r}\left(f_1+\frac{\beta}{r}f_2 n\right)\right]-\frac{n}{r^2}[\lambda\Delta+\mu(1-\beta^2)](f_3\nu-f_1')+$$

$$+\left[\frac{\mu\tau\beta^2 f_2\nu n}{r^2}\right]+\frac{\mu}{r}\left[(1-\alpha)f_3 n\frac{\nu}{r}-\beta^2\frac{\nu}{r}(\tau f_2 n+f_2\nu)-2f_1\beta\tau\nu\right]-$$

$$-\mu\tau\beta^2(f_1'\nu-f_3\nu^2)-\frac{f_2\nu^2}{r^2}\left[\lambda\Delta+2\mu(\alpha-\tfrac{1}{2}\alpha^2-\tfrac{1}{2}r^2\tau^2\beta^2)\right]+$$

$$+\frac{3\mu}{r^2}\left[-\frac{nf_1}{r}(r\beta'+\beta)+r\beta^2\left(\frac{f_2}{r^2}\right)'\right]+$$

$$+\frac{2}{r^3}\left(f_2'-\frac{f_2}{r}\right)\left[\lambda\Delta+\mu\{1-(r\beta'+\beta)^2\}\right]+$$

$$+\frac{\mu\tau}{r}\beta^2 f_1\nu-\frac{1}{r^3}\left(nf_1+\frac{f_2}{r}\right)\left[\lambda\Delta+\mu(1-\beta^2)\right]=0,$$

$$(3.12)\quad \frac{d}{dr}\left[\mu\left\{f_1\nu(r\beta'+\beta)+(1-\alpha)f_3'-\tau\beta^2\left(f_2'-\frac{2f_2}{r}\right)\right\}+\right.$$

$$\left.+f_3'\{\lambda\Delta+\mu-\mu(r\beta'+\beta)^2\}\right]+$$

$$+\frac{\mu}{r}\left[-f_3\frac{n^2}{r}(1-\alpha)-\frac{\beta^2 n\nu f_2}{r}-\frac{\beta^2\tau f_2 n^2}{r}-4\beta\tau f_1 n\right]+n\nu f_3\mu\tau\beta^2+$$

$$+\frac{1}{r^2}f_3(-n^2)\left[\lambda\Delta+\mu(1-\beta^2)\right]+\left[\lambda\delta_1\nu+2\mu\{-\nu^2 f_3(1-\alpha)+\tau\beta^2 f_2\nu^2+\right.$$

$$\left.+r\beta\tau^2 f_1\nu\}\right]+\left[\mu\tau\beta^2 f_3 n\nu-f_3\nu^2\{\lambda\Delta+2\mu(\alpha-\tfrac{1}{2}\alpha^2-\tfrac{1}{2}r^2\tau^2\beta^2)\}\right]+$$

$$+\frac{\mu}{r}\left[f_1\nu(r\beta'+\beta)+(1-\alpha)f_3'-\tau\beta^2\left(f_2'-\frac{2f_2}{r}\right)\right]+$$

$$+\frac{f_3'}{r}\left[\lambda\Delta+\mu\{1-(r\beta'+\beta)^2\}\right]=0,$$

where dash denotes differentiation with respect to r and

$$\delta_1 \equiv f_1'(r\beta' + \beta) + \beta\left(\frac{f_1}{r} + \frac{nf_2\beta}{r^2}\right) - \nu f_3(1 - \alpha) + \tau\beta^2 f_2\nu + f_1 r\beta\tau^2.$$

§ 4. Solution of the equilibrium equations

The equations (3.10)–(3.12) can be solved in general for f_1, f_2, f_3. For the time being, however, we limit our analysis to the case when n is equal to unity and $\lambda = 2\mu$ (or Poisson's ratio $= \frac{1}{3}$), because only in that case will we expect finite non-zero displacements on the axis of the cylinder. It corresponds to the problem of the twisted Euler strut in classical elasticity in which the axis of the cylinder is deformed into a helix of pitch $1/\nu$. The solution is as follows:

$$(4.1)\quad f_1 = \sum_{i=1}^{3}\left[A_i I_0(k_i r) + \left\{-\frac{A_i}{k_i} - \frac{\tau A_i}{24\nu k_i^3}(4k_i^2 - \nu^2)\right\}\frac{1}{r} I_1(k_i r)\right],$$

$$(4.2)\quad f_2 = \sum_{i=1}^{3}\left[\frac{-\tau A_i}{24\nu k_i^2}(4k_i^2 - \nu^2) r I_0(k_i r) + \left\{-\frac{A_i}{k_i} - \frac{A_i\tau}{24\nu k_i^3}(4k_i^2 - \nu^2)\right\} I_1(k_i r)\right],$$

$$(4.3)\quad f_3 = \sum_{i=1}^{3}\left[-\frac{A_i}{3\nu k_i}(4k_i^2 - \nu^2 - \tau\nu) I_1(k_i r)\right],$$

where k_i satisfies

$$(4.4)\quad \left[24\nu k_i^2 + \tau(4k_i^2 - \nu^2)\right]\left[k_i^2 - \nu^2 - \nu\tau\right] + 72k_i^4\nu + 24\nu k_i^2(4k_i^2 - \nu^2 - \nu\tau) = 0,$$

and we have retained terms only up to τ^2 in the series for β.

The boundary conditions (3.9) are satisfied, if $T'^{11} = T'^{12} = T'^{13} = 0$ when $r = a$. These give the following three equations which determine A_1, A_2, A_3:

$$(4.5)\quad \sum_{i=1}^{3}\left[P_i A_i I_1(k_i a) + Q_i A_i I_0(k_i a)\right] = 0,$$

$$(4.6)\quad \sum_{i=1}^{3}\left[R_i A_i I_1(k_i a) + S_i A_i I_0(k_i a)\right] = 0,$$

$$(4.7)\quad \sum_{i=1}^{3}\left[L_i A_i I_1(k_i a) + M_i A_i I_0(k_i a)\right] = 0,$$

where

(4.8)

$$P_i = 2k_i + \frac{2}{a^2 k_i}\left\{1 + \frac{\tau(4k_i^2 - \nu^2)}{24\nu k_i^2}\right\} + \frac{1}{3k_i}(4k_i^2 - \nu^2 - 4\tau\nu),$$

$$Q_i = -\frac{1}{a} - \frac{\tau}{8a\nu k_i^2}(4k_i^2 - \nu^2),$$

$$R_i = \frac{1}{k_i}\left[-\frac{\nu}{a}\left\{1 + \frac{\tau(4k_i^2 - \nu^2)}{24\nu k_i^2}\right\} + \frac{1}{3a\nu}\{4k_i^2 - \tau\nu - \nu^2\} - \frac{3\tau}{a}\right],$$

$$S_i = \left[\nu + \tau - \frac{1}{3\nu}(4k_i^2 - \nu^2 - \nu\tau)\right]$$

$$L_i = \frac{1}{k_i}\left[-\frac{\tau a(4k_i^2 - \nu^2)}{24\nu} + \frac{4}{a} + \frac{4\tau(4k_i^2 - \nu^2)}{24a\nu k_i^2}\right],$$

$$M_i = -2.$$

$$(i = 1, 2, 3)$$

In order that an adjacent equilibrium position may exist, it must be possible to find constants A_i, not all zero, which satisfy equations (4.5) to (4.7) simultaneously. This can be done, if and only if

(4.9)

$$\begin{vmatrix} P_1 I_1(k_1 a) + Q_1 I_0(k_1 a) & P_2 I_1(k_2 a) + Q_2 I_0(k_2 a) & P_3 I_1(k_3 a) + Q_3 I_0(k_3 a) \\ R_1 I_1(k_1 a) + S_1 I_0(k_1 a) & R_2 I_1(k_2 a) + S_2 I_0(k_2 a) & R_3 I_1(k_3 a) + S_3 I_0(k_3 a) \\ L_1 I_1(k_1 a) + M_1 I_0(k_1 a) & L_2 I_1(k_2 a) + M_2 I_0(k_2 a) & L_3 I_1(k_3 a) + M_3 I_0(k_3 a) \end{vmatrix} = 0.$$

§ 5. Remarks

The equation (4.9) gives the frequency ν in terms of τ. Its roots are the values of the frequency ν for which the boundary conditions at the curved surface of B' can be satisfied by displacements of the form (3.3'), with $n = 1$ and for given values of τ. It can be solved numerically and the different interesting cases corresponding to the nature of its roots may be discussed in detail.

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ON YIELD SURFACES FOR IDEALLY PLASTIC SHELLS

G. S. SHAPIRO
Moscow, U.S.S.R.

This paper deduces the equation of the yield surface for ideally plastic shells with consideration of the influence of shear forces. It is assumed that the material of the shell obeys the von Mises flow condition. For the case of shells of revolution, subject to axi-symmetric loading, a method is established which permits approximation of the yield surface by polyhedra.

§ 1. Deduction of the equation of the yield surface

In certain cases it is essential in the construction of yield surfaces for plates and shells to consider the influence of shear forces. In contrast, in the case of beams, this is only important when the thickness of the structure is commensurable with its length. Application of concentrated forces causes in plates and shells the occurrence of infinitely large shear forces which the ideally plastic material cannot sustain. In a number of cases consideration of shear forces has significance in dynamic problems. Here infinitely large shear forces may arise also in the case of distributed loading as, for example, in the case of a constant velocity shock [1].

It is necessary to keep in mind that, strictly speaking, when taking into consideration shear forces the equation of the yield surface will not be unique, since it may depend on the boundary conditions, the loading, etc. For beams, this position has been demonstrated by D. C. Drucker [2], and it may be retained for plates and shells.

In order to obtain upper bounds on the load carrying capacity of shells, we will give the kinematically admissible velocity filed in the form

$$\varepsilon_1 = e_1 + k_1 z, \quad \varepsilon_2 = e_2 + k_2 z, \quad \varepsilon_3 = -(\varepsilon_1 + \varepsilon_2), \tag{1.1}$$
$$\varepsilon_{12} = e_{12} + k_{12} z, \quad \varepsilon_{13} = e_{13}, \quad \varepsilon_{23} = e_{23},$$

where e_1, e_2, e_3 and k_1, k_2, k_3 represent the strain rates and the rates of curvature of the middle surface respectively.

Using the von Mises yield condition and the associated flow rule, we may express the forces and moments in the following manner in terms of

the strain and curvature rates:

$$T_1 = \int_{-h/2}^{h/2} \sigma_1\, dz = \frac{\sigma_s}{\sqrt{3}}[(2e_1 + e_2)I_1 + (2k_1 + k_2)I_2],$$

$$T_2 = \int_{-h/2}^{h/2} \sigma_2\, dz = \frac{\sigma_s}{\sqrt{3}}[(e_1 + 2e_2)I_1 + (k_1 + 2k_2)I_2], \tag{1.2}$$

$$T_{12} = \int_{-h/2}^{h/2} \sigma_{12}\, dz = \frac{\sigma_s}{2\sqrt{3}}[e_{12}I_1 + k_{12}I_2];$$

$$Q_{13} = \int_{-h/2}^{h/2} \sigma_{13}\, dz = \frac{\sigma_s e_{13}}{2\sqrt{3}} I_1, \tag{1.3}$$

$$Q_{23} = \int_{-h/2}^{h/2} \sigma_{23}\, dz = \frac{\sigma_s e_{23}}{2\sqrt{3}} I_1;$$

$$M_1 = \int_{-h/2}^{h/2} \sigma_1 z\, dz = \frac{\sigma_s}{\sqrt{3}}[(2e_1 + e_2)I_2 + (2k_1 + k_2)I_3],$$

$$M_2 = \int_{-h/2}^{h/2} \sigma_2 z\, dz = \frac{\sigma_s}{\sqrt{3}}[(e_1 + 2e_2)I_2 + (k_1 + 2k_2)I_3], \tag{1.4}$$

$$M_{12} = \int_{-h/2}^{h/2} \sigma_{12} z\, dz = \frac{\sigma_s}{2\sqrt{3}}[e_{12}I_2 + k_{12}I_3].$$

Here

$$I_1 = \int_{-h/2}^{h/2} \frac{dz}{P}, \quad I_2 = \int_{-h/2}^{h/2} \frac{z\,dz}{P}, \quad I_3 = \int_{-h/2}^{h/2} \frac{z^2 dz}{P}.$$

$$P = P_e + 2zP_{ek} + z^2 P_k, \tag{1.5}$$

$$P_e = e_1^2 + e_2^2 + e_1 e_2 + \tfrac{1}{4}e_{13}^2 + \tfrac{1}{4}e_{23}^2 + \tfrac{1}{4}e_{12}^2,$$

$$P_{ek} = e_1 k_1 + e_2 k_2 + \tfrac{1}{2}e_1 k_2 + \tfrac{1}{2}e_2 k_1 + \tfrac{1}{4}e_{12}k_{12},$$

$$P_k = k_1^2 + k_2^2 + k_1 k_2 + \tfrac{1}{4}k_{12}^2.$$

The expressions (1.2)—(1.4) may be considered as parametric relationships for the determination of the yield surface. Setting the strain rates e_{13} and e_{23} equal to zero in these relations, we arrive at the usual formulae for shells when shear forces are neglected, obtained first by A. A. Il'iushin [3] and deduced recently by a new method by A. Sawczuk and I. Rychlewski [4]. Setting the quantities e_1, e_2, e_{12}, e_{13}, e_{23} and k_{12} equal to zero, we obtain formulae for the axi-symmetric bending of plates. If, in addition, we set k_2 equal to zero, and keep non-zero only e_{13} and k_1, we arrive at the formulae for the bending of beams with consideration of shear forces, deduced by A. R. Rzhanitsyn [5] and independently from him by D. C. Drucker [2] and P. G. Hodge [6].

In order to obtain lower bounds for the carrying capacity of shells we may start from the solution of the known variational problem for a conditional extreme value for holonomic relations. For this purpose one may seek the conditional extreme value for one of the forces (or moments), assuming that the remaining forces and moments maintain given constant values. After the required computations we obtain, for example, for the force T_2

$$(1.6)\quad T_2 = \frac{\sigma_s}{\sqrt{3}} \int_{-h/2}^{h/2} \frac{[\lambda + 2\mu + z(1+2\xi)]dz}{\left[\left(1+\xi+\xi^2+\frac{\alpha^2}{4}\right)z^2 + 2\left(\lambda+\mu\xi+\frac{\mu}{2}+\frac{\xi\lambda}{2}+\frac{\alpha\beta}{4}\right)z + \right.}$$
$$\left. + \left(\lambda^2+\mu^2+\mu\lambda+\frac{\delta^2}{4}+\frac{\gamma^2}{4}+\frac{\beta^2}{4}\right)\right]^{\frac{1}{2}}$$

where λ, μ, ξ, α, β, γ and δ are Lagrange multipliers. It is readily verified that we arrive at (1.6), if we set in the second of the formulae (1.2)

$$\frac{e_1}{k_1} = \lambda, \quad \frac{e_2}{k_1} = \mu, \quad \frac{k_2}{k_1} = \xi, \quad \frac{k_{12}}{k_1} = \alpha,$$

$$\frac{e_{12}}{k_1} = \beta, \quad \frac{e_{32}}{k_1} = \gamma, \quad \frac{e_{13}}{k_1} = \delta.$$

An analogous agreement is also obtained for the remaining formulae (1.2)—(1.4). Since the upper and lower bounds for the load carrying capacity coincide, we have obtained the exact solution of the problem. The above method for obtaining lower bounds for the load carrying capacity was applied to beams by P. F. Papkovich [7], A. R. Rzhanitsyn [5] and P. G. Hodge [6], and to shells without consideration of shear forces by V. V. Rozhdectvenskii [8].

§ 2. Approximation of the yield surface by a polyhedron

The yield surface, determined by the formulae (1.2)—(1.4), has very complicated outlines and is inconvenient for the solution of concrete problems. One may achieve a significant simplification, if one assumes, for example, following V. I. Rozenblium [9], that the stresses vary linearly through the thickness of the shell and that the von Mises yield criterion is not satisfied at each point, but on an average through the thickness of the shell. In this case the dimensionless yield condition for shells of revolution under axi-symmetric loading will be

$$\delta_m^2 + \delta_n^2 + \delta_q^2 = 1, \tag{2.1}$$

where

$$\delta_m^2 = m_1^2 - m_1 m_2 + m_2^2, \tag{2.2}$$

$$\delta_n^2 = n_1^2 - n_1 n_2 + n_2^2, \tag{2.3}$$

$$\delta_q^2 = 3q_{13}^2. \tag{2.4}$$

The equation (2.1) is the equation of a sphere and the equations (2.2) and (2.3) are the equations of ellipses. As has been stated in [9], one may consider inscribed or circumscribed hexagons in place of the ellipses. Then the yield surface in the force and moment space will be described by a polyhedron.

When shear forces are absent, this method was applied by the author [10] and quite recently by V. I. Rozenblium [11]; the yield polyhedron for cylindrical shells, corresponding to the approximation of a circle by a square, which was obtained in [10] has been found in another manner by D. C. Drucker and R. T. Shield [12]. It is of interest to note that a yield polyhedron of that shape is obtained also in the case of axi-symmetric bending of plates with consideration of shear forces.

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THE SOLUTION OF THE EQUATIONS OF CERTAIN CONTACT PROBLEMS OF THE THEORY OF ELASTICITY (EQUATIONS OF THE PRANDTL TYPE)

M. P. SHEREMET'EV
L'vov, U.S.S.R.

The paper [1] deals with certain contact problems for the infinite plate in which a circular hole has been filled with an absolutely rigid or an elastic disc. The problems were solved under the assumption that the shear stresses vanish along the entire contour of the hole in the plate.

In particular, if there acts on an elastic disc a concentrated force applied to the centre of the disc, the equation for the determination of the normal pressure P will have the form

$$(1)\quad \frac{1}{\pi i}\int_{\Gamma_1}\frac{\widehat{rr}'(t)dt}{t-t_0}=\frac{2[(\nu-1)E_1-(\nu_1-1)E]}{(1+\kappa)(\nu+1)E_1+(1+\kappa_1)(\nu_1+1)E}\,\frac{\widehat{rr}}{t_0}-$$

$$-\frac{4[\kappa(1+\kappa_1)E_1-(1+\kappa)E]+(1+\kappa)(1+\kappa_1)[\kappa(1+\nu)E_1+3E(1+\nu_1)]}{2(1+\kappa)(1+\kappa_1)[(1+\kappa)(1+\nu)E_1+E(1+\nu_1)(1+\kappa_1)]}\cdot$$

$$\cdot\frac{P}{\pi}\left(\frac{1}{R^2}+\frac{1}{t_0^2}\right)-\frac{4E_1b_1}{R^2[(1+\kappa)(1+\nu)E_1+(1+\kappa_1)(1+\nu_1)E]}\,\frac{1}{t_0}.$$

We will write this equation in the form

$$\frac{1}{\pi i}\int_{\Gamma_1}\frac{\widehat{rr}'(t)dt}{t-t_0}=-k\frac{\widehat{rr}}{t_0}-f(t_0),$$

where

$$f(t_0)=\frac{hP}{\pi}\left(\frac{1}{R^2}+\frac{1}{t_0^2}\right)+\frac{g}{t_0}.$$

The coefficients k, h and g in the case $\kappa=(3-\nu)/(1+\nu)$ are easily brought into the form

$$(2)\quad k=\frac{(1-\nu)E_1-(1-\nu_1)E}{2(E_1+E)},$$

$$h=\frac{\kappa(1+\kappa_1)E_1+(1+\kappa)E}{(1+\kappa)(1+\kappa_1)(E_1+E)}=\frac{\kappa(1+\nu)E_1+(1+\nu_1)E}{4(E_1+E)},$$

$$g=-\frac{E_1}{E_1+E}\,\frac{1}{2\pi i}\int_{\Gamma_1}\frac{\widehat{rr}dt}{t}=\frac{E_1}{E_1+E}\,\frac{b}{R^2},$$

where E_1, ν_1 are the elastic constants of the disc, E, ν are the elastic constants of the plate and R is the radius of the hole in the plate.[1]

As $E \to \infty$ the numerical values of the coefficients k, h and g coincide, for the same values of κ, with the values of these coefficients, presented in the equation (3.12) of [1]. If, in (2), one replaces E_1, ν_1, E, ν, κ, by other elastic constants E_1^*, ν_1^*, E^*, ν^*, κ^*, related to the original constants by the equations

$$E_1^* = \frac{E_1}{1-\nu_1^2}, \quad E^* = \frac{E}{1-\nu^2},$$

$$\nu_1^* = \frac{\nu_1}{1-\nu_1}, \quad \nu^* = \frac{\nu}{1-\nu},$$

$$\kappa_1^* = 3 - 4\nu_1, \quad \kappa^* = 3 - 4\nu,$$

the equation (1) (and also the equations (4.13) and (2.21) of [1]) will be true for a state of plane strain. In the paper [4], A. I. Kalandiia presented a somewhat different deduction of the equation (1).

We will now introduce the function

$$z = R\frac{b\zeta - 1}{\zeta - b}, \tag{3}$$

where

$$b = \frac{\cos\varphi}{1 + \sin\varphi}.$$

This function maps conformally the region bounded by the unit circle γ in the ζ plane onto the outside of the circle with radius R in the z plane, so that the points $\zeta_1 = i$, $\zeta_2 = -1$ and $\zeta_3 = -i$ become the points $z_1 = Re^{i\varphi}$, $z_2 = R$ and $z_3 = Re^{-i\varphi}$.

The boundary condition (2.3) of [1], as the result of the transformation (3), assumes the form

$$\Phi_0^*(\sigma, b) + \overline{\Phi_0^*(\sigma, b)} - \frac{P}{2\pi R}\left(\frac{b\sigma - 1}{\sigma - b} + \frac{\sigma - b}{b\sigma - 1}\right) - \frac{b_1}{R^2} = \begin{cases} \sigma_r \text{ on } \gamma_1, \\ 0 \text{ on } \gamma_2, \end{cases} \tag{4}$$

where γ_1 and γ_2 are the halves of the circumference of the circle γ. If b is considered to be a parameter, the function

$$\Phi_0^*(\zeta, b) = \Phi_0\left(\frac{R(b\zeta - 1)}{\zeta - b}\right)$$

[1] The expression for the coefficient h, deduced in [1, formula (4.11)], contains a misprint.

will be analytic in ζ as well as in b. This is seen from the fact that, by (3), the variable z is an analytic function of ζ and b, and the function

$$\Phi_0(z) = \sum_{\nu=1}^{\infty} a_\nu z^{-(\nu+1)}$$

is an analytic function of z.

Proceeding analogously as was done in [1], we obtain from the boundary condition (4) for the determination of σ_r the equation

$$(5) \qquad \frac{1}{\pi i}\int_{\gamma_1}\frac{\sigma_r' d\sigma}{\sigma-\sigma_0} = (1-b^2)\left\{\frac{k\sigma_r}{(\sigma_0-b)(b\sigma_0-1)} + \right.$$

$$\left. + \frac{hP}{\pi R}\left[\frac{1}{(b\sigma_0-1)^2}+\frac{1}{(\sigma_0-b)^2}\right] + \frac{g}{(\sigma_0-b)(b\sigma_0-1)}\right\}.$$

This same equation is obtained from the equation (1), if one introduces there the substitution (3); in fact,

$$\frac{1}{\pi i}\int_{\Gamma_1}\frac{\widehat{rr'}(t)dt}{t-t_0} = -\frac{\sigma_0-b}{i\pi R(1-b^2)}\int_{\gamma_1}\frac{(\sigma-b)\sigma_r' d\sigma}{\sigma-\sigma_0} =$$

$$= -\frac{\sigma_0-b}{i\pi R(1-b^2)}\int_{\gamma_1}\sigma_r' d\sigma - \frac{(\sigma_0-b)^2}{i\pi R(1-b^2)}\int_{\gamma_1}\frac{\sigma_r' d\sigma}{\sigma-\sigma_0}.$$

Since σ_r vanishes at the end γ_1, one has

$$\int_{\gamma_1}\sigma_r' d\sigma = 0.$$

Thus,

$$(6) \qquad \frac{1}{\pi i}\int_{\Gamma_1}\frac{\widehat{rr'}(t)dt}{t-t_0} = -\frac{(\sigma_0-b)^2}{i\pi R(1-b_2^2)}\int_{\gamma_1}\frac{\sigma_r' d\sigma}{\sigma-\sigma_0}.$$

Substituting now the corresponding expressions from (3) and (6) in the equation (1) for t and the integral and executing certain obvious transformations, we obtain the equation (5). It follows from the equality (4) that the normal stress σ_r may be represented in the form of the series

$$\sigma_r = \sum_{\nu=0}^{\infty} b^\nu \sigma_{r,\nu}.$$

This follows from the fact that the left hand side of (4) may be expanded

in a series of powers of b. Then

$$\Phi_0^*(\sigma, b) = \sum_{\nu=0}^{\infty} b^\nu \Phi_0^{(\nu)}(\sigma),$$

and

$$\Phi_0^{(0)}(\sigma) + \Phi_0^{(0)}\left(\frac{1}{\sigma}\right) + \frac{P}{2\pi R}\left(\sigma + \frac{1}{\sigma}\right) - \frac{b_{10}}{R^2} = \begin{cases} \sigma_{r,0} & \text{on } \gamma_1, \\ 0 & \text{on } \gamma_2. \end{cases} \tag{7}$$

The conjugate sign on $\Phi_0^{(0)}$ has been omitted, because the coefficients of these functions will be real.

The function $\Phi_0^{(0)}(\sigma)$ may be represented by the series

$$\Phi_0^{(0)}(\sigma) = \sum_{\nu=1}^{\infty} a_\nu' \sigma^{\nu+1}. \tag{8}$$

In fact, for $b = 0$,

$$\Phi_0^{(0)}(\sigma) = \Phi_0^*(\sigma, b),$$

while

$$\Phi_0^*(\sigma, b) = \Phi_0(z) = \sum_{\nu=1}^{\infty} a_\nu \left[\frac{\sigma - b}{R(b\sigma - 1)}\right]^{\nu+1}.$$

Setting $b = 0$ in the last equality, we find

$$\Phi_0^{(0)}(\sigma) = \sum_{\nu=1}^{\infty} a_\nu \sigma^{\nu+1} R^{-(\nu+1)}(-1)^{\nu+1}. \tag{9}$$

From the equality (9) follows the correctness of the expansion of the function $\Phi_0^{(0)}(\sigma)$ in the series (8), where

$$a_\nu = a_\nu' R^{\nu+1}(-1)^{\nu+1}. \tag{10}$$

In order to determine the coefficients of the expansion of the function $\Phi_0^{(0)}(z)$, one must therefore find the coefficients of the function $\Phi_0^{(0)}(\sigma)$, and then use the formula (10). We obtain the equation for the determination of the function $\Phi_0^{(0)}(\zeta)$ from the equality (5) by setting $b = 0$ in it:

$$\frac{1}{\pi i}\int_{\gamma_1} \frac{\sigma_{r,0}' d\sigma}{\sigma - \sigma_0} = \frac{k\sigma_{r,0}(\sigma_0)}{\sigma_0} + \frac{hP}{\pi R}\left(1 + \frac{1}{\sigma_0^2}\right) - \frac{g_0}{\sigma_0}, \tag{11}$$

where

$$g_0 = \frac{E_1}{E + E_1}\frac{b_{10}}{R^2} = \frac{E_1}{E_1 + E}\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \sigma_{r0}\, d\vartheta.$$

The equation (11) may be solved by the method of successive approximations, described in [5], representing $\sigma_{r,0}$ in the form of the series

$$\sigma_{r,0} = \sum_{\nu=0}^{\infty} k^{\nu} \sigma_{r,0}^{(\nu)}.$$

We believe that the calculations will be simpler, if the solution of the problem is reduced to the solution of an infinite system of equations by a method different from that employed in [2], where also an equation of the Prandtl type had to be solved with a known region of integration, obtained for the study of another problem.

As follows from the equality (7), the function $\sigma_{r,0}$ may be represented on γ_1 by the convergent series

$$\sigma_{r,0} = \sum_{\nu=1}^{\infty} a_{\nu}'[\sigma^{\nu+1} + \sigma^{-(\nu+1)}] + \frac{P}{2\pi R}\left(\sigma + \frac{1}{\sigma}\right) - \frac{b_{10}}{R^2}. \tag{12}$$

We will consider the function

$$\Omega(\zeta) = \frac{1}{2\pi i}\int_{\gamma_1} \frac{\sigma_{r,0}' d\sigma}{\sigma - \zeta}$$

and represent it in the form [3]

$$\frac{1}{2\pi i}\int_{\gamma_1} \frac{\sigma_{r,0}' d\sigma}{\sigma - \zeta} = \frac{1}{\sqrt{1+\zeta^2}} \frac{1}{2\pi i}\int_{\gamma_1} \frac{\omega(\sigma)\sqrt{1+\sigma^2}d\sigma}{\sigma - \zeta} + \frac{d}{2\sqrt{1+\zeta^2}}, \tag{13}$$

where

$$\omega(\sigma) = -\frac{k}{\sigma}\sigma_{r,0} + \frac{hP}{\pi R}\left(1 + \frac{1}{\sigma^2}\right) - g_0 \frac{1}{\sigma_0}. \tag{14}$$

On the other hand, it follows from the equality (4) that

$$\frac{1}{2\pi i}\int_{\gamma_1} \frac{\sigma_r d\sigma}{\sigma - \zeta} = \Phi_0^*(\zeta, b) - \frac{P}{2\pi R}\left(b + \frac{\zeta + b}{b\zeta - 1}\right) + \frac{b_1}{R^2} + \bar{a}_0, \tag{15}$$

where

$$a_0 = \Phi_0^*(0, b).$$

Differentiating (15) with respect to ζ and taking into consideration that $\sigma_r(i) = \sigma_r(-i)$, we obtain

$$\frac{1}{2\pi i}\int_{\gamma_1} \frac{\sigma_r'(\sigma)d\sigma}{\sigma - \zeta} = \Phi_0^{*\prime}(\zeta, b) - \frac{P}{2\pi R}\frac{b^2 - 1}{(b\zeta - 1)^2}. \tag{16}$$

Setting in (15) the quantity $b = 0$, we find that

$$\frac{1}{2\pi i}\int_{\gamma_1}\frac{\sigma'_{r,0}(\sigma)d\sigma}{\sigma-\zeta} = \Phi_0^{(0)\prime}(\zeta) + \frac{P}{2\pi R}. \tag{17}$$

Thus, it follows from (13) and (17) that

$$\Phi_0^{(0)}(\zeta) + \frac{P}{2\pi R} = \frac{1}{\sqrt{1+\zeta^2}}\,\frac{1}{2\pi i}\int_{\gamma_1}\frac{\omega(\sigma)\sqrt{1+\sigma^2}d\sigma}{\sigma-\zeta} + \frac{A}{\sqrt{1+\zeta^2}}. \tag{18}$$

We will now proceed to the evaluation of the integral on the right hand side of (18), taking into consideration for this purpose the equalities (12) and (14). We find

$$\begin{aligned}\frac{1}{2\pi i}\int_{\gamma_1}\frac{\omega(\sigma)\sqrt{1+\sigma^2}d\sigma}{\sigma-\zeta} &= \left(\frac{kb_{10}}{R^2} - g_0\right)\frac{1}{2\pi i}\int_{\gamma_1}\frac{\sqrt{1+\sigma^2}d\sigma}{\sigma(\sigma-\zeta)} + \\ &+ \left(\frac{hP}{\pi R} - \frac{kP}{2\pi R}\right)\frac{1}{2\pi i}\int_{\gamma_1}\frac{(1+\sigma^{-2})\sqrt{1+\sigma^2}d\sigma}{\sigma-\zeta} - \\ &- \frac{k}{2\pi i}\int_{\gamma_1}\frac{\sum_{\alpha=1}^{\infty} a_\alpha'[\sigma^2+\sigma^{-(\alpha+2)}]\sqrt{1+\sigma^2}d\sigma}{\sigma-\zeta}.\end{aligned} \tag{19}$$

If we substitute the values of the constants g_0, k, h on the right hand side of (19), we obtain

$$\frac{kb_{10}}{R^2} - g_0 = -\frac{(1+\nu)E_1 + (1-\nu_1)E}{2(E_1+E)}\,\frac{b_{10}}{R^2} = c_0, \tag{20}$$

$$\frac{hP}{\pi R} - \frac{kP}{2\pi R} = \frac{P}{2\pi R}. \tag{21}$$

Consequently,

$$\begin{aligned}\frac{1}{2\pi i}\int_{\Gamma_1}\frac{\omega(\sigma)\sqrt{1+\sigma^2}d\sigma}{\sigma-\zeta} &= \\ = c_0\frac{1}{2\pi i}\int_{\gamma_1}\frac{\sqrt{1+\sigma^2}d\sigma}{\sigma(\sigma-\zeta)} + \frac{P}{2\pi R}\,\frac{1}{2\pi i}\int_{\gamma_1}\frac{(1+\sigma^{-2})\sqrt{1+\sigma^2}d\sigma}{\sigma-\zeta} - \\ - \frac{k}{2\pi i}\int_{\gamma_1}\frac{\sum_{\alpha=1}^{\infty} a_\alpha'[\sigma^\alpha+\sigma^{-(\alpha+2)}]\sqrt{1+\sigma^2}d\sigma}{\sigma-\zeta}.\end{aligned} \tag{22}$$

The integrals on the right hand side of (22) are readily evaluated, if one takes as line of integration a contour Δ, enveloping the arc γ_1 so closely that z is outside this contour and uses the well known properties of Cauchy type integrals.

For the evaluation of the integrals on the right hand side of (22) we will choose that branch of the function $\sqrt{1+\zeta^2}$ which is determined near the point at infinity by the series

$$(23)\qquad \sqrt{1+\zeta^2} = \zeta\left(1 + \frac{1}{2}\frac{1}{\zeta^2} - \frac{1}{2\cdot 4}\frac{1}{\zeta^4} + \frac{1\cdot 3}{2\cdot 4\cdot 6}\frac{1}{\zeta^6} - \right.$$

$$\left. - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}\frac{1}{\zeta^8} + \cdots\right) = \sum_{\nu=0}^{\infty} d_{2\nu}\zeta^{1-2\nu}.$$

Then

$$(24)\qquad \frac{1}{2\pi i}\int_{\gamma_1} \frac{\sqrt{1+\sigma^2}d\sigma}{\sigma(\sigma-\zeta)} = \frac{1}{2}\left[\frac{1}{\zeta}\sqrt{1+\zeta^2} - \frac{1}{\zeta} - 1\right],$$

$$(25)\qquad \frac{1}{2\pi i}\int_{\gamma_1} \frac{(1+\sigma^{-2})\sqrt{1+\sigma^2}d\sigma}{\sigma-\zeta} = \frac{1}{2}\left[\left(1+\frac{1}{\zeta^2}\right)\sqrt{1+\zeta^2} - \zeta - \frac{1}{\zeta^2}\right].$$

For the calculation of the last integral on the right hand side of the equation (22) we consider the integral

$$(26)\qquad F_\alpha(\zeta) = -\frac{ka_\alpha'}{2\pi i}\int_{\gamma_1} \frac{\sigma^\alpha\sqrt{1+\sigma^2}d\sigma}{\sigma-\zeta} =$$

$$= -\frac{ka_\alpha'}{2}\left[\zeta^\alpha\sqrt{1+\zeta^2} - \sum_{\gamma=0}^{1+\alpha}{}^{*} d_{1+\alpha-\gamma}\zeta^\gamma\right],$$

where $d_{1+\alpha-\gamma}$ are the coefficients of the expansion (23). The asterisk on the sum in (26) indicates that in this sum the coefficients with odd indices are equal to zero.

Further, we will consider the function

$$(27)\qquad \frac{F_\alpha(\zeta)}{\sqrt{1+\zeta^2}} = -\frac{ka_\alpha'}{2}\left[\zeta^\alpha\frac{1}{\sqrt{1+\zeta^2}}\sum_{\gamma=0}^{1+\alpha}{}^{*} d_{1+\alpha-\gamma}\zeta^\gamma\right].$$

We expand the right hand side of (27) in a series about the origin

$$\frac{F_\alpha(\zeta)}{\sqrt{1+\zeta^2}} = -\frac{ka_\alpha'}{2}\left[\zeta^\alpha - \sum_{\beta=0}^{\infty}\left(\sum_{\gamma=0}^{\beta} d^{*}_{\beta-\gamma}d_{1+\alpha-\gamma}\right)\zeta^\beta\right],$$

where $d^*_{\beta-\gamma}$ are the coefficients in the expansion about the origin of the function

$$(28)\qquad \frac{1}{\sqrt{1+\zeta^2}} = \left(1 - \tfrac{1}{2}\zeta^2 + \frac{1\cdot 3}{2\cdot 4}\zeta^4 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\zeta^6 + \right.$$

$$\left. + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\zeta^8 + \cdots\right) = \sum_{\nu=0}^{\infty} d^*_{2\nu}\zeta^{2\nu}.$$

We will now consider the integral

$$F_\alpha^*(\zeta) = -\frac{ka_\alpha'}{2\pi i}\int_{\gamma_1} \frac{\sigma^{-(\alpha+2)}\sqrt{1+\sigma^2}d\sigma}{\sigma-\zeta} =$$

$$= -\frac{ka_\alpha'}{2}\left[\zeta^{-(\alpha+2)}\sqrt{1+\zeta^2} - \sum_{\gamma=1}^{\alpha+2}{}^{*} d_{\alpha+2-\gamma}\zeta^{-\gamma}\right].$$

We expand the function $(1+\zeta^2)^{-\frac{1}{2}}F_\alpha^*(\zeta)$ about zero. For this purpose one must only evaluate the coefficients of the positive powers of ζ in the products

$$\left(\sum_{\nu=0}^{\infty} d^*_{2\nu}\zeta^{2\nu}\right)\left(\sum_{\gamma=1}^{\alpha+2} d_{\alpha+2-\gamma}\zeta^{-\gamma}\right) = \sum_{\beta=0}^{\infty}\left(\sum_{\nu=1}^{[(\alpha+\beta+2)/2]} d^*_{2\nu}d_{\alpha+2+\beta-2\nu}\right)\zeta^\beta.$$

Consequently

$$\frac{F_\alpha^*(\zeta)}{\sqrt{1+\zeta^2}} = \frac{ka_\alpha'}{2}\sum_{\beta=0}^{\infty}\left(\sum_{\nu=1}^{[(\alpha+\beta+2)/2]} d^*_{2\nu}d_{\alpha+2+\beta-2\nu}\right)\zeta^\beta.$$

After these preliminary calculations the last integral on the right hand side of (22) may be represented in the form

$$-\frac{k}{2\pi i}\int_{\gamma_1} \frac{\sum_{\alpha=1}^{\infty} a_\alpha'[\sigma^\alpha + \sigma^{-(\alpha+2)}]\sqrt{1+\sigma^2}d\sigma}{\sigma-\zeta} =$$

$$= -\frac{k}{2}\sum_{\alpha=1}^{\infty} a_\alpha'\zeta^\alpha + \frac{k}{2}\sum_{\beta=1}^{\infty}\left[\sum_{\alpha=1}^{\infty}(c_{\alpha\beta} + c^*_{\alpha\beta})a_\alpha'\right]\zeta^\beta,$$

where

$$c^*_{\alpha\beta} = \sum_{\gamma=0}^{\beta} d^*_{\beta-\gamma}d_{1+\alpha-\beta},\quad c_{\alpha\beta} = \sum_{\nu=1}^{[(\alpha+\beta+2)/2]} d^*_{2\nu}d_{\alpha+\beta+2-2\nu}.$$

As a result of these calculations the equality (18) may be written in

the form

$$(29)\quad 2\Phi_0^{(0)\prime}(\zeta)+k\sum_{\alpha=1}^{\infty}a_\alpha'\zeta^\alpha=c_0\left[\frac{1}{\zeta}-\left(1+\frac{1}{\zeta}\right)\frac{1}{\sqrt{1+\zeta^2}}\right]+$$
$$+\frac{P}{2\pi R}\left[1+\frac{1}{\zeta^2}-\left(\zeta+\frac{1}{\zeta^2}\right)\frac{1}{\sqrt{1+\zeta^2}}\right]-$$
$$-\frac{P}{\pi R}+k\sum_{\beta=0}^{\infty}\left[\sum_{\alpha=1}^{\infty}(c_{\alpha\beta}+c_{\alpha\beta}^*)a_\alpha'\right]\zeta^\beta+\frac{A}{\sqrt{1+\zeta^2}}.$$

Having in mind the expansions (8) and (28), we find from (29) after obvious transformations that

$$(30)\quad \sum_{\beta=1}^{\infty}[2(1+\beta)+k]a_\beta'\zeta^\beta=$$
$$=-\frac{P}{2\pi R}+(2A-c_0)\sum_{\beta=0}^{\infty}d_{2\beta}^*\zeta^{2\beta}-c_0\sum_{\beta=1}^{\infty}d_{2\beta}^*\zeta^{2\beta-1}-$$
$$-\frac{P}{2\pi R}\left[\sum_{\beta=0}^{\infty}d_{2\beta}\zeta^{2\beta+1}+\sum_{\beta=1}^{\infty}d_{2\beta}^*\zeta^{2\beta-1}\right]+$$
$$+k\sum_{\beta=0}^{\infty}\left[\sum_{\alpha=1}^{\infty}(c_{\alpha\beta}+c_{\alpha\beta}^*)a_\alpha'\right]\zeta^\beta.$$

Comparing in (30) the coefficients of equal powers of ζ, we obtain for the determination of the a_α' the infinite system of equations

$$(31)\quad \begin{aligned}
&k\sum_{\alpha=1}^{\infty}(c_{\alpha0}+c_{\alpha0}^*)a_\alpha'=\frac{P}{4\pi R}+(c_0-2A),\\
&\cdots\cdots\cdots

and the system (31) assumes a very simple form. In this case

$$2A - c_0 = \frac{P}{4\pi R},$$

and it follows from (31) that

$$a'_{2m} = \frac{P}{8\pi R} \frac{(2m-1)!!}{(2m)!!} \frac{3m+2}{(2m+1)(m+1)} (-1)^m,$$

where $(2m-1)!! = 1\cdot 3\cdot 5\cdot\cdots\cdot(2m-1)$, $2m)!! \equiv 2\cdot 4\cdot 6\cdot\cdots\cdot(2m)$. For the case $k = 0$,

$$g_0 = \frac{1}{2}\frac{b_{10}}{R^2} = -c_0 = \frac{1}{4\pi}\int_{-\pi/2}^{\pi/2} \sigma_{r,0}\, d\vartheta. \tag{32}$$

We will now substitute in (32) for $\sigma_{r,0}$ its value (12). As a result of the integration we will have

$$\tfrac{3}{2}g_0 = \frac{1}{\pi}\left\{\frac{P}{2\pi R} + \frac{P}{8\pi R}\sum_{m=1}^{\infty}\frac{(2m-1)!!}{(2m)!!}\left[\frac{1}{(2m+1)^2} + \frac{2}{2m+1} - \frac{1}{m+1}\right]\right\}. \tag{33}$$

The series (33) is readily summed. In fact,

$$\sum_{m=1}^{\infty}\frac{(2m-1)!!}{(2m)!!}\frac{1}{(2m+1)^2} = \int_0^1 \frac{\arcsin x}{x}\,dx - 1 = \frac{\pi}{2}\log 2 - 1, \tag{34}$$

$$\sum_{m=1}^{\infty}\frac{2(m-1)!!}{(2m)!!}\frac{1}{2m+1} = \arcsin 1 - 1 = \frac{\pi}{2} - 1. \tag{35}$$

For the summation of the series

$$\sum_{m=1}^{\infty}\frac{(2m-1)!!}{(2m)!!}\frac{1}{m+1}$$

we will use the known formula

$$B(p, q) = \frac{\Gamma(p)\cdot\Gamma(q)}{\Gamma(p+q)}.$$

Then

$$\frac{(2m-1)!!}{(2m)!!} = \frac{\Gamma(m+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(m+1)} = \frac{1}{\pi}B(m+\tfrac{1}{2}, \tfrac{1}{2}) = \tag{36}$$

$$= \frac{1}{\pi}\int_0^1 x^{m+1}x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}\,dx.$$

It follows from this equation that

$$(37)\qquad \sum_{m=1}^{\infty} \frac{2(m-1)!!}{(2m)!!}\,\frac{1}{m+1} = \frac{1}{\pi}\sum_{m=1}^{\infty}\frac{1}{m+1}\int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}x^{m+1}\,dx.$$

Interchanging the summation and the integral sign on the right hand side of this equality and taking into consideration that

$$\Sigma\frac{x^{m+1}}{m+1} = -x - \log(1-x),$$

we obtain

$$(38)\qquad \sum_{m=1}^{\infty}\frac{(2m-1)!!}{(2m)!!}\,\frac{1}{m+1} = -\frac{1}{\pi}\int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}[x+\log(1-x)]dx = 1.$$

Substituting in (38) the values of the sums (34), (35) and (38), we find

$$(39)\qquad g_0 = \frac{P}{24\pi R}(2+\log 2) = -c_0.$$

In this case we obtain for the constant A

$$(40)\qquad A = \frac{P(4-\log 2)}{48\pi R}.$$

From the system (31) we find

$$(41)\qquad a'_{2m-1} = \frac{P}{2\pi R}\,\frac{(2m-1)!!}{2(m-1)!!}\,\frac{2m(14+\log 2)-(2+\log 2)}{2\cdot 48m^2}(-1)^m.$$

Once the coefficients a'_{2m-1} and a'_{2m} are known, we find from the formulae (10) the coefficients a_r:

$$(C)\qquad a_{2m-1} = a'_{2m-1}R^{2m},$$
$$a_{2m} = a'_{2m}R^{2m+1}(-1)^{2m+1}.$$

The condition for the determination of $\widehat{rr}(t)$ in terms of the function $\Phi_0(z)$ will have the form

$$(42)\qquad \Phi_0(t) + \overline{\Phi_0(t)} - \frac{P}{2\pi}\left(\frac{t}{R^2}+\frac{1}{t}\right)\frac{b_1}{R^2} = \begin{cases}\widehat{rr} & \text{on } \Gamma_1,\\ 0 & \text{on } \Gamma_2.\end{cases}$$

We can determine from this equality the quantity $\widehat{rr}$ at any point of

Γ_1 only in the case when the constant

$$\frac{b_1}{R^2} = -\frac{1}{2\pi i}\int_{\Gamma_1}\frac{\widehat{rr}dt}{t} \tag{43}$$

is known as well as the angle φ subtended by the arc Γ_1 which must satisfy the condition

$$\int_{\Gamma_1}\widehat{rr}dt = -iP = -\int_{\Gamma_1} t\widehat{rr}'(t)\,dt. \tag{44}$$

We will turn for the determination of the subtended angle φ to the equation (1). We consider the function

$$\Omega(z) = \frac{1}{2\pi i}\int_{\Gamma_1}\frac{\widehat{rr}'(t)dt}{t-z},$$

which, as is known, may be represented in the form [3]

$$\Omega(z) = \frac{1}{\sqrt{(z-\alpha)(z-\beta)}}\,\frac{1}{2\pi i}\int_{\Gamma_1}\frac{\omega(t)\sqrt{(t-\alpha)(t-\beta)}dt}{t-z} + \frac{d}{2\sqrt{(z-\alpha)(z-\beta)}}, \tag{45}$$

where α, β are the coordinates of the ends of the arc Γ_1 and

$$\omega(t) = -\frac{k\widehat{rr}}{t} - \frac{hP}{\pi i}\left(\frac{1}{R^2}+\frac{1}{t^2}\right) - \frac{g}{t}.$$

On the other hand, one obtains readily from the equality (4) that [1]

$$\frac{1}{2\pi i}\int_{\Gamma_1}\frac{\widehat{rr}'(t)dt}{t-z} = -\frac{P}{2\pi}\,\frac{1}{z^2} - \Phi_0'(z), \quad |z| > R, \tag{46}$$

$$\frac{1}{2\pi i}\int_{\Gamma_1}\frac{\widehat{rr}'(t)dt}{t-z} = -\bar{\Phi}_0'\left(\frac{R^2}{z}\right)\frac{R^2}{z^2} - \frac{P}{2\pi R^2}, \quad |z| < R. \tag{47}$$

It follows from (46), (47) that

$$-\frac{P}{2\pi}\,\frac{1}{z^2} - \Phi_0'(z) = \frac{1}{\sqrt{(z-\alpha)(z-\beta)}}\,\frac{1}{2\pi i}\int_{\Gamma_1}\frac{\omega(t)\sqrt{(t-\alpha)(t-\beta)}dt}{t-z} + \frac{d}{2\sqrt{(z-\alpha)(z-\beta)}}. \tag{48}$$

We will expand the right and left hand sides of (48) about the point at infinity. We will assume that for $|z| > R$

$$\frac{1}{\sqrt{(z-\alpha)(z-\beta)}} = \frac{1}{z} + \frac{\alpha+\beta}{2}\frac{1}{z^2} + \tag{49}$$

$$+ \frac{3(\alpha^2+\beta^2)+2\alpha\beta}{8}\frac{1}{z^3} + \cdots = \sum_{\nu=1}^{\infty} d_\nu' z^{-\nu},$$

where $d_\nu' = P_{\nu-1}R^{\nu-1}$. Then

$$\frac{1}{\sqrt{(z-\alpha)(z-\beta)}} = \sum_{\nu=1}^{\infty} P_{\nu-1}R^{\nu-1}z^{-\nu}, \tag{50}$$

where the P_ν are Legendre polynomials. Further,

$$\frac{1}{\sqrt{(z-\alpha)(z-\beta)}}\frac{1}{2\pi i}\int_{\Gamma_1}\frac{\omega(t)\sqrt{(t-\alpha)(t-\beta)}}{t-z}\,dt =$$

$$= \sum_{\nu=1}^{\infty} d_\nu' z^{-\nu}\sum_{k=1}^{\infty} b_k' z^{-k} = \sum_{s=2}^{\infty}\Big(\sum_{k=1}^{s-1} d_{s-k}' b_k'\Big)z^{-s},$$

where the coefficients

$$b_k' = -\frac{1}{2\pi i}\int_{\Gamma_1}\frac{\omega(t)\sqrt{(t-\alpha)(t-\beta)}}{t^{1-k}}\,dt$$

are unknown numbers, since the function

$$\omega(t) = -\frac{k\widehat{rr}}{t} + f(t)$$

is unknown.

It follows from the equality (48) that $d = 0$; this is easily verified, if one compares in (48) the coefficients of the first negative power of z. Thus the formula (48) may be written in the form

$$\sum_{\nu=1}^{\infty}(\nu+1)a_\nu z^{-(\nu+2)} - \frac{P}{2\pi}\frac{1}{z^2} = \sum_{s=2}^{\infty}\Big(\sum_{k=1}^{s-1} d_{s-k}' b_k'\Big)z^{-s}. \tag{51}$$

Comparing in (51) coefficients of equal powers of z, we obtain

$$b_1' = -\frac{P}{\pi},$$

$$d_2'b_1' + b_2' = 2a_1, \tag{52}$$

$$d_3'b_1' + d_2'b_2' + b_3' = 3a_2,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$d_n'b_1' + d_{n-1}b_2' + \cdots + b_n' = na_{n-1},$$

$$b_n' = na_{n-1} - \sum_{k=2}^{n} d_k'b_{n+1-k}'. \tag{52}$$

Next, we will write down the infinite system of equations which is obtained as the result of comparison of the coefficients in the expansions about zero of the equality

$$-\Phi_0'\left(\frac{R^2}{z}\right)\frac{R^2}{z^2} - \frac{P}{2\pi R^2} = \tag{53}$$

$$= \frac{1}{\sqrt{(z-\alpha)(z-\beta)}}\,\frac{1}{2\pi i}\int_{\Gamma_1}\frac{\omega(t)\sqrt{(t-\alpha)(t-\beta)}}{t-z}$$

which follows from the relations (45) and (47).

This system has the form

$$d_0^*b_0^* = -\frac{P}{2\pi R^2}, \quad b_0^* = \frac{P}{2\pi R},$$

$$\frac{2a_1}{R^4} = d_1^*b_0^* + d_0^*b_1^*, \tag{54}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{na_{n-1}}{R^{2n}} = d_{n-1}^*b_0^* + \cdots + d_0^*b_{n-1}^*,$$

where the unknown numbers

$$b_n^* = \frac{1}{2\pi i}\int_{\Gamma_1}\frac{\omega(t)\sqrt{(t-\alpha)(t-\beta)}}{t^{n+1}}\,dt, \quad (n = 1, 2, 3, \cdots), \tag{55}$$

and d_n^* are the coefficients in the expansion of the function

$$\frac{1}{\sqrt{(z-\alpha)(z-\beta)}}$$

about zero. The branch of this function is determined by the series (49). Therefore its expansion about zero has the form

$$\frac{1}{\sqrt{(z-\alpha)(z-\beta)}} = \tag{56}$$

$$= -\frac{1}{\sqrt{\alpha\beta}}\left(1 + \frac{\alpha+\beta}{2R^2}z + \frac{3(\alpha^2+\beta^2)+2\alpha\beta}{8R^4}z^2 + \cdots\right).$$

Comparing the coefficients in the expansions (49) and (56) we see that

$$d_0^* = -\frac{d_1'}{R}, \cdots, \quad d_{n-1}^* = -\frac{d_n'}{R^{2n-1}} = -P_{n-1}R^{-n}. \tag{57}$$

Performing on the basis of (56) the corresponding substitution in the system, one may express the coefficients b_n^* in terms of the b_n' and conversely; we obtain

$$\begin{aligned} b_0^* &= \frac{P}{2\pi R} = -\frac{b_1'}{R}, \\ b_1^* &= -\frac{2a_1}{R^3} + \frac{d_2'b_1'}{R^3}, \\ b_2^* &= -\frac{3a_2}{R^5} + \frac{d_3'b_1'}{R^5} + \frac{d_2'b_2'}{R^5}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ b_{n-1}^* &= -\frac{na_{n-1}}{R^{2n-1}} + \frac{d_n'b_1'}{R^{2n-1}} + \cdots + \frac{d_2'b_{n-1}'}{R^{2n-1}}. \end{aligned} \tag{58}$$

It is now seen from (52) and (58) that one has between the coefficients b_n^* and b_n' the relations

$$\begin{aligned} b_0^* &= -\frac{1}{R}b_1', \quad b_1^* = \frac{b_2'}{R^3}, \\ b_{n-1}^* &= -\frac{b_n'}{R^{2n-1}}, \quad b_n^* = -\frac{b_{n+1}'}{R^{2n-1}}, \quad (n = 0, 1, 2, \cdots). \end{aligned} \tag{59}$$

We will now consider the function

$$F(z) = \frac{1}{2\pi i}\int_{\Gamma_1} \frac{\omega(t)\sqrt{(t-\alpha)(t-\beta)}}{t-z}\,dt.$$

Its expansion about the point at infinity and about zero has been shown to be determined by the series $\sum_{n=1}^{\infty} b_n'z^{-n}$ and $\sum_{n=0}^{\infty} b_n^*z^n$ respectively. We will denote the limiting values of the function $F(z)$ on the arc Γ_1 by

$$F^+(t) = \sum_{n=0}^{\infty} b_n^*t^n \text{ and } F^-(t) = \sum_{n=1}^{\infty} b_n't^{-n}. \tag{60}$$

On the basis of (45) and Sokhotskii's formula we have

$$\widehat{rr}'(t) = \frac{1}{\sqrt{(t-\alpha)(t-\beta)}}\left(\sum_{n=1}^{\infty} b_n't^{-n} + \sum_{n=0}^{\infty} b_n^*t^n\right). \tag{61}$$

Substituting the expression for $\widehat{rr}'(t)$ from (61) in (44) we find

$$(62)\qquad \int_{\Gamma_1} \frac{1}{\sqrt{(t-\alpha)(t-\beta)}} \{\sum_{n=1}^{\infty} b_n' t^{-(n-1)} + \sum_{n=0}^{\infty} b_n^* t^{n+1}\} dt = iP.$$

The integrals in (62) are readily evaluated. We obtain

$$I_k = \int_{\Gamma_1} \frac{t^k dt}{\sqrt{(t-\alpha)(t-\beta)}} = -\pi i d'_{k+1},$$

where the d'_{k+1} are the coefficients of the terms with $z^{-(k+1)}$ in the expansion of the function $\{(z-\alpha)(z-\beta)\}^{-\frac{1}{2}}$ about the point at infinity,

$$I_k^* = \int_{\Gamma_1} \frac{dt}{t^k \sqrt{(t-\alpha)(t-\beta)}} = \pi i d^*_{k-1},$$

where d^*_{k-1} are the coefficients in the expansion of the function $\{(z-\alpha)(z-\beta)\}^{-\frac{1}{2}}$ about zero.

Substituting these values for the integrals in (62) we obtain

$$(63)\qquad \sum_{n=2}^{\infty} b_n' d^*_{n-2} - \sum_{n=0}^{\infty} b_n^* d'_{n+2} - d_1' b_1' = \frac{P}{\pi}.$$

By virtue of the fact that $d_1' = 1$, $b_1' = -P/2\pi$, $b_0^* = P/2\pi R$ and by (57), (59) and (50) it follows from (63) that

$$(64)\qquad \sum_{n=1}^{\infty} R^{-n} b'_{n+1}(P_{n+1} - P_{n-1}) = \frac{P}{2\pi}(1 + \cos\varphi).$$

In the case that the materials of the plate and the disc are the same, one has to retain in (64) only b_1' and b_2'; the remaining coefficients will be equal to zero.

Thus, for the same materials, the subtended angle is determined by the equation

$$b_1' P_1 + \frac{b_2' P_2}{R} - \frac{b_2'}{R} = \frac{P}{2\pi}$$

which in expanded form becomes

$$\left(\cos\varphi - \frac{26 + \log 2}{48}\right) 3\cos 2\varphi - 7\cos\varphi = 4 - \frac{26 + \log 2}{16}.$$

After finding the subtended angle φ, all coefficients b_n', b_n^* will be determined. We obtain $\widehat{rr}$ by integration, i.e., we determine the law of distribution of the pressure of the disc on the plate.

If one need know not only the values of $\widehat{rr}$ on the common boundary of the disc and the plate, but also other stress components on the boundary and inside the contacting bodies, one must find at the start one of the quantities b_1 or g which are related to each other by the last of the formulae (2). Then one has to find the function $\Psi(z)$ the coefficients of the expansion of which can be determined by solution of the system (31) and the equality (10). The function $\Psi(z)$ is given by the formula (3.2) of [1]. Thus the problem is solved.

The question now is only how cumbersome the calculations will be and what will be their accuracy. The computations may be simplified, if one proceeds in the following manner. We will consider side by side with the equation (1) the equation

$$\frac{1}{\pi i}\int_{\Gamma_1}\frac{\widehat{rr}(t)dt}{t-t_0} = -k\int_{\alpha}^{t_0}\frac{\widehat{rr}(t)dt}{t} - h\left(\frac{t_0}{R^2}-\frac{1}{t_0}\right) - g\log t_0 + A \tag{65}$$

which is obtained from the equation (1) on the basis of the known identity

$$\frac{1}{\pi i}\int_{\Gamma_1}\frac{\widehat{rr}'(t)}{t-t_0}\,dt = \frac{d}{dt_0}\int_{\Gamma_1}\frac{\widehat{rr}(t)dt}{t-t_0}, \tag{66}$$

correct for the condition $\widehat{rr}(\alpha) = \widehat{rr}(\beta) = 0$.

We will denote by $\omega_1(t)$ the right hand side of the equation (65) and consider the function

$$\Omega_1(z) = \frac{1}{2\pi i}\int_{\Gamma_1}\frac{\widehat{rr}(t)dt}{t-z} \tag{67}$$

which we will represent in the form

$$\Omega_1(z) = \sqrt{(z-\alpha)(z-\beta)}F_1(z), \tag{68}$$

where

$$F_1(z) = \frac{1}{2\pi i}\int_{\Gamma_1}\frac{\omega_1(t)dt}{\sqrt{(t-\alpha)(t-\beta)}(t-z)}. \tag{69}$$

Since the function $\Omega_1(z)$ must vanish at infinity, such a representation is possible if

$$\frac{1}{2\pi i}\int_{\Gamma_1}\frac{\omega_1(t)dt}{\sqrt{(t-\alpha)(t-\beta)}} = 0. \tag{70}$$

The constant A on the right hand side of the equation may be determined from this condition, but we do not require its value. The function $F_1(z)$ for $|z| > R$ may be expanded in the series

$$F_1(z) = \sum_{n=1}^{\infty} c_n z^{-n},$$

where

$$C_n = -\frac{1}{2\pi i} \int_{\Gamma_1} \frac{\omega_1(t) t^{n-1} dt}{\sqrt{(t-\alpha)(t-\beta)}}, \quad (n = 1, 2, 3, \cdots).$$

We will select that branch of the root for which for $|z| > R$

$$\sqrt{(z-\alpha)(z-\beta)} = z - \frac{\alpha+\beta}{2} - \frac{(\alpha-\beta)^2}{8}\frac{1}{z} + \cdots = \tag{71}$$

$$= z - \frac{\alpha+\beta}{2} + \sum_{n=1}^{\infty} c_n z^{-n}.$$

On the basis of the formulae (46), (47) and (68) we find now that for $|z| > R$

$$-\Psi_0(z) + \frac{P}{2\pi}\frac{1}{z} = \sqrt{(z-\alpha)(z-\beta)} F_1(z), \tag{72}$$

and for $|z| \leq R$

$$\Phi_0\left(\frac{R^2}{z}\right) + \frac{P}{2\pi}\frac{z}{R^2} - \frac{b_1}{R^2} = \sqrt{(z-\alpha)(z-\beta)} F_1(z). \tag{73}$$

Replacing the functions in the equality (72) by the corresponding series, we obtain

$$-\sum_{n=1}^{\infty} a_n z^{-(n+1)} = \frac{P}{2\pi}\frac{1}{z} + \left(z - \frac{\alpha+\beta}{2}\right)\sum_{n=1}^{\infty} c_n z^{-n} + \sum_{\nu=2}^{\infty}\left(\sum_{n=1}^{\nu-1} l_n c_{\nu-n}\right) z^{-\nu}.$$

Equating the coefficients of equal powers of z, we find $c_1 = 0$, in agreement with (70); further, we obtain

$$\begin{aligned} c_2 &= \frac{P}{2\pi}, \\ -a_1 &= c_3 - \frac{\alpha+\beta}{2} c_2, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ -a_k &= c_{k+2} - \frac{\alpha+\beta}{2} c_{k+1} + \sum_{n=1}^{k-1} l_n c_{k+1-n}. \end{aligned} \tag{74}$$

This system determines all the coefficients c_k.

We will now consider the equality (73). We expand $F_1(z)$ for $|z| < R$ in the series

$$F_1(z) = \sum_{n=0}^{\infty} c_n^* z^n$$

and, taking into consideration (71), we obtain for $|z| < R$ the expansion

$$\sqrt{(z-\alpha)(z-\beta)} = \tag{75}$$

$$= -\sqrt{\alpha\beta}\left(1 - \frac{\alpha+\beta}{2\alpha\beta}z - \frac{(\alpha-\beta)^2}{8\alpha^2\beta^2}z^2 - \cdots\right) = -\sum_{n=0}^{\infty} l_n^* z^n.$$

Hence we now find

$$\sum_{k=1}^{\infty} a_k R^{-2(k+1)} z^{k+1} = \frac{Pz}{2\pi R^2} - \sum_{\nu=0}^{\infty}\left(\sum_{n=0}^{\nu} l_n^* c_{\nu-n}^* z^\nu\right) + \frac{b_1}{R^2}. \tag{76}$$

As the result of the comparison of the coefficients of equal powers of z in (76) we will have

$$Rc_0 = \frac{b_1}{R^2},$$

$$Rc_1 - \frac{\alpha+\beta}{2}c_0 = \frac{P}{2\pi R^2},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_k R^{-2(k+1)} = \sum_{n=0}^{k+1} l_n^* c_{k-n+1}^*.$$

We may determine from this system all the c_k exactly, apart from an unknown term,

$$c_k^* = \gamma_k \frac{P}{\pi} + \gamma_k^* \frac{b_1}{R^2}, \tag{77}$$

where γ_k and γ_k^* are known quantities.

By use of Sokhotskii's formula and the formulae (68), (67), the stress $\widehat{rr}$ on the boundary of contact may be represented in the form

$$\widehat{rr} = \sqrt{(t-\alpha)(t-\beta)}\left[\sum_{n=1}^{\infty} c_{n+1} t^{-(n+1)} + \sum_{n=0}^{\infty}\left(\gamma_n \frac{P}{\pi} + \gamma_n^* \frac{b_1}{R^2}\right)t^n\right]. \tag{78}$$

The equation for the determination of b_1/R^2 is now obtained from (43), if we substitute for $\widehat{rr}$ the expression (78),

$$-\frac{1}{2\pi i}\int_{\Gamma_1} \frac{\sqrt{(t-\alpha)(t-\beta)}}{t}\left\{\sum_{n=1}^{\infty} c_{n+1} t^{-(n+1)} + \right. \tag{79}$$

$$\left. + \sum_{n=0}^{\infty}\left(\gamma_n \frac{P}{\pi} + \gamma_n^* \frac{b_1}{R^2}\right)t^n\right\} dt = \frac{b_1}{R^2}.$$

The integrals in (79) are readily evaluated. If one selects the contour of integration shown in Fig. 1, one finds

$$(80)\quad 2I_k' = \int_{\Delta} \frac{\sqrt{(t-\alpha)(t-\beta)}dt}{t^k} = \int_{\delta} \frac{\sqrt{(t-\alpha)(t-\beta)}dt}{t^k} - \int_{\Gamma} \frac{\sqrt{(t-\alpha)(t-\beta)}dt}{t^k},$$

where

$$I_k' = \int_{\Gamma_1} \frac{\sqrt{(t-\alpha)(t-\beta)}dt}{t^k}$$

and Δ is a contour enveloping the arc Γ_1 which may be replaced by twice

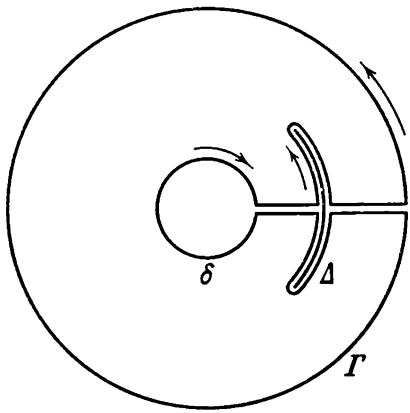

Fig. 1

its passage. Applying the residue theorem to the right hand side of (79), we obtain

$$I_1' = -\pi i(l_0^* - R\cos\varphi),\ I_2' = -\pi i(1+l_1^*),\ I_3' = -\pi i l_2^*,\ \text{etc.},$$

where l_0^*, l_1^*, l_2^* are determined by (75). Further, we have

$$(81)\qquad \int_{\Gamma_1} t^k\sqrt{(t-\alpha)(t-\beta)}dt = -\pi i l_{k+1},$$

where l_{k+1} is determined by (71).

In the equality (79) we take m terms, selecting the number m in accordance with the computational accuracy desired and we compute the integrals in the expressions obtained by use of the formulae (80)

and (81). Then b_1/R^2 will be determined by

$$\frac{b_1}{R^2} = -\frac{\sum_{n=1}^{m}\left(c_{n+1}l_n^* - \frac{P}{\pi}\gamma_n l_n\right)}{-\cos\varphi + 1 + \sum_{n=1}^{m}\gamma_n^* l_n}. \tag{82}$$

If we set the coefficient $c_m = 0$ in (77), then

$$\frac{b_1}{R^2} = -\frac{\gamma_k P/\pi}{\gamma_k^*}. \tag{83}$$

For sufficiently large m, the values of b_1/R^2, obtained from (82) and (83), must coincide with the adopted degree of accuracy.

Thus all the necessary quantities for the evaluation have been determined and the problem is solved.

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THE ELASTIC, HEAVY HALF-PLANE, WEAKENED BY HOLES OF ELLIPTIC SHAPE LYING SUFFICIENTLY CLOSE TO ITS BOUNDARY

D. I. SHERMAN
Moscow, S.S.R.

In various papers, devoted to the study of the state of stress in a heavy medium occupying the half-plane with one or several holes, it is, in essence, assumed that these holes lie sufficiently far from the boundary of the half-plane. The problem is seriously complicated in the study of so-called "floating" holes, lying close to the boundary of the half-plane; naturally, in this case, not only the volume of numerical work is increased in a significant measure, but also the principal difficulties become more severe. The method of solution to be proposed here for the problem mentioned in the title has been found to be useful, generally speaking, for any relative dimensions of the region. However, the numerical results of the solution are given for the limiting case when the elliptical hole reduces to a crack.

The results of the paper may be used for deductions and recommendations by specialists in mining engineering.

§ 1. We denote the heavy, doubly connected, semi-infinite medium bounded by a straight boundary L_2 and an ellipse L_1 by S, the major and minor axes of the ellipse by a and b respectively and the distance of its center from the straight boundary by h. We place the origin of the Cartesian coordinate system at the center of the ellipse and direct the abscissa and the ordinate along the major and the minor axes of the ellipse respectively.

Let S_1 and S_2 be the finite and the semi-infinite, simply connected regions, composed of the inside of the ellipse and the half-plane, bounded by the contours L_1 and L_2 respectively, so that $S = S_2 - S_1$. As usual, we select as positive directions of L_1 and L_2 those for which the region S lies on the left.

We assume that the initial stresses, acting in the unweakened heavy medium [1] and taken in the generally accepted form which sufficiently correctly reflects their occurrence in mountain massifs, have been allowed for in a proper manner. Then the additional stresses X_x, X_y, X_y

[1] Cf., for example, Ref. [1].

are determined by certain functions $\varphi_1(z)$ and $\psi_1(z)$ of the complex variable $z = x + iy$, regular in the region S under consideration and satisfying the conditions

$$\varphi_1(t) + t\overline{\varphi_1'(t)} + \overline{\psi_1(t)} = 2f(t) + 2C_j \text{ on } L_j, \qquad j = 1, 2. \tag{1.1}$$

Here t is the coordinate of the point on the curve L_j and $f(t)$ is some function

$$f(t) = h_1 t + h_2 \bar{t} \quad \text{on } L_1, \qquad f(t) = 0 \text{ on } L_2, \tag{1.2}$$

where h_1 and h_2 are known constant quantities. Further, C_1 and C_2 are constants to be determined during the process of solution. We assume that the functions $\varphi_1(z)$ and $\psi_1(z)$ vanish at infinity, in which case the constants C_1 and C_2 must be strictly determined.

Note: Having in mind the applications for which the present study was undertaken, we take $b/h \ll 1$. Proceeding just as in [2] and adding to the unknown functions $\varphi_1(z)$ and $\psi_1(z)$ the supplementary logarithmic terms, specified there, we consider the more general case $b/h < 1$. As regards the approach of the ratio b/h to unity, the problem under consideration (just as in other problems of a similar type) presents more specific, serious difficulties; however, on the whole (especially in the presence of sufficiently effective computers), they bear a technical character (these difficulties and methods for overcoming them will be described in detail below).

The presently adopted scheme for solving the formulated problem somewhat idealizes the formulation of the question, dictated by the practice of mining engineering. In reality, the hole damaging the medium (the mining shaft) is bounded by an extended curvilinear rectangle; on the whole, the influence of the stress field in the neighbourhood of the elongated sides of the rectangle (walls of the mining shaft) is of interest, and, mainly, at points sufficiently removed from its other two opposite sides; therefore it must be assumed that near those sides observation of the local stress concentrations does not give any essential information regarding the magnitudes of the stress components at the mentioned points. Hence, the legality of the approximation of the rectangular hole by a corresponding hole of elliptic shape is clear.

We introduce on the curve L_1 the new unknown function $g(t)$ in accordance with the equality

$$\varphi_1(t) - t\overline{\varphi_1'(t)} - \overline{\psi_1(t)} = 2g(t) - 2f(t). \tag{1.3}$$

Adding and subtracting (1.1) and (1.3) we obtain

$$\varphi_1(t) = g(t) + C_1, \quad \psi_1(t) = -\{\overline{g(t)} + \bar{t}g'(t)\} + 2\overline{f(t)} + \bar{C}_1 \text{ on } L_1. \tag{1.4}$$

We introduce the new functions $\varphi(z)$ and $\psi(z)$, regular in the region S,

by the conditions

$$\varphi(z) = \varphi_1(z) - \frac{1}{2\pi i}\int_{L_1} \frac{g(t)}{t-z}\,dt,$$

$$\psi(z) = \psi_1(z) + \frac{1}{2\pi i}\int_{L_1} \frac{\overline{g(t)} + \bar{t}g'(t) - 2\overline{f(t)}}{t-z}\,dt. \tag{1.5}$$

Then we have inside the curve L_1

$$\varphi(z) = -\frac{1}{2\pi i}\int_{L_1} \frac{g(t)}{t-z}\,dt + C_1,$$

$$\psi(z) = \frac{1}{2\pi i}\int_{L_1} \frac{\overline{g(t)} + \bar{t}g'(t) - 2\overline{f(t)}}{t-z}\,dt + \bar{C}_1. \tag{1.6}$$

It is not difficult to see that the functions $\varphi(z)$ and $\psi(z)$ will be regular in the half-plane S_2, bounded by the contour L_2.

§ 2. The outside of the ellipse L_1 is mapped onto the outside of the circle γ with radius ρ in the ζ plane by the function

$$z = A\left(\zeta + \frac{1}{\zeta}\right),\quad \zeta = \frac{z + \sqrt{z^2 - 4A^2}}{2A},\quad \rho = \sqrt{\frac{a+b}{a-b}}. \tag{2.1}$$

We denote by τ the coordinate of the point γ, corresponding by the transformation (2.1) to the coordinate t of a point of L_1. We choose the function $g(t)$ in the form of a complex Fourier series in powers of the variable τ with coefficients subject to determination, so that

$$g^*(\tau) = g(t),\quad g^*(\tau) = \sum_{k=-\infty}^{+\infty} \alpha_k \tau^k. \tag{2.2}$$

Having in mind the relation (as everywhere, by the strength of (2.1), the points $z(t)$ and $\zeta(\tau)$ are mapped into each other)

$$\frac{dt}{t-z} = \left[-\frac{1}{\tau} + \frac{1}{\tau-\zeta} + \frac{1}{\tau - \frac{1}{\zeta}}\right] d\tau,$$

we find after some simple manipulations for the point z, lying outside

the contour L_1 (with clockwise direction)

$$\frac{1}{2\pi i}\int_{L_1}\frac{g(t)}{t-z}\,dt = -\sum_{k=1}^{\infty}(\alpha_k - \alpha_{-k})\frac{1}{\zeta^k},$$

$$\frac{1}{2\pi i}\int_{L_1}\frac{\overline{g(t)}}{t-z}\,dt = \sum_{k=1}^{\infty}(\rho^{2k}\bar{\alpha}_k - \rho^{-2k}\bar{\alpha}_{-k})\frac{1}{\zeta^k}. \tag{2.3}$$

We evaluate now for the same point z the integral

$$I(z) = \frac{1}{2\pi i}\int_{L_1}\frac{\bar{t}g'(t)}{t-z}\,dt = -\frac{1}{2\pi i}\int_{L_1} g(t)\,d\,\frac{\bar{t}}{t-z}. \tag{2.4}$$

Taking into consideration the formula (for the point t, belonging to L_1)

$$\frac{\bar{t}}{t-z} = A(\zeta) + \frac{B(\zeta)}{\tau-\zeta} + \frac{C(\zeta)}{\tau - \frac{1}{\zeta}},$$

where

$$A(\zeta) = \rho^{-2},\ B(\zeta) = \frac{(\rho^2 + \rho^{-2}\zeta^2)\zeta}{\zeta^2 - 1},$$

$$C(\zeta) = -\frac{(\rho^{-2} + \rho^2\zeta^2)}{\zeta(\zeta^2 - 1)},$$

we write the integral (2.4) in the form

$$I(z) = B(\zeta)\frac{\partial}{\partial\zeta}\frac{1}{2\pi i}\int_{\gamma} g^*(\tau)\frac{d\tau}{\tau-\zeta} + C(\zeta)\frac{\partial}{\partial\frac{1}{\zeta}}\frac{1}{2\pi i}\int\frac{g^*(\tau)d\tau}{\tau - \frac{1}{\zeta}}. \tag{2.5}$$

Hence we find without difficulty

$$I(z) = \frac{1}{\zeta^2 - 1}\sum_{k=1}^{\infty} k[(\rho^{-2} + \rho^2\zeta^2)\alpha_k - (\rho^2 + \rho^{-2}\zeta^2)\alpha_{-k}]\frac{1}{\zeta^k}. \tag{2.6}$$

Further, adding the second of the integrals (2.3) and (2.4) and transforming the right hand side of the equality (2.6), we obtain

$$\frac{1}{2\pi i}\int_{L_1}\frac{\overline{g(t)} + \bar{t}g'(t)}{t-z}\,dt = \sum_{k=1}^{\infty}\left\{\delta_k - \frac{1}{1-\zeta^{-2}}\gamma_k\right\}\frac{1}{\zeta^k}, \tag{2.7}$$

where the following notation has been introduced:

$$\begin{aligned}&\delta_k = \bar{\alpha}_k\rho^{2k} - \bar{\alpha}_{-k}\rho^{-2k},\ \varepsilon_1 = 0,\ \varepsilon_k = 1\ (k \neq 1),\\ &\gamma_k = k(\rho^{-2}\alpha_{-k} - \rho^2\alpha_k) + \varepsilon_k(k-2)(\rho^2\alpha_{-(k-2)} - \rho^{-2}\alpha_{k-2}).\end{aligned} \tag{2.8}$$

In addition to the above we note yet the equality

$$-\frac{1}{\pi i}\int_{L_1}\frac{\overline{f(t)}}{t-z}\,dt=-E\,\frac{1}{\zeta}, \tag{2.9}$$

where E is the real constant

$$E=(\rho^2c-\rho^{-2}d),\ c=A(2h_1+2h_2\rho^{-2}),\ d=A(2h_1+2h_2\rho^2). \tag{2.10}$$

It is seen that it vanishes for $\rho=1$.

The formulae (1.5) may now be written in the following form which, in essence, will be used below:

$$\begin{aligned}\varphi_1(z)&=\varphi(z)+\sum_{k=1}^{\infty}(\alpha_{-k}-\alpha_k)\,\frac{1}{\zeta^k},\\ \psi_1(z)&=\psi(z)-\sum_{k=1}^{\infty}\left\{\delta_k-\frac{1}{1-\zeta^{-2}}\,\gamma_k\right\}\frac{1}{\zeta^k}-E\,\frac{1}{\zeta}.\end{aligned} \tag{2.11}$$

Differentiating the first of these equalities with respect to z, we obtain

$$\varphi_1'(z)=\varphi'(z)-\frac{1}{A}\,\frac{1}{1-\zeta^{-2}}\sum_{k=1}^{\infty}(k-1)[\alpha_{-(k-1)}-\alpha_{k-1}]\,\frac{1}{\zeta^k}. \tag{2.12}$$

We introduce the expressions for the functions $\varphi_1(z)$, $\varphi_1'(z)$ and $\psi_1(z)$ from (2.11) and (2.12) into the boundary condition (1.1) on the contour L_2. After obvious transformations we will have

$$\varphi(t)+(\bar{t}+2ih)\overline{\varphi'(t)}+\overline{\psi(t)}=G(g(t))+E\,\frac{1}{\bar{\tau}}+C_2 \text{ on } L_2, \tag{2.13}$$

where with the help of the quantities

$$\begin{aligned}\Delta_k=&-(\rho^2-1)\{k[\alpha_k+\rho^{-2}\alpha_{-k}]-\varepsilon_k(k-2)[\rho^{-2}\alpha_{k-2}+\alpha_{-(k-2)}]\}-\\&-2i\,\frac{h}{A}\,(k-1)(\alpha_{k-1}-\alpha_{-(k-1)})\end{aligned} \tag{2.14}$$

the first term on the right hand side may be given the more compact and convenient form

$$G(g(t))=\sum_{k=1}^{\infty}\left\{(\alpha_k-\alpha_{-k})\,\frac{1}{\tau^k}+\left(\bar{\delta}_k-\frac{1}{1-\bar{\tau}^{-2}}\,\bar{\Delta}_k\right)\frac{1}{\bar{\tau}^k}\right\}. \tag{2.15}$$

§ 3. The function which is conjugate on the rectangular boundary L_2 to $1/\zeta$ and defined by (2.1) is analytically continued into the region S_2 and its modulus vanishes at the point at infinity. It has the form

$$\frac{1}{\chi(z)}=\frac{z-2ih-\sqrt{(z-2ih)^2-4A^2}}{2A},\quad \xi(z-2ih)=\chi(z). \tag{3.1}$$

Observing the variation of the argument of the root in (2.1), we convince ourselves that the root in the equality assumes at the origin of coordinates the purely imaginary, negative value $-2i(A^2+h^2)^{\frac{1}{2}}$. We will assume that the large semi-axis of the ellipse L_1 does not exceed $2(A^2+h^2)^{\frac{1}{2}}$. Then, assuming that the modulus of the variable z as it changes on the circumference of the curve L_1 likewise does not exceed $2(A^2+h^2)^{\frac{1}{2}}$ and writing the root (3.1) in the form

$$\sqrt{(z-2ih)^2-4A^2}=-2i\sqrt{A^2+h^2}\sqrt{1-\frac{z}{2A+2ih}}\sqrt{1+\frac{z}{2A-2ih}},$$

we easily arrive at the expansion

$$\sqrt{(z-2ih)^2-4A^2}=-2iAq\sum_{n=0}^{\infty}a_n\left(\frac{z}{2A}\right)^n, \tag{3.2}$$

where the coefficients a_n are given by

$$a_n=\frac{1}{q^n}\sum_{\nu=0}^{n}(-1)^\nu C_{\frac{1}{2}}^{\nu}C_{\frac{1}{2}}^{n-\nu}e^{i(n-2\nu)\vartheta} \tag{3.3}$$

and the constant q and the argument ϑ are to be found from the formulae

$$A+ih=Aqe^{i\vartheta},\quad q=\sqrt{1+\left(\frac{h}{A}\right)^2}. \tag{3.4}$$

Introducing the new quantities

$$b_0=i\left(q-\frac{h}{A}\right),\quad b_1=1+iqa_1,\quad b_n=iqa_n\ (n>1), \tag{3.5}$$

we obtain from (3.1) the formula

$$\frac{1}{\chi(z)}=\sum_{n=0}^{\infty}b_n\left(\frac{z}{2A}\right)^n. \tag{3.6}$$

The coefficients b_n take either real or purely imaginary values, depending on whether the index n is an odd or an even number. This property of these coefficients is readily confirmed by seeing from (3.3) that

$$a_n=\frac{1}{q^n}\{(-1)^{n/2}(C_{\frac{1}{2}}^{n/2})^2+2\sum_{\nu=0}^{(n/2)-1}(-1)^\nu C_{\frac{1}{2}}^{\nu}C_{\frac{1}{2}}^{n-\nu}\cos(n-2\nu)\vartheta\} \tag{3.7}$$

for even n and

$$a_n=i\frac{2}{q^n}\sum_{\nu=0}^{(n-1)/2}(-1)^\nu C_{\frac{1}{2}}^{\nu}C_{\frac{1}{2}}^{n-\nu}\sin(n-2\nu)\vartheta \tag{3.8}$$

for odd n. Taking this into consideration one may rewrite (3.6) in the form

$$\frac{1}{\chi(z)} = i\sum_{e=0}^{\infty} r_e\left(\frac{iz}{2A}\right)^e, \quad b_e = i^{e+1}r_e, \tag{3.9}$$

where the r_e are real quantities.

Raising the series (3.9) to some power k, we will have

$$\frac{1}{\chi^k(z)} = i^k\sum_{e=0}^{\infty} r_e^{(k)}\left(\frac{iz}{2A}\right)^e \quad (k = 1, 2, 3, \cdots), \tag{3.10}$$

where the quantities $r_e^{(k)}$ $(k = 2, 3, \cdots)$ are to be found for given r_e with the help of recurrence relations.

It is not difficult to find new expressions for the r_e by somewhat modifying the expansion process (3.2); they will be needed in what follows for the comparison of these quantities in cases which are unfavorable for computations, when the semi-axis of the elliptical hole exceeds in a significant measure the distance of its centre from the straight boundary L_2.

We take the root (3.1) in the form of the product

$$\sqrt{(z-2ih)^2 - 4A^2} = -i\sqrt{4(h^2+A^2) - z^2}\sqrt{1 + \frac{4ihz}{4(h^2+A^2)-z^2}}. \tag{3.11}$$

At least for $|z| \leq 2(-h + \sqrt{2h^2 + A^2})$ one has the equality

$$\sqrt{1 + \frac{4ihz}{4(h^2+A^2) - z^2}} =$$

$$= \sum_{\nu=0}^{\infty} i^\nu C_{\frac{1}{2}}^\nu \left(\frac{2Ah}{h^2+A^2}\right)^\nu \sum_{e=\nu}^{\infty}{}^*(-1)^{(e-\nu)/2} C_{-\nu}^{(e-\nu)/2} \left(\frac{A^2}{h^2+A^2}\right)^{(e-\nu)/2}\left(\frac{z}{2A}\right)^e$$

or, after changing the order of summation,

$$\sqrt{1 + \frac{4ihz}{4(h^2+A^2) - z^2}} = \sum_{e=0}^{\infty} \sigma_e\left(\frac{iz}{2A}\right)^e, \tag{3.12}$$

where the coefficients under the sum sign on the right hand side are given by

$$\sigma_e = \sum_{\nu=e-2[e/2]}^{e}{}^* C_{\frac{1}{2}}^\nu C_{-\nu}^{(e-\nu)/2} \left(\frac{2Ah}{h^2+A^2}\right)^{(e+\nu)/2}\left(\frac{A}{2h}\right)^{(e-\nu)/2} \tag{3.13}$$

with $[e/2]$ denoting the greatest integer contained in $e/2$; the asterisk, attached to the sum sign [just as in the formula preceding (3.12)] indicates

that adjacent values of the summation index differ by 2. Taking still into consideration the root multiplier in (3.11) which likewise depends on z, we obtain after a number of completely similar manipulations

$$-\frac{\sqrt{(z-2ih)^2-4A^2}}{2A}=i\sum_{e=0}^{\infty}\lambda_e\left(\frac{iz}{2A}\right)^e, \tag{3.14}$$

where the coefficients in the expansion have the values

$$\lambda_e=\sum_{k=0}^{[e/2]}C_{\frac{1}{2}}^{k}q^{-(2k-1)}\sigma_{e-2k}. \tag{3.15}$$

With the last equality in mind, we find from (3.1)

$$r_0=-\left(\frac{h}{A}-\lambda_0 q\right),\quad r_1=-(1-\lambda_1 q),\quad r_e=\lambda_e q\ (e=2,3,\cdots). \tag{3.16}$$

One may find a formula which is more convenient than (3.15) for the quantities λ_e. In fact, it is not difficult to see that the left hand side of (3.2) may be written in the form of the double series

$$iq\sqrt{1+\frac{iz}{2\sqrt{h^2+A^2}}\left(\frac{iz}{2\sqrt{h^2+A^2}}+\frac{2h}{\sqrt{h^2+A^2}}\right)}=$$

$$=iq\sum_{\nu=0}^{\infty}C_{\frac{1}{2}}^{\nu}\sum_{n=\nu}^{2\nu}C_{\nu}^{n-\nu}\left(\frac{2Ah}{h^2+A^2}\right)^n\left(\frac{\sqrt{h^2+A^2}}{2h}\right)^{2(n-\nu)}\left(\frac{iz}{2A}\right)^n.$$

Reversing in this expansion the order of the summations one arrives at the form

$$iq\sum_{n=0}^{\infty}\left(\frac{iz}{2A}\right)^n\left(\frac{2Ah}{h^2+A^2}\right)^n\sum_{\nu=(n+\varepsilon)/2}^{n}C_{\frac{1}{2}}^{\nu}C_{\nu}^{n-\nu}\left(\frac{\sqrt{h^2+A^2}}{2h}\right)^{2(n-\nu)},$$

where $\varepsilon=0$ or 1 [$(n+\varepsilon)/2$ is a non-negative integer]. Hence we obtain for the coefficients λ_n the required formula

$$\lambda_n=q\left(\frac{2Ah}{h^2+A^2}\right)^n\sum_{\nu=(n+\varepsilon)/2}^{n}C_{\frac{1}{2}}^{\nu}C_{\nu}^{n-\nu}\left(\frac{\sqrt{h^2+A^2}}{2h}\right)^{2(n-\nu)}. \tag{3.17}$$

Proceeding in an analogous manner, we obtain the formula

$$\frac{2A}{\sqrt{(z-2ih)^2-4A^2}}=i\sum_{n=0}^{\infty}\mu_n\left(\frac{iz}{2A}\right)^n, \tag{3.18}$$

where the coefficients under the sum sign are given by

$$\mu_n=\frac{1}{q}\left(\frac{2Ah}{h^2+A^2}\right)^n\sum_{\nu=(n+\varepsilon)/2}^{n}C_{-\frac{1}{2}}^{\nu}C_{\nu}^{n-\nu}\left(\frac{\sqrt{h^2+A^2}}{2h}\right)^{2(n-\nu)}.$$

The equalities (3.10) and (3.18) readily lead to the expansions (which are valid near the circumference of the ellipse L_1)

$$\frac{1}{1-\chi^{-2}(z)} = \sum_{n=0}^{\infty} d_n \left(\frac{iz}{2A}\right)^n,$$

$$\frac{1}{1-\chi^{-2}(z)}\,\frac{1}{\chi^k(z)} = i^k \sum_{\nu=0}^{\infty} h_\nu^{(k)} \left(\frac{iz}{2A}\right)^\nu, \tag{3.19}$$

where the real quantities d_n and $h_\nu^{(k)}$ are given by

$$d_0 = \tfrac{1}{2}\left(1 + \frac{h}{A}\,\mu_0\right),$$

$$d_n = \tfrac{1}{2}\left(\mu_{n-1} + \frac{h}{A}\,\mu_n\right), \quad n \geq 1, \tag{3.20}$$

$$h_\nu^{(k)} = \sum_{n=0}^{\nu} d_{\nu-n} r_n^{(k)} \quad (\nu = 0, 1, \cdots;\ k = 1, 2, \cdots).$$

The quantities $r_\nu^{(k)}$ and $h_\nu^{(k)}$ have been evaluated for the cases $h/A = 4$ and $h/A = 2$, selected with a view to practical needs; their values are given in Tables 1 and 2.

NOTE: The quantities r_e are in a certain sense quite fundamental; they are employed for the construction of the intermediate quantities in (3.10) and (3.19), and then, later on, also for quantities which enter directly into the formation of the Fourier coefficients α_n. Therefore it should not come as a surprise that the author has taken such intense interest in them. In the following considerations a formula was deduced for the quantities r_e which was structurally somewhat different. It has not been included and it is even probable that for very contracted boundaries of the region S every one of these formulae may be found to involve terms which are numerically very similar and of opposite signs, so that, generally speaking, they will not guarantee the desired accuracy for the computed quantities. However, this unfavorable property which affects the accuracy of the computations and, as a rule, seriously complicates the performance of the numerical operations, does not necessarily appear in each of the formulae in an equal manner and then only for one or the other of the values of the indices by which the found quantity was supplied. The last circumstance gives for the direction of the computations more or less reliable means of control and it permits in doubtful cases establishment of their values with an accuracy not exceeding the admitted limits by comparison of these quantities obtained from different formulae.

What has been said here applies even more so to the determination of other quantities occurring below in the formulae (5.4)–(5.8) and, in particular, of the simplest of them $\xi_{e,k}^{(1,1)}$ in terms of which all the others will be seen to be easily determined.

If by proceeding in this manner we nevertheless do not succeed in overcoming to any essential degree the difficulties about which we have spoken, then we must

TABLE 1.

ν \ k	$\frac{h}{A}$	1	2	3	4	5	6	7	8	9	10
0	$h/A = 4$	0.1231056	0.01515499	$0.0^{2}186566$	$0.0^{3}229674$	$0.0^{4}282741$	$0.0^{5}34807$	$0.0^{6}4285$	$0.0^{5}96449$	$0.0^{5}22768$	$0.0^{6}5375$
	$h/A = 2$	0.2360680	0.05572810	0.01315562	$0.0^{2}310562$	$0.0^{3}733138$	$0.0^{3}173070$	$0.0^{4}40856$			
1	$h/A = 4$	−0.02985754	$-0.0^{2}735126$	$-0.0^{2}135747$	$-0.0^{3}222817$	$-0.0^{4}34287$	$-0.0^{5}50652$	$-0.0^{6}7275$	$-0.0^{4}34507$	$-0.0^{5}91641$	$-0.0^{5}24037$
	$h/A = 2$	−0.1055728	−0.04984472	−0.01765011	$-0.0^{2}555550$	$-0.0^{2}163935$	$-0.0^{3}464396$	$-0.0^{3}127901$			
2	$h/A = 4$	$0.0^{2}713340$	$0.0^{2}264780$	$0.0^{3}653556$	$0.0^{3}134296$	$0.0^{4}24824$	$0.0^{5}42814$	$0.0^{6}7031$	$0.0^{4}68628$	$0.0^{4}20275$	$0.0^{5}58556$
	$h/A = 2$	0.04472136	0.03226018	0.01537008	$0.0^{2}608009$	$0.0^{2}216071$	$0.0^{3}715932$	$0.0^{3}225776$			
3	$h/A = 4$	$-0.0^{2}167845$	$-0.0^{3}839224$	$-0.0^{3}260246$	$-0.0^{4}64366$	$-0.0^{4}13908$	$-0.0^{5}27454$	$-0.0^{6}5078$	$-0.0^{4}99915$	$-0.0^{4}32548$	$-0.0^{4}10275$
	$h/A = 2$	−0.01788854	−0.01788854	−0.01085473	$-0.0^{2}520979$	$-0.0^{2}217576$	$-0.0^{3}828169$	$-0.0^{3}294952$			
4	$h/A = 4$	$0.0^{3}388758$	$0.0^{3}246831$	$0.0^{4}92561$	$0.0^{4}26831$	$0.0^{5}66461$	$0.0^{5}14791$	$0.0^{6}3044$	$0.0^{3}118586$	$0.0^{4}42294$	$0.0^{4}14506$
	$h/A = 2$	$0.0^{2}670820$	$0.0^{2}894427$	$0.0^{2}670820$	$0.0^{2}382091$	$0.0^{2}184413$	$0.0^{3}795915$	$0.0^{3}316808$			
5	$h/A = 4$	$-0.0^{4}88569$	$-0.0^{4}68967$	$-0.0^{4}30491$	$-0.0^{4}10164$	$-0.0^{5}28438$	$-0.0^{6}7052$	$-0.0^{6}1600$	$-0.0^{3}121361$	$-0.0^{4}47119$	$-0.0^{4}17473$
	$h/A = 2$	$-0.0^{2}232551$	$-0.0^{2}411436$	$-0.0^{3}375659$	$-0.0^{2}250440$	$-0.0^{2}138083$	$-0.0^{3}669318$	$-0.0^{3}295393$			
6	$h/A = 4$	$0.0^{4}19825$	$0.0^{4}18534$	$0.0^{5}94910$	$0.0^{5}35872$	$0.0^{5}11210$	$0.0^{6}3069$	$0.0^{7}762$	$0.0^{3}110764$	$0.0^{4}46599$	$0.0^{4}18608$
	$h/A = 2$	$0.0^{3}724486$	$0.0^{2}175308$	$0.0^{2}194091$	$0.0^{2}150264$	$0.0^{3}939148$	$0.0^{3}507713$	$0.0^{3}246934$			
7	$h/A = 4$	$-0.0^{5}43535$	$-0.0^{5}48243$	$-0.0^{5}28260$	$-0.0^{5}11982$	$-0.0^{6}4145$	$-0.0^{6}1243$	$-0.0^{7}335$	$-0.0^{4}92162$	$-0.0^{4}41856$	$-0.0^{4}17937$
	$h/A = 2$	$-0.0^{3}189619$	$-0.0^{3}690498$	$-0.0^{3}933782$	$-0.0^{3}837184$	$-0.0^{3}590322$	$-0.0^{3}354193$	$-0.0^{3}188975$			
8	$h/A = 4$	$0.0^{6}9357$	$0.0^{5}12216$	$0.0^{6}8113$	$0.0^{6}3828$	$0.0^{6}1455$	$0.0^{7}475$	$0.0^{7}139$	$0.0^{4}70982$	$0.0^{4}34695$	$0.0^{4}15912$
	$h/A = 2$	$0.0^{4}32691$	$0.0^{3}248472$	$0.0^{3}41980$	$0.0^{3}436838$	$0.0^{3}346814$	$0.0^{3}230226$	$0.0^{3}134298$			
9	$h/A = 4$	$-0.0^{6}1962$	$-0.0^{6}3017$	$-0.0^{6}2258$	$-0.0^{6}1178$	$-0.0^{7}489$	$-0.0^{7}173$	$-0.0^{8}54$	$-0.0^{4}51161$	$-0.0^{4}26851$	$-0.0^{4}13149$
	$h/A = 2$	$0.0^{5}3488$	$-0.0^{4}79336$	$-0.0^{3}17611$	$-0.0^{3}214484$	$-0.0^{3}19185$	$-0.0^{3}140693$	$-0.0^{4}8953$			
10	$h/A = 4$	$0.0^{7}399$	$0.0^{7}728$	$0.0^{7}612$	$0.0^{7}351$	$0.0^{7}159$	$0.0^{8}61$	$0.0^{8}20$	$0.0^{4}34790$	$0.0^{4}19569$	$0.0^{4}10214$
	$h/A = 2$	$-0.0^{5}69488$	$0.0^{4}20818$	$0.0^{4}684478$	$0.0^{4}99253$	$0.0^{3}10040$	$0.0^{4}81346$	$0.0^{4}56405$			
11	$h/A = 4$	$-0.0^{8}785$	$-0.0^{7}171$	$-0.0^{7}162$	$-0.0^{7}102$	$-0.0^{8}50$	$-0.0^{8}20$	$-0.0^{9}73$	$-0.0^{4}22455$	$-0.0^{4}13520$	$-0.0^{5}75114$
	$h/A = 2$	$0.0^{5}42936$	$-0.0^{5}32768$	$-0.0^{4}24236$	$-0.0^{4}43227$	$-0.0^{4}49836$	$-0.0^{4}42692$	$-0.0^{4}33762$			
12	$h/A = 4$	$0.0^{8}147$	$0.0^{8}395$	$0.0^{8}417$	$0.0^{8}29$	$0.0^{8}15$	$0.0^{9}67$	$0.0^{9}26$	$0.0^{4}13821$	$0.0^{5}89010$	$0.0^{5}58178$
	$h/A = 2$	$-0.0^{5}19632$	$-0.0^{6}7344$	$0.0^{5}75281$	$0.0^{4}17626$	$0.0^{4}23475$	$0.0^{4}23396$	$0.0^{4}19275$			
13	$h/A = 4$	$-0,0^{9}26$	$-0.0^{9}89$	$-0.0^{8}10$	$-0.0^{9}79$	$-0.0^{9}45$	$-0.0^{9}21$	$-0.0^{10}9$	$-0.0^{5}81405$	$-0.0^{5}56059$	$-0.0^{5}34803$
	$h/A = 2$	$-0.0^{8}8066$	$0.0^{8}2863$	$-0.0^{5}21041$	$-0.0^{5}67359$	$-0.0^{4}10504$	$-0.0^{4}11694$	$-0.0^{4}10526$			
14	$h/A = 4$	$0.0^{10}10$	$0.0^{9}19$	$0.0^{9}26$	$0.0^{9}21$	$0.0^{9}13$	$0.0^{10}66$	$0.0^{10}3$	$0.0^{5}46011$	$0.0^{5}33885$	$0.0^{5}23236$
	$h/A = 2$	$0.0^{6}4080$	$0.0^{8}76$	$0.0^{6}5360$	$0.0^{5}24154$	$0.0^{5}44680$	$0.0^{5}55890$	$0.0^{5}55107$			
15	$h/A = 2$	$-0.0^{8}1636$	$-0.0^{7}827$	$-0.0^{6}1392$	$-0.0^{6}8165$	$-0.0^{5}18091$	$-0.0^{5}25578$	$-0.0^{5}27708$	$-0.0^{5}25010$	$-00^{5}19706$	$-0.0^{5}14356$
16	$h/A = 2$	$0.0^{7}519$	$0.0^{7}678$	$0.0^{7}497$	$0.0^{6}2642$	$0.0^{6}6988$	$0.0^{5}11224$	$0.0^{5}13400$	$0.0^{6}13097$	$0.0^{5}11050$	$0.0^{6}8553$
17	$h/A = 2$	$-0.0^{7}108$	$-0.0^{7}379$	$-0.0^{7}269$	$-0.0^{7}851$	$-0.0^{6}2588$	$-0.0^{6}4729$	$-0.0^{6}6241$	$-0.0^{6}6618$	$-0.0^{6}5985$	$-0.0^{6}4923$
18	$h/A = 2$	$-0.0^{8}74$	$0.0^{7}167$	$0.0^{7}163$	$0.0^{7}293$	$0.0^{7}929$	$0.0^{6}1919$	$0.0^{6}2804$	$0.0^{6}3231$	$0.0^{6}3135$	$0.0^{6}2743$

TABLE 2.

e \ k	h/A	1	2	3	4	5	6	7	8	9	10
0	$h/A = 4$	0.1212678	0.01492874	$0.0^{2}183781$	$0.0^{3}226245$	$0.0^{4}27852$	$0.0^{5}34287$	$0.0^{6}4221$			
	$h/A = 2$	0.2236068	0.05278641	0.01246118	$0.0^{2}294168$	$0.0^{3}694438$	$0.0^{3}163635$	$0.0^{4}386997$	$0.0^{5}91358$	$0.0^{5}21567$	$0.0^{6}50912$
1	$h/A = 4$	-0.02853364	$-0.0^{2}713341$	$-0.0^{3}132390$	$-0.0^{3}217852$	$-0.0^{4}33574$	$-0.0^{5}49647$	$-0.0^{6}7136$			
	$h/A = 2$	-0.08944270	-0.04472136	-0.01613008	$-0.0^{2}512336$	$-0.0^{2}152002$	$-0.0^{3}432143$	$-0.0^{3}119322$	$-0.0^{4}322538$	$-0.0^{5}85786$	$-0.0^{5}22528$
2	$h/A = 4$	$0.0^{3}650398$	$0.0^{2}251767$	$0.0^{3}629418$	$0.0^{3}130123$	$0.0^{4}241373$	$0.0^{5}41725$	$0.0^{6}6864$			
	$h/A = 2$	0.03130496	0.02683281	0.01341641	$0.0^{2}542736$	$0.0^{2}195367$	$0.0^{3}652728$	$0.0^{3}207042$	$0.0^{4}632039$	$0.0^{4}187341$	$0.0^{5}54246$
3	$h/A = 4$	$-0.0^{2}143162$	$-0.0^{3}777516$	$-0.0^{3}246831$	$-0.0^{4}61708$	$-0.0^{4}13415$	$-0.0^{5}26584$	$-0.0^{6}49302$			
	$h/A = 2$	$-0.0^{2}894428$	-0.01341641	$-0.0^{2}894428$	$-0.0^{2}447214$	$-0.0^{2}191046$	$-0.0^{3}737652$	$-0.0^{3}265305$	$-0.0^{4}905166$	$-0.0^{4}29647$	$-0.0^{5}93986$
4	$h/A = 4$	$0.0^{3}302549$	$0.0^{3}221422$	$0.0^{4}86209$	$0.0^{4}25409$	$0.0^{5}63523$	$0.0^{5}14219$	$0.0^{6}2938$			
	$h/A = 2$	$0.0^{2}156525$	$0.0^{2}581377$	$0.0^{2}514295$	$0.0^{3}313049$	$0.0^{3}156525$	$0.0^{3}690417$	$0.0^{3}278882$	$0.0^{3}105497$	$0.0^{4}37925$	$0.0^{4}13089$
5	$h/A = 4$	$-0.0^{4}60768$	$-0.0^{4}59476$	$-0.0^{4}27800$	$-0.0^{5}94910$	$-0.0^{5}269037$	$-0.0^{6}6726$	$-0.0^{6}1535$			
	$h/A = 2$	$0.0^{3}304105$	$-0.0^{3}217346$	$-0.0^{3}262962$	$-0.0^{3}194091$	$-0.0^{3}112698$	$-0.0^{3}563489$	$-0.0^{3}253856$	$-0.0^{3}105829$	$-0.0^{4}41537$	$-0.0^{4}15533$
6	$h/A = 4$	$0.0^{4}11383$	$0.0^{4}15237$	$0.0^{5}84426$	$0.0^{5}32970$	$0.0^{5}10485$	$0.0^{6}2901$	$0.0^{7}725$			
	$h/A = 2$	$-0.0^{3}483885$	$0.0^{3}663665$	$0.0^{3}120837$	$0.0^{3}108941$	$0.0^{3}732536$	$0.0^{3}413225$	$0.0^{3}206613$	$0.0^{4}94487$	$0.0^{4}40321$	$0.0^{4}16277$
7	$h/A = 4$	$-0.0^{5}19103$	$-0.0^{5}37425$	$-0.0^{5}24432$	$-0.0^{6}10818$	$-0.0^{6}3828$	$-0.0^{6}1164$	$-0.0^{7}317$			
	$h/A = 2$	$0.0^{3}307325$	$-0.0^{3}130765$	$-0.0^{3}496944$	$-0.0^{3}559732$	$-0.0^{3}436838$	$-0.0^{3}277451$	$-0.0^{3}153484$	$-0.0^{4}767419$	$-0.0^{4}35491$	$-0.0^{4}15420$
8	$h/A = 4$	$0.0^{2}2569$	$0.0^{3}8829$	$0.0^{6}6788$	$0.0^{6}3387$	$0.0^{6}1325$	$0.0^{7}4405$	$0.0^{7}130$			
	$h/A = 2$	$-0.0^{3}145814$	$-0.0^{4}15697$	$0.0^{3}178505$	$0.0^{3}264169$	$0.0^{3}241294$	$0.0^{3}172669$	$0.0^{3}105520$	$0.0^{4}57556$	$0.0^{4}28778$	$0.0^{4}13425$
9	$h/A = 4$	$-0.0^{7}142$	$-0.0^{6}1998$	$-0.0^{6}1819$	$-0.0^{6}1019$	$-0.0^{7}439$	$-0.0^{7}159$	$-0.0^{8}50$			
	$h/A = 2$	$0.0^{4}55535$	$0.0^{4}34744$	$-0.0^{4}52047$	$-0.0^{3}11408$	$-0.0^{3}124066$	$-0.0^{3}100404$	$-0.0^{4}67789$	$-0.0^{4}40289$	$-0.0^{4}21744$	$-0.0^{4}10872$
10	$h/A = 4$	$-0.0^{8}72$	$0.0^{7}431$	$0.0^{7}472$	$0.0^{7}296$	$0.0^{7}140$	$0.0^{8}548$	$0.0^{8}19$			
	$h/A = 2$	$-0.0^{4}15960$	$-0.0^{4}23615$	$0.0^{5}90113$	$0.0^{4}44433$	$0.0^{4}59437$	$0.0^{4}54819$	$0.0^{4}40967$	$0.0^{4}26527$	$0.0^{4}15438$	$0.0^{5}82625$
11	$h/A = 4$	$0.0^{8}40$	$-0.0^{8}88$	$-0.0^{7}118$	$-0.0^{8}84$	$-0.0^{8}43$	$-0.0^{8}18$	$-0.0^{9}67$			
	$h/A = 2$	$0.0^{5}20904$	$0.0^{4}11779$	$0.0^{5}22032$	$-0.0^{4}15056$	$-0.0^{4}26440$	$-0.0^{4}28171$	$-0.0^{4}23396$	$-0.0^{4}16521$	$-0.0^{4}10366$	$-0.0^{5}59340$
12	$h/A = 4$	$-0.0^{8}14$	$0.0^{8}17$	$0.0^{8}29$	$0.0^{8}23$	$0.0^{8}13$	$0.0^{9}59$	$0.0^{9}23$			
	$h/A = 2$	$0.0^{6}132335$	$-0.0^{5}47371$	$-0.0^{5}32865$	$0.0^{5}40027$	$0.0^{4}10815$	$0.0^{4}13624$	$0.0^{4}12661$	$0.0^{5}97723$	$0.0^{5}66138$	$0.0^{5}45790$
13	$h/A = 4$	$0.0^{9}42$	$-0.0^{9}28$	$-0.0^{9}68$	$-0.0^{9}61$	$-0.0^{9}38$	$-0.0^{9}18$	$-0.0^{10}77$			
	$h/A = 2$	$-0.0^{6}27857$	$0.0^{6}8149$	$0.0^{5}18936$	$-0.0^{5}5488$	$-0.0^{5}40025$	$-0.0^{5}61883$	$-0.0^{5}65014$	$-0.0^{5}55055$	$-0.0^{5}40245$	$-0.0^{5}25736$
14	$h/A = 4$	$-0.0^{9}11$	$0.0^{10}36$	$0.0^{9}15$	$0.0^{9}16$	$0.0^{9}11$	$0.0^{10}56$	$0.0^{10}25$			
	$h/A = 2$	$0.0^{5}12724$	$0.0^{6}2540$	$-0.0^{6}7534$	$-0.0^{6}2112$	$0.0^{5}12999$	$0.0^{6}26295$	$0.0^{6}31689$	$0.0^{5}29597$	$0.0^{5}23419$	$0.0^{5}16890$
15	$h/A = 2$	$-0.0^{6}4479$	$-0.0^{6}3371$	$0.0^{6}1928$	$0.0^{6}2172$	$-0.0^{6}3454$	$-0.0^{5}10382$	$-0.0^{5}14650$	$-0.0^{5}15200$	$-0.0^{5}13058$	$-0.0^{5}10093$
16	$h/A = 2$	$0.0^{6}1082$	$0.0^{6}2056$	$0.0^{8}35$	$-0.0^{6}1076$	$0.0^{7}592$	$0.0^{6}3768$	$0.0^{6}6415$	$0.0^{6}7462$	$0.0^{6}6987$	$0.0^{6}5799$
17	$h/A = 2$	$-0.0^{8}22$	$-0.0^{7}926$	$-0.0^{7}420$	$0.0^{7}338$	$0.0^{8}454$	$-0.0^{6}1236$	$-0.0^{6}2653$	$-0.0^{6}3501$	$-0.0^{6}3591$	$-0.0^{6}3207$
18	$h/A = 2$	$-0.0^{7}168$	$0.0^{7}319$	$0.0^{7}324$	$-0.0^{8}25$	$-0.0^{8}87$	$0.0^{7}357$	$0.0^{6}1033$	$0.0^{6}1569$	$0.0^{6}1774$	$0.0^{6}1710$

search for new formulae for those quantities in which the stated deficiencies are inherent to a possibly weakened degree (there is almost nothing which can completely prevent the success of the numerical process) or are completely absent (which, apparently, is always observed in a case which, unfortunately, is rarely encountered, when the found formula consists of terms with constant signs).

It is possible that we will also find ourselves forced for the determination of the $r_\nu^{(k)}$ and $h_\nu^{(k)}$ to select more convenient recurrence relations.

Having at one's disposal the modern digital computers, it presents no difficulty to retain right from the beginning in the intermediate quantities larger numbers of terms than at a first glance might be required to guarantee certain decimals and to arrive in the end at the desired result, so that in the values of the stress components, evaluated at certain points of the medium and, especially, in the region of their concentration, one is guaranteed an accuracy of several (for example, two to three) significant digits. Thus, the process of taking by the proposed method the influence of one of the two closely spaced boundary regions into account will bear straight away a more complete and final character and it does not entail in itself anything new, having mainly the technical character of a complication. We note for the information of the reader that these complications which, as a rule, arise in the application of mechanical computers with insufficiently large registers (again this is true mainly for the case discussed when the boundaries of the regions are close to each other) have all been overcome by us in the papers published up to date; at times this success came after a number of attempts in the direction of combined utilization of several of the quoted methods, which realized in a simpler and quicker manner the goal in each separate case of non-simply connected regions of a special type.

The first of the numerical examples studied here ($h = 4A$) did not encounter difficulties of the stated type; however, in the second example ($h = 2A$), such difficulties manifest themselves to a minor degree and their resolution does not require any major effort.

§ 4. Following the well known method of N. I. Muskhelishvili [3] and remembering that the functions $\varphi(z)$ and $\psi(z)$, regular in the half-plane S_2, vanish at infinity, we obtain for them from the boundary conditions (2.13)

$$\varphi(z) = \sum_{k=1}^{\infty} \left\{ \delta_k - \frac{1}{1 - \chi^{-2}(z)} \bar{A}_k \right\} \frac{1}{\chi^k(z)} + E \frac{1}{\chi(z)}, \tag{4.1}$$

$$(z - 2ih)\varphi'(z) + \psi(z) = \sum_{k=1}^{\infty} (\bar{\alpha}_k - \bar{\alpha}_{-k}) \frac{1}{\chi^k(z)}. \tag{4.2}$$

It is not difficult to show that the constant C_2 on the right hand side of the equality (1.1) must be set equal to zero. We form now an expression which is important for the following work:

$$-[\bar{z}\varphi'(z) + \psi(z)] = [(z - \bar{z}) - 2ih]\varphi'(z) - \sum_{k=1}^{\infty} (\bar{\alpha}_k - \bar{\alpha}_{-k}) \frac{1}{\chi^k(z)}. \tag{4.3}$$

Differentiating with respect to the abscissa x we obtain

$$(4.4)\quad -[\bar{z}\varphi''(z)+\psi'(z)]= \\ =[z-\bar{z}-2ih]\varphi''(z)+\varphi'(z)-\sum_{k=1}^{\infty}(\bar{\alpha}_k-\bar{\alpha}_{-k})\frac{\partial}{\partial z}\frac{1}{\chi^k(z)}.$$

Further, taking into consideration the relations

$$\frac{\partial}{\partial z}\frac{1}{1-\chi^{-2}(z)}=-\frac{2}{A}\frac{1}{\chi^3(z)}\frac{1}{[1-\chi^{-2}(z)]^3},$$

$$\frac{\partial}{\partial z}\frac{1}{\chi^k(z)}=-\frac{1}{A}k\frac{1}{1-\chi^{-2}(z)}\frac{1}{\chi^{k+1}(z)},$$

$$(4.5)\quad \frac{\partial}{\partial z}\frac{1}{1-\chi^{-2}(z)}\frac{1}{\chi^k(z)}= \\ =-\frac{1}{A}\frac{1}{[1-\chi^{-2}(z)]^2\chi^{k+1}(z)}\left[k+\frac{2}{[1-\chi^{-2}(z)]\chi^2(z)}\right],$$

we bring the derivative of (4.7) into the form

$$(4.6)\quad \varphi'(z)=\frac{1}{A}\sum_{k=1}^{\infty}\frac{1}{[1-\chi^{-2}(z)]\chi^{k+1}(z)}\omega_k(z)-\frac{1}{A}E\frac{1}{[1-\chi^{-2}(z)]\chi^2(z)},$$

where we have under the sum sign on the right hand side the multiplier

$$(4.7)\quad \omega_k(z)=-k\delta_k+\frac{1}{1-\chi^{-2}(z)}\left[k+\frac{2}{[1-\chi^{-2}(z)]\chi^2(z)}\right]\bar{\Delta}_k.$$

Using now side by side with (4.6) the relations

$$\omega_k'(z)=-\frac{2}{A}\bar{\Delta}_k\frac{1}{\chi^3(z)[1-\chi^{-2}(z)]^3}\left\{(k+2)+4\frac{1}{\chi^2(z)[1-\chi^{-2}(z)]}\right\},$$

$$(4.8)$$

$$\theta_k(z)=A\omega_k'(z)-\frac{1}{[1-\chi^{-2}(z)]\chi(z)}\left\{(k+1)+\frac{2}{\chi^2(z)[1-\chi^{-2}(z)]}\right\}\omega_k(z),$$

we write the second derivative of (4.1) in the form

$$(4.9)\quad \varphi''(z)=\frac{1}{A^2}\left\{\sum_{k=1}^{\infty}\frac{1}{[1-\chi^{-2}(z)]\chi^{k+1}(z)}\theta_k(z)+2E\frac{1}{\chi^3(z)[1-\chi^{-2}(z)]^3}\right\}.$$

We may express on the basis of the formulae (3.10) and (3.19) the

function $\varphi(z)$, determined by (4.1), near the ellipse in the form

$$\varphi(z) = \sum_{\nu=0}^{\infty} \varphi_\nu \left(\frac{z}{2A}\right)^\nu + E\frac{1}{\chi(z)} \tag{4.10}$$

with the coefficients

$$\varphi_\nu = \sum_{k=1}^{\infty} (i)^{k+\nu}[\gamma_\nu^{(k)}\delta_k - h_\nu^{(k)}\Delta_k]. \tag{4.11}$$

They are expressible directly in terms of the unknowns α_n by the formula

$$\varphi_\nu = \sum_{k=1}^{\infty} (i)^{k+\nu}\{\varphi_{\nu,k}^{(1)}\alpha_k + \varphi_{\nu,-k}^{(1)}\alpha_{-k} + \varphi_{\nu,k}^{(2)}\bar{\alpha}_k + \varphi_{\nu,-k}^{(2)}\bar{\alpha}_{-k}\}, \tag{4.12}$$

where we have

$$\begin{aligned}
\varphi_{\nu,k}^{(1)} &= \rho^{2k}\gamma_\nu^{(k)},\\
\varphi_{\nu,-k}^{(1)} &= -\rho^{-2k}\gamma_\nu^{(k)},\\
\varphi_{\nu,k}^{(2)} &= k\left\{(\rho^2-1)(h_\nu^{(k)} + \rho^{-2}h_\nu^{(k+2)}) + 2\frac{h}{A}h_\nu^{(k+1)}\right\},\\
\varphi_{\nu,-k}^{(2)} &= k\left\{(\rho^2-1)(\rho^{-2}h_\nu^{(k)} + h_\nu^{(k+2)}) - 2\frac{h}{A}h_\nu^{(k+1)}\right\}.
\end{aligned} \tag{4.13}$$

Further, using the formula (valid at some distance from the straight cut between the foci of the ellipse)

$$\left(\frac{z}{2A}\right)^\nu = \frac{1}{2^\nu}\sum_{e=N(\nu)}^{\nu *} \varepsilon_e^{(1)} C_\nu^{(\nu-e)/2}(\zeta^e + \zeta^{-e}), \tag{4.14}$$

$$\varepsilon_0^{(1)} = \tfrac{1}{2},\ \varepsilon_e^{(1)} = 1\ \ (e \neq 0),$$

where $N(\nu) = 0$ for even ν and $N(\nu) = 1$ for odd ν, we obtain

$$\sum_{\nu=0}^{\infty} \varphi_\nu \left(\frac{z}{2A}\right)^\nu = \sum_{e=0}^{\infty} \varepsilon_e(\zeta^e + \zeta^{-e}) \sum_{\nu=e}^{\infty *} \varphi_\nu \frac{1}{2^\nu} C_\nu^{(\nu-e)/2}. \tag{4.15}$$

In the last formula we have for the internal sum, by (4.12) and (4.13),

$$\sum_{\nu=|e|}^{\infty *} \varphi_\nu \frac{1}{2^\nu} C_\nu^{(\nu-|e|)/2} = i^{|e|+1} q_{|e|}, \tag{4.16}$$

$$q_{|e|} = \sum_{k=1}^{\infty} (i)^{k-1}\{\alpha_k q_{|e|,k}^{(1)} + \alpha_{-k} q_{|e|,-k}^{(1)} + \bar{\alpha}_k q_{|e|,k}^{(2)} + \bar{\alpha}_{-k} q_{|e|,-k}^{(2)}\},$$

where the following notation has been introduced:

$$
\begin{aligned}
q^{(1)}_{|e|,\,k} &= \rho^{2k}\xi^{(1,1)}_{|e|,\,k} \quad , \\
q^{(1)}_{|e|,\,-k} &= -\rho^{-2k}\xi^{(1,1)}_{|e|,\,k} \quad , \\
q^{(2)}_{|e|,\,k} &= k\left\{2\frac{h}{A}\,\xi^{(2,2)}_{|e|,\,k} + (\rho^2-1)\xi^{(2,1)}_{|e|,\,k} - \rho^{-2}(\rho^2-1)^2\xi^{(2,2)}_{|e|,\,k+1}\right\}, \\
q^{(2)}_{|e|,\,-k} &= k\left\{-2\frac{h}{A}\,\xi^{(2,2)}_{|e|,\,k} + (\rho^2-1)\xi^{(2,1)}_{|e|,\,k} - \rho^{-2}(\rho^2-1)^2\xi^{(2,2)}_{|e|,\,k-1}\right\},
\end{aligned}
\tag{4.17}
$$

with the quantities

$$
\begin{aligned}
\xi^{(1,1)}_{|e|,\,k} &= \sum_{\nu=|e|}^{\infty}{}^{*}\,(-1)^{(\nu-|e|)/2}\,\frac{1}{2^\nu}\,C_\nu^{(\nu-|e|)/2}\gamma_\nu^{(k)}, \\
\xi^{(2,1)}_{|e|,\,k} &= \sum_{\nu=|e|}^{\infty}{}^{*}\,(-1)^{(\nu-|e|)/2}\,\frac{1}{2^\nu}\,C_\nu^{(\nu-|e|)/2}(h_\nu^{(k)} + h_\nu^{(k+2)}), \\
\xi^{(2,2)}_{e,k} &= \sum_{\nu=|e|}^{\infty}{}^{*}\,(-1)^{(\nu-|e|)/2}\,\frac{1}{2^\nu}\,C_\nu^{(\nu-|e|)/2}h_\nu^{(k+1)}.
\end{aligned}
\tag{4.18}
$$

We find with the help of (4.13) the following expansion, important for the subsequent work (which is true for any ellipse in the half-plane S_2 and confocal with S_1):

$$
\frac{1}{\chi^k(z)} = \sum_{e=0}^{\infty}(i)^{k+e}\varepsilon_e^{(1)}(\zeta^e + \zeta^{-e})\xi^{(1,1)}_{e,k}. \tag{4.19}
$$

An analogous expansion with $\xi^{(1,1)}_{e,k}$ replaced by $\xi^{(2,2)}_{e,k-1}$ is true for the second function (3.19).

On the basis of (4.19) the expression

$$
I(z) = -\sum_{k=1}^{\infty}(\bar{\alpha}_k - \bar{\alpha}_{-k})\,\frac{1}{\chi^k(z)} \tag{4.20}
$$

after some simple manipulations assumes the form of a Laurent series (converging at least in the ring $1 < |\zeta| < \frac{1}{2}\{h + (h^2 + 4A^2)^{\frac{1}{2}}\}$ in the transformed ζ plane)

$$
\begin{aligned}
I(z) &= -\sum_{e=-\infty}^{\infty}(i)^{|e|+1}I_{|e|}\zeta^e, \\
I_{|e|} &= \sum_{k=1}^{\infty}(i)^{k-1}(\bar{\alpha}_k - \bar{\alpha}_{-k})\xi^{(1,1)}_{|e|,\,k}.
\end{aligned}
\tag{4.21}
$$

§ 5. We consider now on the ellipse L_1 the expression

$$\Lambda = [2ih - (z - \bar{z})]\varphi'(z). \tag{5.1}$$

After several manipulations and transformations (which have been deleted for the sake of brevity, but which will be readily reproduced by the reader) one obtains the Fourier series

$$\Lambda = \sum_{e=-\infty}^{\infty} (i)^{|e|+1}(p_e - E\Delta_{2,e})\zeta^e. \tag{5.2}$$

The first of the two terms in the brackets under the sum sign gives, apart from the multiplier $(i)^{|e|+1}$, the values of the coefficients of the expansion so that

$$p_e = \sum_{k=1}^{\infty} (i)^{k+|e|}\{p^{(1)}_{e,k}\alpha_k + p^{(1)}_{e,-k}\alpha_{-k} + p^{(2)}_{e,k}\bar{\alpha}_k + p^{(2)}_{e,-k}\bar{\alpha}_{-k}\}. \tag{5.3}$$

The quantities in braces on the right hand side of this equality, as multipliers of the unknowns α_k, are given by $(e \geq 0)$

$$\begin{aligned}
p^{(1)}_{e,k} &= \rho^{2k}\left\{-\frac{h}{A}\eta^{(1,1)}_{e,k} - (\rho^2-1)\eta^{(1,2)}_{e,k} + \rho^{-2}(\rho^2-1)^2\eta^{(1,3)}_{e,k}\right\},\\
p^{(1)}_{e,-k} &= \rho^{-2k}\left\{\frac{h}{A}\eta^{(1,1)}_{e,k} + (\rho^2-1)\eta^{(1,2)}_{e,k} - \rho^{-2}(\rho^2-1)^2\eta^{(1,3)}_{e,k}\right\},\\
p^{(1)}_{-e,k} &= \rho^{2k}\left\{-\frac{h}{A}\eta^{(1,1)}_{e,k} + (\rho^2-1)\eta^{(1,2)}_{e,k} + \rho^{-2}(\rho^2-1)^2\eta^{(1,3)^*}_{e,k}\right\},\\
p^{(1)}_{-e,-k} &= \rho^{-2k}\left\{\frac{h}{A}\eta^{(1,1)}_{e,k} - (\rho^2-1)\eta^{(1,2)}_{e,k} - \rho^{-2}(\rho^2-1)^2\eta^{(1,3)^*}_{e,k}\right\},
\end{aligned} \tag{5.4}$$

with the additional notation

$$\begin{aligned}
\eta^{(1,1)}_{e,k} &= \sum_{\nu=e}^{\infty}{}^{*}\,(-1)^{(\nu-e)/2}\frac{1}{2^\nu}(\nu+1)C^{(\nu-e)/2}_{\nu}\gamma^{(k)}_{\nu+1},\\
\eta^{(1,2)}_{e,k} &= \sum_{\nu=e}^{\infty}{}^{*}\,(-1)^{(\nu-e)/2}\frac{1}{2^\nu}\nu[C^{(\nu-e)/2}_{\nu-1} - C^{[(\nu-e)/2]-1}_{\nu-1}]\gamma^{(k)}_{\nu},\\
\eta^{(1,3)}_{e,k} &= \sum_{\nu=e}^{\infty}{}^{*}\,(-1)^{(\nu-e)/2}\frac{1}{2^\nu}\nu C^{(\nu-e)/2}_{\nu-1}\gamma^{(k)}_{\nu},\\
\eta^{(1,3)^*}_{e,k} &= \sum_{\nu=e}^{\infty}{}^{*}\,(-1)^{(\nu-e)/2}\frac{1}{2^\nu}C^{[(\nu-e)/2]-1}_{\nu-1}\gamma^{(k)}_{\nu}.
\end{aligned} \tag{5.5}$$

The second, more complicated group of multipliers of the same α_k in (5.3) may conveniently be given in the following form, to avoid, as far as possible, a repetition of unnecessary cumbersomeness:

$$p^{(2)}_{\pm e,\,\pm k} = k \sum_{\nu=1}^{5} (\rho^2 - 1)^{\nu-1} p^{(2,\nu)}_{\pm e,\,\pm k}, \tag{5.6}$$

where the values of the $p^{(2,\nu)}_{\pm e,\pm k}$ are given by the following group of formulae (5.7):

(5.7′) $(e \geq 0)$

$$\begin{aligned}
p^{(2,1)}_{e,k} &= -2\left(\frac{h}{A}\right)^2 \eta^{(2,1)}_{e,k},\\
p^{(2,2)}_{e,k} &= -\frac{h}{A}\left[2\eta^{(2,2)}_{e,k} + (\eta^{(2,1)}_{e,k-1} + \eta^{(2,1)}_{e,k+1})\right],\\
p^{(2,3)}_{e,k} &= \rho^{-2}\frac{h}{A}\left[\eta^{(2,1)}_{e,k+1} + 2\eta^{(2,3)}_{e,k-1}\right] - (\eta^{(2,2)}_{e,k-1} + \eta^{(2,2)}_{e,k+1}),\\
p^{(2,4)}_{e,k} &= \rho^{-2}[\eta^{(2,3)}_{e,k-2} + \eta^{(2,3)}_{e,k} + \eta^{(2,2)}_{e,k+1}],\\
p^{(2,5)}_{e,k} &= -\rho^{-4}\eta^{(2,3)}_{e,k},
\end{aligned}$$

$(e \geq 0)$

$$\begin{aligned}
p^{(2,1)}_{e,-k} &= 2\left(\frac{h}{A}\right)^2 \eta^{(2,1)}_{e,k},\\
p^{(2,2)}_{e,-k} &= \frac{h}{A}\left[2\eta^{(2,2)}_{e,k} - (\eta^{(2,1)}_{e,k-1} + \eta^{(2,1)}_{e,k+1})\right],\\
p^{(2,3)}_{e,-k} &= \rho^{-2}\frac{h}{A}(\eta^{(2,1)}_{e,k-1} - 2\eta^{(2,3)}_{e,k-1}) - (\eta^{(2,2)}_{e,k-1} + \eta^{(2,2)}_{e,k+1}),\\
p^{(2,4)}_{e,-k} &= \rho^{-2}[\eta^{(2,2)}_{e,k-1} + \eta^{(2,3)}_{e,k-2} + \eta^{(2,3)}_{e,k}],\\
p^{(2,5)}_{e,-k} &= -\rho^{-4}\eta^{(2,3)}_{e,k-2},
\end{aligned}$$

(5.7″) $(e > 0)$

$$\begin{aligned}
p^{(2,1)}_{-e,k} &= -2\left(\frac{h}{A}\right)^2 \eta^{(2,1)}_{e,k},\\
p^{(2,2)}_{-e,k} &= \frac{h}{A}\left[2\eta^{(2,2)}_{e,k} - (\eta^{(2,1)}_{e,k-1} + \eta^{(2,1)}_{e,k+1})\right],\\
p^{(2,3)}_{-e,k} &= \rho^{-2}\frac{h}{A}\left[\eta^{(2,1)}_{e,k+1} + 2(\eta^{(2,3)}_{e,k-1} - \eta^{(2,2)}_{e,k})\right] + (\eta^{(2,2)}_{e,k-1} + \eta^{(2,2)}_{e,k+1}),\\
p^{(2,4)}_{-e,k} &= \rho^{-2}[(\eta^{(2,3)}_{e,k-2} - \eta^{(2,2)}_{e,k-1}) + (\eta^{(2,3)}_{e,k} - \eta^{(2,2)}_{e,k+1}) - \eta^{(2,2)}_{e,k+1}],\\
p^{(2,5)}_{-e,k} &= -\rho^{-4}(\eta^{(2,3)}_{e,k} - \eta^{(2,2)}_{e,k+1}),
\end{aligned}$$

$$(5.7'')\quad (e>0)\quad \begin{cases} p_{-e,-k}^{(2,1)} = 2\left(\dfrac{h}{A}\right)^2 \eta_{e,k}^{(2,1)}, \\ p_{-e,-k}^{(2,2)} = -\dfrac{h}{A}\left[2\eta_{e,k}^{(2,2)} + (\eta_{e,k-1}^{(2,1)} + \eta_{e,k+1}^{(2,1)})\right], \\ p_{-e,-k}^{(2,3)} = \rho^{-2}\dfrac{h}{A}\left[\eta_{e,k-1}^{(2,1)} - 2(\eta_{e,k-1}^{(2,3)} - \eta_{e,k}^{(2,2)})\right] + (\eta_{e,k-1}^{(2,2)} + \eta_{e,k+1}^{(2,2)}), \\ p_{-e,-k}^{(2,4)} = \rho^{-2}[\eta_{e,k-2}^{(2,3)} - 2\eta_{e,k-1}^{(2,2)}] + \eta_{e,k}^{(2,3)} - \eta_{e,k+1}^{(2,2)}, \\ p_{-e,-k}^{(2,5)} = -\rho^{-4}(\eta_{e,k-2}^{(2,3)} - \eta_{e,k-1}^{(2,2)}). \end{cases}$$

In these formulae

$$(5.8)\quad \begin{aligned} \eta_{e,k}^{(2,1)} &= \sum_{\nu=e}^{\infty}{}^{*}\,(-1)^{(\nu-e)/2}\frac{1}{2^\nu}(\nu+1)C_\nu^{(\nu-e)/2}h_{\nu+1}^{(k+1)}, \\ \eta_{e,k}^{(2,2)} &= \sum_{\nu=e}^{\infty}{}^{*}\,(-1)^{(\nu-e)/2}\frac{1}{2^\nu}\nu(C_{\nu-1}^{(\nu-e)/2} - C_{\nu-1}^{[(\nu-e)/2]-1})h_\nu^{(k+1)}, \\ \eta_{e,k}^{(2,3)} &= \sum_{\nu=e}^{\infty}{}^{*}\,(-1)^{(\nu-e)/2}\frac{1}{2^\nu}\nu C_{\nu-1}^{(\nu-e)/2}h_\nu^{(k+2)}. \end{aligned}$$

The second of the above mentioned terms in (5.2) is determined by the following relations (it must not be overlooked that it contains the factor E)

$$(5.9)\quad \begin{aligned} \Delta_{2,e} &= \frac{h}{A}\eta_{e,1}^{(1,1)} + (\rho^2-1)\eta_{e,1}^{(1,2)} - \rho^{-2}(\rho^2-1)^2\eta_{e,1}^{(1,3)}\quad (e=0,1,\cdots), \\ \Delta_{2,-e} &= \frac{h}{A}\eta_{e,1}^{(1,1)} - (\rho^2-1)\eta_{e,1}^{(1,2)} - \rho^{-2}(\rho^2-1)^2\eta_{e,1}^{(1,3)*}\quad (e=1,2,\cdots). \end{aligned}$$

By the strength of the relations (4.14), (4.20), (5.1) and (5.2), we will have for the function (4.3), considered on the contour L_1, the Fourier expansion

$$(5.10)\qquad \bar{t}\varphi'(t) + \psi(t) = \sum_{e=-\infty}^{\infty}(i)^{|e|+1}[p_e + I_{|e|} - E\Delta_{2,e}]\tau^e.$$

We construct now on the same contour L_1 the expression

$$(5.11)\qquad \Omega(t) = \varphi(t) - [\overline{t\varphi'(t)} + \overline{\psi(t)}].$$

On the basis of (4.15), (4.19) and (5.10), it assumes the form

$$(5.12)\qquad \Omega(t) = \sum_{e=-\infty}^{\infty}(i)^{|e|+1}\{\Omega_e + G_e\}\tau^e,$$

where

$$\Omega_e = q_{|e|} - (-1)^{|e|+1}(\bar{p}_{-e} + \bar{I}_{|e|})\rho^{-2e},$$
$$G_e = E[\delta_{|e|}^{(1)} + (-1)^{|e|+1}\Delta_{2,-e}\rho^{-2e}]. \tag{5.13}$$

The formulae (5.3)—(5.8) look cumbersome enough and at first they may even scare the reader by creating the false impression that the evaluation of the quantities with which we are concerned and, more so, the deduction of numerical results from the solution is associated with burdensome computations. However, they are far from complicated, as will be shown by the first impression gained in the course of the calculations and it is not too difficult to complete the study of the problem in a proper manner (in the sense of desired accuracy). The numerical side of the task is essentially simplified due to the fact that the quantities on the right hand side of (5.3)—(5.8) which are interrelated are easily ascertained by use of simple relations. Hence there is no need to find each of the quantities, with the exception of $\xi_{e,k}^{(1,1)}$; at least this concerns in full measure the above mentioned equalities.

In fact, differentiating (3.9) with respect to z and comparing the result with the second formula in (3.19), we find

$$h_\nu^{(k+1)} = -\frac{1}{2k}(\nu+1)\gamma_{\nu+1}^{(k)}.$$

Taking this relation into consideration we obtain from (4.18) and (5.5)

$$\xi_{e,k}^{(2,2)} = -\frac{1}{2k}\eta_{e,k}^{(1,1)}. \tag{5.14}$$

The expression in brackets under the sum sign in the second of the formulae (5.5) and (5.8) is equal to $e/\nu C_\nu^{(\nu-e)/2}$; therefore

$$\eta_{e,k}^{(1,2)} = e\xi_{e,k}^{(1,1)},$$
$$\eta_{e,k}^{(2,2)} = e\xi_{e,k}^{(2,2)}, \tag{5.15}$$

In the same manner we find

$$\eta_{0,k}^{(1,2)} = 0,$$
$$\eta_{e,k}^{(1,3)} - \eta_{e,k}^{(1,3)*} = \eta_{e,k}^{(1,2)}, \tag{5.16}$$
$$\eta_{e+2,k}^{(1,3)} = -\eta_{e,k}^{(1,3)*} \quad (e = 1, 2, \cdots).$$

It is likewise not difficult to establish the formulae

$$\eta_{e,k}^{(1,2)} = \tfrac{1}{2}[\eta_{e-1,k}^{(1,1)} + \eta_{e+1,k}^{(1,1)}],$$
$$\eta_{e,k}^{(1,3)} = \tfrac{1}{2}\eta_{e-1,k}^{(1,1)} \qquad (e = 1, 2, \cdots). \tag{5.17}$$

The first of these may be obtained in the following manner. We set

$$\eta_{e,k}^{(1,2)} = (\eta_{e,k}^{(1,2)})_1 + (\eta_{e,k}^{(1,2)})_2 \quad ,$$

where

$$(\eta_{e,k}^{(1,2)})_1 = \sum_{\nu=e}^{\infty *} (-1)^{(\nu-e)/2} \frac{1}{2^\nu} \nu C_{\nu-1}^{(\nu-e)/2} \gamma_\nu^{(k)},$$

$$(\eta_{e,k}^{(1,2)})_2 = -\sum_{\nu=e}^{\infty *} (-1)^{(\nu-e)/2} \frac{1}{2^\nu} \nu C_{\nu-1}^{(\nu-e)/2-1} \gamma_\nu^{(k)}.$$

Hence we readily find

$$(\eta_{1+e,k}^{(1,2)})_1 = \tfrac{1}{2}\eta_{e,k}^{(1,1)}, \quad (\eta_{e-1,k}^{(1,2)})_2 = \tfrac{1}{2}\eta_{e,k}^{(1,1)},$$

which for an obvious replacement of the first subscripts leads to (5.17). No difficulties will be encountered in verifying by the same method the truth of the relations

$$\begin{aligned} \xi_{e,k}^{(2,1)} &= \xi_{e,k-1}^{(2,2)} + \xi_{e,k+1}^{(2,2)}, \\ \eta_{e,k}^{(2,2)} &= \tfrac{1}{2}(\eta_{e-1,k}^{(2,1)} + \eta_{1+e,k}^{(2,1)}) \qquad (e = 1, 2, \cdots). \end{aligned} \tag{5.18}$$

Finally, we note the equality

$$\eta_{e,k}^{(2,2)} = \eta_{e,k-1}^{(2,3)} + \tfrac{1}{2}\eta_{1+e,k}^{(2,1)}. \tag{5.19}$$

We arrive at it in an analogous manner as this was done for (5.17) by decomposing $\eta_{e,k}^{(2,2)}$ into two terms; each of these is then determined by a part of the expression (5.8) for the first and second terms respectively in the brackets under the sum sign. It is clear from this that all the remaining quantities in (5.3)—(5.8) are simply evaluated for known $\xi_{e,k}^{(1,1)}$ and initial (i.e., $e = 0$ and $e = 1$) values $\eta_{0,k}^{(1,1)}$, $\eta_{1,k}^{(1,1)}$, $\eta_{0,k}^{(2,1)}$, $\eta_{1,k}^{(2,1)}$ (for certain required k) on the basis of the elementary relations, deduced in this section. Therefore one might limit, for applied work only, the table of the quantities $\xi_{e,k}^{(1,1)}$, $\eta_{0,k}^{(1,1)}$, $\eta_{1,k}^{(1,1)}$, $\eta_{0,k}^{(2,1)}$, $\eta_{1,k}^{(2,1)}$, found from the first formulae (4.18), (5.5) and (5.8); nevertheless we consider it useful to present the tables of all the quantities entering into the expressions for the coefficients of the infinite system of equations constructed below for the unknowns α_n in the limiting case $\rho \to 1$; we note that the computations of these quantities were performed by use of the formulae (5.3)–(5.8) [without employment of the auxiliary relations (5.14)—(5.19)].

The quantities $\xi_{e,k}^{(1,1)}$, $\xi_{e,k}^{(2,2)}$ and $\eta_{e,k}^{(1,1)}$, $\eta_{e,k}^{(2,1)}$ are presented in the same sequence in Tables 3—6.

TABLE 3.

e \ k	h/A	1	2	3	4	5	6	7	8	9	10
0	$h/A = 4$	0.1196788	0.01391817	$0.0^{3}157084$	$0.0^{3}171563$	$0.0^{4}180405$					
	$h/A = 2$	0.2160057	0.04246693	$0.0^{4}74792$	$0.0^{3}112738$	$0.0^{3}11703$	$0.0^{5}214$	$-0.0^{5}415$	$-0.0^{5}1615$	$-0.0^{6}43$	$0.0^{7}15$
1	$h/A = 4$	-0.01432589	$-0.0^{2}3381224$	$-0.0^{3}589952$	$-0.0^{4}90146$	$-0.0^{4}12714$					
	$h/A = 2$	-0.04675332	-0.0193298	$-0.0^{2}571143$	$-0.0^{2}142209$	$-0.0^{3}31172$	$-0.0^{4}6052$	$-0.0^{4}1017$	$-0.0^{5}14$	$-0.0^{6}11$	$0.0^{7}15$
2	$h/A = 4$	$0.0^{2}169061$	$0.0^{3}604332$	$0.0^{3}142307$	$0.0^{4}27630$	$0.0^{5}47783$					
	$h/A = 2$	$0.0^{2}966486$	$0.0^{2}618990$	$0.0^{2}254121$	$0.0^{3}838767$	$0.0^{3}24752$	$0.0^{4}6166$	$0.0^{4}1440$	$0.0^{5}307$	$0.0^{6}60$	$0.0^{7}1$
3	$h/A = 4$	$-0.0^{3}196650$	$-0.0^{4}94871$	$-0.0^{4}28196$	$-0.0^{5}66365$	$-0.0^{5}13549$					
	$h/A = 2$	$-0.0^{2}190359$	$-0.0^{2}169405$	$-0.0^{3}89777$	$-0.0^{3}36809$	$-0.0^{3}12821$	$-0.0^{4}3965$	$-0.0^{4}1115$	$-0.0^{5}288$	$-0.0^{6}68$	$-0.0^{6}15$
4	$h/A = 4$	$0.0^{4}22537$	$0.0^{4}13815$	$0.0^{5}49774$	$0.0^{5}13787$	$0.0^{6}3245$					
	$h/A = 2$	$0.0^{3}355452$	$0.0^{3}419319$	$0.0^{3}27605$	$0.0^{3}13595$	$0.0^{4}5083$	$0.0^{4}2005$	$0.0^{5}651$	$0.0^{5}194$	$0.0^{6}53$	$0.0^{6}12$
5	$h/A = 4$	$-0.0^{5}254299$	$-0.0^{5}19113$	$-0.0^{6}8130$	$-0.0^{6}2596$	$-0.0^{7}693$					
	$h/A = 2$	$-0.0^{4}62460$	$-0.0^{4}96098$	$-0.0^{4}76931$	$-0.0^{4}44598$	$-0.0^{4}2112$	$-0.0^{5}867$	$-0.0^{5}320$	$-0.0^{5}108$	$-0.0^{6}35$	$-0.0^{6}10$
6	$h/A = 2$	$0.0^{4}10120$	$0.0^{4}20578$	$0.0^{4}19842$	$0.0^{4}13375$	$0.0^{5}968$	$0.0^{5}334$	$0.0^{5}137$	$0.0^{6}52$	$0.0^{6}18$	$0.0^{7}34$
7	$h/A = 2$	$-0.0^{5}14055$	$-0\ 0^{5}4099$	$-0.0^{5}47814$	$-0.0^{5}37256$	$-0.0^{5}227$	$-0.0^{5}117$	$0.0^{6}53$	$-0.0^{6}21$	$-0.0^{7}76$	$-0.0^{7}26$
8	$h/A = 2$	$0.0^{5}1563$	$0.0^{6}7566$	$0.0^{5}10816$	$0.0^{6}974$	$-0.0^{6}22$	$0.0^{6}383$	$0.0^{6}19$	$0.0^{7}84$	$0.0^{7}33$	$0.0^{7}19$

TABLE 4.

e \ k	h/A	1	2	3	4	5	6	7	8
1	$h/A = 4$	$-0.0^{3}329275$	$-0.0^{3}577449$	$-0.0^{4}88479$	$-0.0^{4}12502$	$-0.0^{5}16673$			
	$h/A = 2$	-0.01796686	$-0.0^{2}54098$	$-0.0^{2}13629$	$-0.0^{3}30166$	$-0.0^{4}5915$	$-0.0^{4}1006$	$-0.0^{5}138$	$-0.0^{6}1$
2	$h/A = 4$	$0.0^{3}577449$	$0.0^{3}137642$	$0.0^{4}26883$	$0.0^{5}46656$	$0.0^{6}7471$			
	$h/A = 2$	$0.0^{2}54098$	$0.0^{2}231488$	$0.0^{3}78008$	$0.0^{3}22634$	$0.0^{4}5869$	$0.0^{4}1382$	$0.0^{5}297$	$0.0^{6}58$
3	$h/A = 4$	$-0.0^{4}88479$	$-0.0^{4}26883$	$-0.0^{5}63925$	$-0.0^{5}13128$	$-0.0^{6}24398$			
	$h/A = 2$	$-0.0^{2}13629$	$-0.0^{3}78006$	$-0.0^{3}33118$	$-0.0^{3}11772$	$-0.0^{4}36915$	$-0.0^{4}1049$	$-0.0^{5}274$	$-0.0^{6}66$
4	$h/A = 4$	$0.0^{4}12502$	$0.0^{5}46656$	$0.0^{5}13128$	$0.0^{6}3117$	$0.0^{7}659$			
	$h/A = 2$	$0.0^{3}30161$	$0.0^{3}22631$	$0.0^{3}11771$	$0.0^{4}49735$	$0.0^{4}18240$	$0.0^{5}601$	$0.0^{5}181$	$0.0^{6}51$
5	$h/A = 4$	$-0.0^{5}16673$	$-0.0^{6}74705$	$-0.0^{6}2440$	$-0.0^{7}6591$	$-0.0^{7}1563$			
	$h/A = 2$	$-0.0^{4}5918$	$-0.0^{4}5869$	$-0.0^{4}3692$	$-0.0^{4}1824$	$-0.0^{5}768$	$-0.0^{5}288$	$-0.0^{6}989$	$-0.0^{6}316$
6	$h/A = 2$	$0.0^{4}1008$	$0.0^{4}1388$	$0.0^{4}1050$	$0.0^{5}601$	$0.0^{5}2876$	$0.0^{5}1209$	$0.0^{6}461$	$0.0^{6}163$
7	$h/A = 2$	$-0.0^{5}135$	$-0.0^{5}296$	$-0.0^{5}274$	$-0.0^{5}182$	$-0.0^{6}981$	$-0.0^{6}457$	$-0.0^{6}190$	$-0.0^{7}70$
8	$h/A = 2$	$0.0^{6}1$	$0.0^{6}57$	$0.0^{6}66$	$0.0^{6}53$	$0.0^{6}31$	$0.0^{6}16$	$0.0^{7}73$	$0.0^{7}30$

TABLE 5.

e \ k	h/A	1	2	3	4	5	6	7	8
1	h/A = 4	0.0²658549	0.0²230980	0.0³530874	0.0³100017	0.0⁴16673			
	h/A = 2	0.03593520	0.02164035	0.0²817821	0.0²241348	0.0³591521	0.0³12076	0.0⁴193	0.0⁵18
2	h/A = 4	−0.0²115490	−0.0³550566	−0.0³161297	−0.0⁴37324	−0.0⁵7470			
	h/A = 2	−0.01081618	−0.0²925834	−0.0²468026	−0.0³181066	−0.0³58678	−0.0³16564	−0.0⁴412	−0.0⁵88
3	h/A = 4	0.0³176958	0.0³107531	0.0⁴38355	0.0⁴10502	0.0⁵2440			
	h/A = 2	0.0²272464	0.0²311927	0.0²198665	0.0³94158	0.0³36911	0.0³12586	0.0⁴383	0.0⁴105
4	h/A = 4	−0.0⁴25004	−0.0⁴18663	−0.0⁵7877	−0.0⁵2494	−0.0⁶659			
	h/A = 2	−0.0³60538	−0.0³90595	−0.0³70637	−0.0³39790	−0.0³18245	−0.0⁴7227	−0.0⁴257	−0.0⁵84
5	h/A = 4	0.0⁵3335	0.0⁵2928	0.0⁵1464	0.0⁶527	0.0⁶156			
	h/A = 2	0.0³11905	0.0³23528	0.0³22172	0.0³14598	0.0⁴7685	0.0⁴3456	0.0⁴138	0.0⁵51
6	h/A = 2	−0.0⁴1922	−0.0⁴5503	−0.0⁴6294	−0.0⁴4809	−0.0⁴287	−0.0⁴144	−0.0⁵63	−0.0⁵24
7	h/A = 2	0.0⁵2468	0.0⁴1166	0.0⁴1639	0.0⁴1451	0.0⁵979	0.0⁵548	0.0⁵26	0.0⁵11
8	h/A = 2	−0.0⁶458	−0.0⁵237	−0.0⁵3995	−0.0⁶408	−0.0⁵311	−0.0⁵20	−0.0⁵11	−0.0⁶57

TABLE 6.

e \ k	h/A	1	2	3	4	5	6	7	8
1	h/A = 4	0.0²221228	0.0³514558	0.0⁴97518	0.0⁴16316	0.0⁵2499			
	h/A = 2	0.01934577	0.0²7606	0.0²2294	0.0³5721	0.0³1189	0.0⁴194	0.0⁵19	−0.0⁶15
2	h/A = 4	−0.0³514558	−0.0³154048	−0.0⁴36008	−0.0⁵7249	−0.0⁵1316			
	h/A = 2	−0.0²7605	−0.0²413316	−0.0²16538	−0.0³5471	−0.0³1569	−0.0⁴3969	−0.0⁵88	−0.0⁵15
3	h/A = 4	0.0⁴97518	0.0⁴36008	0.0⁴10014	0.0⁵2346	0.0⁶489			
	h/A = 2	0.0²2294	0.0³16533	0.0³8257	0.0³33328	0.0³1159	0.0⁴3583	0.0⁴100	0.0⁵25
4	h/A = 4	−0.0⁴16316	−0.0⁵7249	−0.0⁵2346	−0.0⁶628	−0.0⁶1473			
	h/A = 2	−0.0³5726	−0.0³5472	−0.0³3329	−0.0³15920	−0.0⁴6461	−0.0⁴2326	−0.0⁵77	−0.0⁵24
5	h/A = 4	0.0⁵2499	0.0⁵1316	0.0⁶489	0.0⁶147	0.0⁷385			
	h/A = 2	0.0³1193	0.0³1571	0.0³1159	0.0⁴6461	0.0⁴3003	0.0⁴122	0.0⁵45	0.0⁵157
6	h/A = 2	−0.0⁴192	−0.0⁴3975	−0.0⁴3587	−0.0⁴232	−0.0⁴122	−0.0⁵553	−0.0⁵22	−0.0⁶8
7	h/A = 2	0.0⁵16	0.0⁵88	0.0⁴100	0.0⁵75	0.0⁵45	0.0⁵23	0.0⁵100	0.0⁶4
8	h/A = 2	0.0⁶19	−0.0⁵17	−0.0⁵26	−0.0⁵22	−0.0⁵151	−0.0⁶9	−0.0⁶5	−0.0⁶2

§ 6. We proceed now to the deduction of the infinite system of linear equations for the α_k $(k = 0, \pm 1, \pm 2, \cdots)$; once these are known one may complete the theoretical study of the problem and then proceed to the qualitative and quantitative analysis of the stress field. We introduce the values of the functions $\varphi_1(z)$ and $\psi_1(z)$ from (1.5) into the auxiliary boundary condition (1.3). After some elementary manipulations we obtain for $g(t)$ the Fredholm integral equation on L_1 (the α_k are known to be determined by $g(t)$ by simple integral relations; it must be recalled that t and τ correspond to points on L_1 and γ respectively)

$$(6.1)\qquad -g(t_0)+\frac{1}{2\pi i}\int_{L_1} g(t)d\log\frac{t-t_0}{\bar{t}-\bar{t}_0}+ \\ +\frac{1}{2\pi i}\int_{L_1}\overline{g(t)}d\frac{t-t_0}{\bar{t}-\bar{t}_0}+\Omega(t_0)=-(c+E\rho^{-2})\tau_0-d\frac{1}{\tau_0}.$$

Instead of α_k and α_{-k} we deduce the unknowns β_k and β_{-k}, related to the former by

$$(6.2)\qquad \beta_k=(i)^{k-1}\alpha_k,\ \beta_{-k}=(i)^{k-1}\alpha_{-k}.$$

On observing the structure of the basic formulae (and the separate, more important intermediate steps leading to them) we arrive at the conclusion that the β_k are real quantities. Running somewhat ahead we note that this conclusion directly confirms the discovered internal lack of contradiction of the mentioned infinite system of equations for the unknowns β_k, written down below.

Transforming the expressions (4.16), (4.20) and (5.3) in correspondence with (6.2) we obtain

$$(6.3)\qquad q_{|e|}=\sum_{k=1}^{\infty}\{[q^{(1)}_{|e|,k}+(-1)^{k-1}q^{(2)}_{|e|,k}]\beta_k+[q^{(1)}_{|e|,-k}+(-1)^{k-1}q^{(2)}_{|e|,-k}]\beta_{-k}\},$$

$$(6.4)\qquad I_{|e|}=\sum_{k=1}^{\infty}(-1)^{k-1}(\beta_k-\beta_{-k})\xi^{(1,1)}_{|e|,k},$$

$$(6.5)\qquad p_e=\sum_{k=1}^{\infty}\{[p^{(1)}_{e,k}+(-1)^{k-1}p^{(2)}_{e,k}]\beta_k+[p^{(1)}_{e,-k}+(-1)^{k-1}p^{(2)}_{e,-k}]\beta_{-k}\}.$$

Using these formulae and writing

$$(6.6)\qquad \begin{aligned}\Omega_{e,k}&=q^{(1)}_{|e|,k}+(-1)^{k-1}q^{(2)}_{|e|,k}- \\ &\quad-(-1)^{|e|+1}\rho^{-2e}[p^{(1)}_{-e,k}+(-1)^{k-1}(p^{(2)}_{-e,k}+\xi^{(1,1)}_{|e|,k})],\\ \Omega_{e,-k}&=q^{(1)}_{|e|,-k}+(-1)^{k-1}q^{(2)}_{|e|,-k}- \\ &\quad-(-1)^{|e|+1}\rho^{-2e}[p^{(1)}_{-e,-k}+(-1)^{k-1}(p^{(2)}_{-e,-k}-\xi^{(1,1)}_{|e|,k})],\end{aligned}$$

we obtain the first of the equations (5.13) in the rather lucid form

$$\Omega_e = \sum_{k=1}^{\infty} [\Omega_{e,k}\beta_k + \Omega_{e,-k}\beta_{-k}].$$

Proceeding to the study of the quantities

$$\begin{aligned} Q_1 &= -G_1 + (c + E\rho^{-2}), \\ Q_{-1} &= -G_{-1} + d, \\ Q_e &= -G_e \ (e \geq 0,\ e \neq 1), \\ Q_{-e} &= -G_{-e} \ (e \neq 1), \end{aligned} \tag{6.7}$$

and executing obvious computations, we compare in the integral equation (6.1) coefficients of the same powers of τ. As a result we have

$$-2\beta_0 + \Omega_0 = Q_0, \tag{6.8}$$

$$(\beta_e + \beta_{-e}) + (-1)^{e+1}\Omega_{-e} = (-1)^{e+1}Q_{-e} \qquad (e = 1, 2, \cdots), \tag{6.9}$$

$$\{[1 - (-1)^{e+1}(\rho^4 - 1)e\rho^{-2(e+1)}]\beta_e + \rho^{-4e}\beta_{-e}\} + (-1)^{e+1}\Omega_e = (-1)^{e+1}Q_e \ (e = 1, 2, \cdots). \tag{6.10}$$

In the second group of equations (6.10) the equation corresponding to the value $e = 1$ drops out, since it may be verified to coincide with the corresponding equation of (6.9); introducing from the first group of equations (6.9) the expression for β_{-e} into (6.10) we obtain

$$\{(1 - \rho^{-4e}) - (-1)^{e+1}(\rho^4 - 1)e\rho^{-2(e+1)}\}\beta_e + (-1)^{e+1}[\Omega_e - \rho^{-4e}\Omega_{-e}] = (-1)^{e+1}[Q_e - \rho^{-4e}Q_{-e}] \ (e = 2, 3, \cdots). \tag{6.11}$$

Here we are mainly interested in the case which is important for applications when the ratio of the semi-axes of the ellipse L_1 is very small. In order to obtain a representation of the state of stress which is close to reality at points of the boundary of the ellipse, not far from the vertices of its minor axis (the walls of the working chamber), it is sufficient to consider the case when the ellipse degenerates into a crack; we arrive at this by going in (6.9) and (6.11) to the limit $\rho \to 1$. The coefficient of β_e in (6.11) tends then to zero like $(\rho - 1)^3$. In spite of this last fact, there arises naturally great concern with regard to the possible complexity of the computations associated with the stated type of transition to the limit; nevertheless its execution is truly simple due to the fact that in the remaining terms of the equality (6.11), depending on β_k and β_{-k}, and also in the free term one can single out a factor which coincides

identically with that contained in β_e. In fact, we write

$$
\begin{aligned}
R_e &= (-1)^{e+1}\{\Omega_e - \rho^{-4e}\Omega_{-e}\},\\
R_e &= \sum_{k=1}^{\infty}\{R_{e,k}\beta_k + R_{e,-k}\beta_{-k}\},
\end{aligned}
\tag{6.12}
$$

so that

$$
\begin{aligned}
R_{e,k} &= (-1)^{e+1}\{\Omega_{e,k} - \rho^{-4e}\Omega_{-e,k}\},\\
R_{e,-k} &= (-1)^{e+1}\{\Omega_{e,-k} - \rho^{-4e}\Omega_{-e,-k}\}.
\end{aligned}
\tag{6.13}
$$

We have by (5.4)—(5.7) then the following set of relations:

$$
\begin{aligned}
&p^{(1)}_{e,k} - p^{(1)}_{-e,k} = -(\rho^4 - 1)\rho^{2(k-1)}e\xi^{(1,1)}_{e,k},\\
&p^{(1)}_{e,-k} - p^{(1)}_{-e,-k} = (\rho^4 - 1)\rho^{-2(k+1)}e\xi^{(1,1)}_{e,k},\\
&p^{(2,1)}_{e,k} - p^{(2,1)}_{-e,k} = 0,\\
&p^{(2,2)}_{e,k} - p^{(2,2)}_{-e,k} = -4\frac{h}{A}e\xi^{(2,2)}_{e,k},\\
&p^{(2,3)}_{e,k} - p^{(2,3)}_{-e,k} = 2e\left[\rho^{-2}\frac{h}{A}\xi^{(2,2)}_{e,k} - (\xi^{(2,2)}_{e,k-1} + \xi^{(2,2)}_{e,k+1})\right],\\
&p^{(2,4)}_{e,k} - p^{(2,4)}_{-e,k} = \rho^{-2}e[\xi^{(2,2)}_{e,k-1} + 3\xi^{(2,2)}_{e,k+1}],\\
&p^{(2,5)}_{e,k} - p^{(2,5)}_{-e,k} = -\rho^{-4}e\xi^{(2,2)}_{e,k+1},\\
&p^{(2,1)}_{e,-k} - p^{(2,1)}_{-e,-k} = 0,\\
&p^{(2,2)}_{e,-k} - p^{(2,2)}_{-e,-k} = 4\frac{h}{A}e\xi^{(2,2)}_{e,k},\\
&p^{(2,3)}_{e,-k} - p^{(2,3)}_{-e,-k} = -2e\left[\rho^{-2}\frac{h}{A}\xi^{(2,2)}_{e,k} + (\xi^{(2,2)}_{e,k-1} + \xi^{(2,2)}_{e,k+1})\right],\\
&p^{(2,4)}_{e,-k} - p^{(2,4)}_{-e,-k} = \rho^{-2}e[3\xi^{(2,2)}_{e,k-1} + \xi^{(2,2)}_{e,k+1}],\\
&p^{(2,5)}_{e,-k} - p^{(2,5)}_{-e,-k} = -\rho^{-4}e\xi^{(2,2)}_{e,k-1}.
\end{aligned}
\tag{6.14}
$$

Substituting in the first formula (6.13) the values of the quantities entering into it from (6.6), we obtain

$$
\begin{aligned}
R_{e,k} &= \sum_{\nu=1}^{2} R^{(\nu)}_{e,k},\\
R^{(\nu)}_{e,k} &= (-1)^{(\nu-1)(k-1)}\{(1-\rho^{-4e})q^{(\nu)}_{|e|,k} - (-1)^{|e|+1}\rho^{-2e}[p^{(\nu)}_{-e,k} - p^{(\nu)}_{e,k}]\}.
\end{aligned}
\tag{6.15}
$$

Hence we find, using (4.17) and (6.14),

$$R_{e,k}^{(1)} = [(1-\rho^{-4e})-(-1)^{e+1}(\rho^4-1)e\rho^{-2(e+1)}]\rho^{2k}\xi_{e,k}^{(1,1)}, \tag{6.16}$$

$$R_{e,k}^{(2)} = [(1-\rho^{-4e})-(-1)^{e+1}(\rho^4-1)e\rho^{-2(e+1)}]\times \tag{6.17}$$

$$\times(-1)^{k-1}k\left[2\frac{h}{A}\xi_{e,k}^{(2,2)}+(\rho^2-1)\xi_{e,k-1}^{(2,2)}+\rho^{-2}(\rho^2-1)\xi_{e,k+1}^{(2,2)}\right].$$

In the same way as in (6.15), the second of the quantities (6.13) can be decomposed into the sum of two terms; executing analogous steps, we have the relations

$$R_{e,-k}^{(1)} = -[(1-\rho^{-4e})-(-1)^{e+1}(\rho^4-1)e\rho^{-2(e+1)}]\rho^{-2k}\xi_{e,k}^{(1,1)}, \tag{6.18}$$

$$R_{e,-k}^{(2)} = [(1-\rho^{-4e})-(-1)^{e+1}(\rho^4-1)e\rho^{-2(e+1)}]\times \tag{6.19}$$

$$\times(-1)^{k-1}k\left[-2\frac{h}{A}\xi_{e,k}^{(2,2)}+\rho^{-2}(\rho^2-1)\xi_{e,k-1}^{(2,2)}+(\rho^2-1)\xi_{e,k+1}^{(2,2)}\right].$$

Adding now (6.16), (6.17) and (6.18), (6.19) respectively we obtain the values of both quantities (6.13),

$$R_{e,\pm k} = [(1-\rho^{-4e})-(-1)^{e+1}(\rho^4-1)e\rho^{-2(e+1)}](-1)^{e+1}q_{e,\pm k}, \tag{6.20}$$

where

$$q_{e,\pm k} = q_{e,\pm k}^{(1)} + (-1)^{k-1}q_{e,\pm k}^{(2)}. \tag{6.21}$$

The free terms in the equations (6.11) transform on the basis of certain of the relations (5.4)—(5.7) to the form (the presence of the factor $(-1)^{e+1}$ on both sides of the equality is implied)

$$(Q_e - \rho^{-4e}Q_{-e}) = -E[(1-\rho^{-4e})-(-1)^{e+1}(\rho^4-1)e\rho^{-2(e+1)}]\xi_{e,1}^{(1,1)}. \tag{6.22}$$

Studying the terms of (6.11) for a general multiplier contained in them, we obtain the modified group of equations (true for $\rho \geq 1$)

$$(-1)^{e+1}\beta_e + \sum_{k=1}^{\infty}(q_{e,k}\beta_k + q_{e,-k}\beta_{-k}) = -E\xi_{e,1}^{(1,1)} \quad (e=2,3,\cdots). \tag{6.23}$$

We must add here one missing equation which was mentioned above and which contains β_1; in addition, besides (6.8), we need, in principle, still an equation for β_0; both of these must be taken from the first equation (1.6) which has not been used hitherto and which must be obeyed in the ellipse S_1. It is readily verified that these equations have the form

$$\beta_0 - q_0 = E\xi_{0,1}^{(1,1)} + C_1, \tag{6.24}$$

$$\beta_1 + q_1 = -E\xi_{1,1}^{(1,1)}. \tag{6.25}$$

Having determined $\beta_{\pm k}$ $(k = 1, 2, \cdots)$ from the systems (6.9), (6.23) and (6.25), we find successively β_0 and C_1 from (6.8) and (6.24).

NOTE 1. Introducing into the deduced infinite system of linear equations for β_k the unknowns $x_k = \rho^k \beta_k$ we obtain a new infinite system for these new variables which is quasi-regular for any relative dimensions of the region. This is readily verified from the estimates (apparently not profound, but sufficient for our purposes) of the quantities $\xi^{(1)}_{e,k}$ and $\xi^{(2,2)}_{e,k}$ to be deduced below.

Obviously, the modulus of the function $\chi^{-1}(z)$ attains its largest value at the point $z = ih$, and, consequently, one has everywhere in the half-plane S_2 the inequality

$$\left|\frac{1}{\chi^k(z)}\right| \leq \left(\frac{\lambda}{\rho}\right)^k, \quad \lambda = \frac{\sqrt{b^2 + 4A^2} + b}{\sqrt{h^2 + 4A^2} + h} < 1. \tag{6.26}$$

Let L_1^* be an ellipse, surrounding L_1, which is confocal with it and entirely contained in the region S; it corresponds in the ζ plane to the circle γ_1^* with radius $\rho^* > \rho$, so that

$$\frac{1}{\rho^*} = \kappa \frac{1}{\rho}, \quad \kappa = \frac{\rho}{\rho^*} < 1. \tag{6.27}$$

By (3.19), (4.14), (4.18) and (4.19) we have

$$i^{k+e}\xi^{(1,1)}_{e,k} = \frac{1}{2\pi i}\int_{L_1^*} \frac{1}{\chi^k(t)} \frac{d\tau}{\tau^{e+1}},$$

$$i^{k+e}\xi^{(1,1)}_{e,k-1} = \frac{1}{2\pi i}\int_{L_1^*} \frac{1}{1 - \chi^{-2}(t)} \frac{1}{\chi^k(t)} \frac{d\tau}{\tau^{e+1}}.$$

Hence, on the basis of (6.26) and (6.27), we find the above mentioned estimates

$$\begin{aligned} |\xi^{(1)}_{e,k}| &< \lambda^k \kappa^e \rho^{-(k+e)}, \\ |\xi^{(2,2)}_{e,k-1}| &< \frac{1}{1 - \lambda^2 \rho^{-2}} \lambda^k \kappa^e \rho^{-(k+e)}, \end{aligned} \tag{6.28}$$

which are readily seen to lead to the required result.

NOTE 2. It is not without interest to elucidate whether it is possible, in retaining from the infinite system the set of equations (6.8) and (6.9), to introduce into the consideration instead of (6.23) and (6.25), for example, the equation (1.6); the last together with the second equation (1.6) remain almost outside the field of interest for the effective study of the problem. The equations (6.24) and (6.25) added to the infinite system, of course, do not exhaust by far the contents of the integral formulae (1.6).

In order to give one or the other answer to the question of interest to us, it is necessary to establish whether the first integral formula (1.6) is properly fulfilled without the equation (6.10), because taking into consideration the one-sided, outside link with the second formula (1.6) accompanying it, it may even detract from the unknown solution.

Putting

$$M(t) = \varphi_1(t) - t\overline{\varphi_1'(t)} - \overline{\psi_1(t)} - 2g(t) + 2f(t), \tag{6.29}$$

we see that the retained equations (6.8) and (6.9) may be written in the form of the integral equations

$$\frac{1}{2\pi i}\int_\gamma \frac{M(t)}{\tau - \zeta}\,d\tau = 0, \quad \frac{1}{2\pi i}\int_\gamma \frac{M(t)}{\tau}\,d\tau = 0, \tag{6.30}$$

where in the first of them the point ζ lies outside γ, so that

$$M(t) = \delta(\tau), \qquad \delta(\zeta) = \sum_{k=1}^{\infty} \lambda_k \zeta^k, \tag{6.31}$$

with the function $\delta(\zeta)$, regular inside γ.

It has to be shown that the auxiliary function $g(t)$ and, in the final computations, the unknown functions $\varphi_1(z)$ and $\psi_1(z)$ expressed in terms of it, determined by the set of equations (1.5) and the first equation of (1.6), coincide identically for the supplementary relations (6.29) and (6.31) and the second boundary condition (1.1) on L_2 with the same functions determined from (1.5) and the same condition (1.1) together with the auxiliary equation (1.3).

In the first equations (1.5) and (1.6), true outside and inside the contour L_1 respectively, we go to the limit as z tends to some point t_0 on L_1. Evaluating these limiting values from each other, we find that also in the given case on the contour L_1 the first relation (1.4) is fulfilled. Taking the last into consideration we obtain from (6.29) on the contour L_1

$$\varphi_1(t) + t\overline{\varphi_1'(t)} + \overline{\psi_1(t)} = 2f(t) - \delta(t) + 2C_1. \tag{6.32}$$

In turn, it follows from this formula by the reasoning given at the end of § 1 that the function

$$\psi^*(z) = \psi_1(z) + \frac{1}{2\pi i}\int_{L_1} \frac{\overline{g(t)} + t\overline{g'(t)} - 2\overline{f(t)} + \overline{\delta(t)}}{t - z}\,dt \tag{6.33}$$

is regular in the half-plane S_2 bounded by L_2, and this means also inside the ellipse L_1. Hence, taking into consideration (1.6), we will have in the region

$$\psi^*(z) - \psi(z) = \frac{1}{2\pi i}\int_{L_1} \frac{\overline{\delta(t)}}{t - z}\,dt.$$

The integral on the right hand side of this equality is a function regular outside L_1 and vanishes at infinity; we continue it analytically into the inside of the ellipse L_1 by the help of the left hand side of the same equality; this integral obviously must be identically equal to zero. Therefore its density $\overline{\delta(t)}$ can be expanded into a series of powers of the two terms $\tau^k + \tau^{-k}$ $(k = 1, 2, \cdots)$ which, by comparison with (6.31), gives the required equality $\lambda_k = 0$ $(k = 1, 2, \cdots)$, leading to the original auxiliary boundary condition (1.3).

NOTE 3. The proposition established in the preceding note remains true for any internal boundary L_1. In fact, the function mapping the outside of any contour L_1 onto the outside of the unit circle in the ζ plane, generally speaking, is given by (we present also its inverse, true outside the circumference enveloping L_1)

$$z = \zeta \sum_{k=0}^{\infty} a_k \frac{1}{\zeta^k}, \qquad \zeta = z \sum_{k=0}^{\infty} b_k \frac{1}{z^k}, \tag{6.34}$$

where a_k and b_k are certain given constants. Raising both sides of each of the equalities to certain powers m we find

$$z^m = \sum_{k=-m}^{\infty} a_{m+k}^{(m)} \frac{1}{\zeta^k}, \qquad \zeta^m = \sum_{k=-m}^{\infty} b_{m+k}^{(m)} \frac{1}{z^k}. \tag{6.35}$$

The coefficients $a_k^{(m)}$ and $b_k^{(m)}$ are readily evaluated in terms of the known a_k and b_k, where $a_0^{(0)} = b_0^{(0)} = 1$ and $a_k^{(0)} = b_k^{(0)} = 0$ $(k \geq 1)$.

Let $f(z)$ be some function, regular inside L_1; we expand it on the contour L_1 in the Fourier series (as before, t and τ are the coordinates of corresponding points of L_1 and γ)

$$f(t) = f_0 + \sum_{k=-\infty}^{\infty}{}' f_k \tau^k, \tag{6.36}$$

where the stroke on the sum indicates omission of the term $k = 0$.

Multiplying both sides of this equation by the kernel $[2\pi i(t - z)]^{-1}$ where z is a point inside L_1 and integrating them along the same contour L_1, we find

$$f(z) = \sum_{k=0}^{\infty} f_k \frac{1}{2\pi i} \int_{L_1} \frac{\tau^k}{t - z}\, dt.$$

Taking as contour of integration instead of L_1 a circle with sufficiently large radius and using (6.35) we obtain

$$f(z) = \sum_{k=0}^{\infty} f_k \chi_k(z), \qquad \chi_k(z) = \sum_{\nu=0}^{k} b_{k-\nu}^{(k)} z^\nu, \tag{6.37}$$

where the $\chi_k(z)$ essentially coincide with the so-called Faber polynomials. It is clear from this that

$$\tau^k + \omega_k(t) = \chi_k(t) \qquad (k = 0, 1, \cdots), \tag{6.38}$$

where the functions $\omega_k(z)$ are regular outside L_1 and vanish at infinity. We go now to the second equality (6.37) on the contour γ and, using (6.38), execute

manipulations which suggest themselves. We will have

$$\chi_k(t) = \tau^k + \sum_{n=1}^{\infty} \frac{1}{\tau^n} \sum_{\nu=1}^{k} a_{\nu+n}^{(\nu)} b_{k-\nu}^{(k)}, \qquad \chi_0(t) = 1, \tag{6.39}$$

where we conclude from what has been said in connection with the $\omega_k(t)$ that the following relations must be satisfied identically:

$$\sum_{\nu=n}^{k} a_{\nu-n}^{(\nu)} b_{k-\nu}^{(k)} = 0 \qquad (n = 0, 1, \cdots, k-1).$$

In the first equation (6.37), taken on γ, we substitute now for the functions $\chi_k(t)$ their expressions from (6.39). Then we obtain

$$f(t) = \sum_{k=0}^{\infty} f_k \tau^k + \sum_{k=1}^{\infty} \frac{1}{\tau^k} \left[\sum_{n=1}^{\infty} f_n \sum_{\nu=1}^{n} a_{\nu+k}^{(\nu)} b_{n-\nu}^{(n)} \right]. \tag{6.40}$$

Comparing the expansions (6.36) and (6.40) we arrive at the equations

$$f_{-k} = \sum_{n=1}^{\infty} f_n \sum_{\nu=1}^{n} a_{\nu+k}^{(\nu)} b_{n-\nu}^{(n)} \qquad (k = 1, 2, \cdots). \tag{6.41}$$

Thus, only if the relations (6.41) between the Fourier coefficients of the expansion (6.36) are fulfilled, does the last indeed determine the function, continued analytically into the inside of the contour L_1.

Reasoning as above we will have on the circle γ simultaneously the expansions

$$\overline{\delta(t)} = \sum_{k=-\infty}^{\infty} \delta_k \tau^k, \quad \delta(t) = \sum_{k=1}^{\infty} \lambda_k \tau^k.$$

For the supplementary relations (6.41), connecting the coefficients δ_k and δ_{-k}, they give

$$\delta_0 = 0, \qquad \lambda_k = \delta_{-k} = \delta_k = 0 \qquad (k = 1, 2, \cdots). \tag{6.42}$$

Consequently, also in the case of a contour L_1 of an arbitrary shape the equality (1.5) and the first equation (1.6) together with (6.29) and (6.31) (for the boundary condition on L_2) ensure fulfillment of the limiting condition (1.3).

Now let $\rho \to 1$ in the system of equations (6.9), (6.23) and (6.25), supplying in them the limiting values of the coefficients and of the free terms with zero as superscript and remembering that for this $E \to 0$, we obtain

$$\begin{aligned} &(\beta_e + \beta_{-e}) + (-1)^{e+1} \sum_{k=1}^{\infty} \Omega_{-e,k}^{(0)} (\beta_k - \beta_{-k}) = (-1)^{e+1} Q_{-e}^{(0)}, \\ &\beta_e + (-1)^{e+1} \sum_{k=1}^{\infty} q_{e,k}^{(0)} (\beta_k - \beta_{-k}) = 0. \end{aligned} \tag{6.43}$$

The values of the quantities entering into these equations are given by

TABLE 7.

s \ k	$\frac{h}{A}$	1	2	3	4	5	6	7	8	$\beta_k - \beta_{-k}$
1	$h/A = 4$	0.8478714	0.01646586	$-0.^{3}492953$	$0.^{3}522115$	$-0.^{3}111744$				-1.1802888
	$h/A = 2$	0.6079890	0.0608296	-0.0330564	$0.^{3}458565$	$-0.^{3}155449$	$0.^{3}151345$	$-0.^{4}261797$	$-0.^{5}10475$	-1.676721
2	$h/A = 4$	0.03293180	0.9933368	$0.^{3}230451$	$-0.^{3}342279$	$0.^{3}484262$				0.01958572
	$h/A = 2$	0.1216964	0.9585254	0.02645564	$-0.^{3}597106$	$0.^{3}251281$	$-0.^{3}456429$	$0.^{3}142062$	$-0.^{4}202802$	0.1090318
3	$h/A = 4$	-0.01478860	$0.^{3}345677$	0.9986754	$0.^{3}225265$	$-0.^{3}460262$				$-0.^{3}587137$
	$h/A = 2$	-0.09920506	0.03969008	0.9704403	$0.^{3}799914$	$-0.^{3}382330$	$0.^{3}860768$	$-0.^{3}315787$	$0.^{4}616216$	-0.06020569
4	$h/A = 4$	$0.^{3}208845$	$-0.^{3}684533$	$0.^{3}300352$	0.9999368	$0.^{3}418854$				$0.^{3}624315$
	$h/A = 2$	0.01830568	-0.01194188	0.01066521	0.9962254	$0.^{3}206727$	$-0.^{3}576606$	$0.^{3}241008$	$-0.^{4}570812$	$0.^{3}886339$
5	$h/A = 4$	$-0.^{3}558654$	$0.^{3}210559$	$-0.^{3}100468$	$0.^{3}423568$	0.9999925				$-0.^{3}133907$
	$h/A = 2$	$-0.^{3}774384$	$0.^{3}627506$	$-0.^{3}636833$	$0.^{3}258451$	0.9984492	$0.^{3}488567$	$-0.^{3}223017$	$0.^{4}605153$	$-0.^{3}313402$
6	$h/A = 4$	$0.^{3}933984$	$-0.^{3}136524$	$0.^{3}171974$	$-0.^{3}867640$	$0.^{3}587807$	0.9997855	$0.^{3}108712$	$-0.^{4}346992$	$0.^{3}362081$
7	$h/A = 2$	$-0.^{3}204167$	$0.^{3}489637$	$-0.^{3}734460$	$0.^{3}429712$	$-0.^{3}315239$	$0.^{3}122913$	0.9999323	$0.^{4}256238$	$-0.^{4}812776$
8	$h/A = 2$	$-0.^{4}101738$	$-0.^{4}675740$	$0.^{3}158672$	$-0.^{3}124682$	$0.^{4}100392$	$-0.^{4}394621$	$0.^{4}247355$	0.9999873	$0.^{4}48752$

the formula

$$q_{e,k}^{(0)} = \xi_{e,k}^{(1,1)} + 2\frac{h}{A}(-1)^{k-1}k\xi_{e,k}^{(2,2)},$$

$$q_{e,k}^{(0)} = -q_{e,-k}^{(0)},$$

$$\Omega_{-e,k}^{(0)} = -\Omega_{-e,-k}^{(0)},$$

$$\Omega_{-e,k}^{(0)} = [1-(-1)^{k+e}]\xi_{e,k}^{(1,1)} + 2\frac{h}{A}(-1)^{k-1}k\xi_{e,k}^{(2,2)} + \tag{6.44}$$

$$+\frac{h}{A}(-1)^{e+1}\eta_{e,k}^{(1,1)} + 2\left(\frac{h}{A}\right)^2(-1)^{k+e}k\eta_{e,k}^{(2,1)},$$

$$Q_{-1}^{(0)} = AC,$$

$$C = 2(h_1 + h_2),$$

$$Q_{-e}^{(0)} = 0 \quad (e \neq 1).$$

Subtracting from the equations of the first group of (6.43) twice the equations of the second group, we arrive at the rather simple, infinite system of equations for the differences $\beta_k - \beta_{-k}$

$$\sum_{k=1}^{\infty} c_{e,k}(\beta_k - \beta_{-k}) = f_e \quad (e = 1, 2, \cdots), \tag{6.45}$$

$$c_{e,e} = 1 + (-1)^e[\Omega_{-e,e}^{(0)} - 2q_{e,e}^{(0)}],$$

$$c_{e,k} = (-1)^e[\Omega_{-e,k}^{(0)} - 2q_{e,k}^{(0)}], \quad k \neq e, \tag{6.46}$$

$$f_e = (-1)^e Q_{-e}^{(0)}.$$

The system (6.45), truncated to five and eight equations in the first and second case respectively ($h/A = 4$ and $h/A = 2$), has been solved by the method of successive approximations. The values of the corresponding coefficients of the system and of $\beta_k - \beta_{-k}$ are given in Table 7. With their help, one may find the unknown functions $\varphi_1(z)$ and $\psi_1(z)$ in sufficient approximation, and hence the stress components. The computations give the following values for the components of the supplementary stress X_x and the total stress $X_x^{(1)}$ at the points $z = 0$ and $z = A$ on the side of the crack (wall of the shaft).

$$\frac{h}{A} = 4, \quad z = 0, \quad X_x = 1.27C, \quad X_x^{(1)} = 0.27C,$$

$$\frac{h}{A} = 2, \quad \begin{matrix} z = 0, & X_x = 2.23C, & X_x^{(1)} = 1.23C, \\ z = A, & X_x = 1.73C, & X_x^{(1)} = 0.73C. \end{matrix}$$

For the "drowned" crack the same stresses at the point $z = 0$ are equal to $X_x = C$, $X_x^{(1)} = 0$. Hence it is seen that the magnitudes of the supplementary stresses X_x and the total stress $X_x^{(1)}$ on the sides of the crack (wall of the shaft) increase significantly as the crack approaches the edge of the half-plane (the surface).

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ESTIMATES OF NATURAL FREQUENCIES OF VIBRATION OF CLAMPED PLATES OF CONSTANT AND VARIABLE THICKNESS

M. G. SLOBODIANSKII
Moscow, U.S.S.R.

The problem stated in the title is reduced to the determination of the eigenvalues for a first boundary value problem for elliptic equations of fourth order with constant and variable coefficients.

Upper bounds for the eigenvalues may be obtained by the Ritz-Galerkin method.

The problem of the determination of lower bounds for the eigenvalues has been studied by many authors. Referring for details to the literature [1–18] which includes also a convenient bibliography, we will only comment on those papers which are devoted directly to the problem under consideration.

In S. Tomotika's paper [11] estimates were obtained from below for the natural frequencies of oscillation of rectangular plates of constant thickness clamped along their contour.

A. Weinstein [12] proposed a method for the determination of lower bounds for the eigenvalues of arbitrary plates of constant thickness, clamped along their contour. A survey of work, devoted to Weinstein's method and its generalization was given in the paper [13] by I. B. Diaz.

In the interesting paper [14] by H. F. Weinberger which appeared recently estimates from below were given for the eigenvalues for a boundary value problem for the biharmonic equation in the case when the eigenvalues are found by the method of finite differences.

We also note here that if rough estimates from below for eigenvalues have been found by use of any method and these estimates must satisfy definite conditions, then one may apply for the further improvement of these estimates the inclusion theorems of Temple, Krylov-Bogoliubov, et al.

The method of obtaining estimates from below for eigenvalues, considered in this paper, is based on the application of known theorems of Courant and Weyl [18] regarding the minimal properties of eigenvalues [10] and on the introduction of a "similar operator". Side by side with plates of constant thickness, plates of variable thickness will be considered. In addition, the case will be discussed when all intermediate integrals, necessary for the calculation of these estimates, may be evalu-

ated in closed form; this is important from the point of view of applications.

The last problem likewise has important significance for two-dimensional statical problems of the theory of elasticity, although there N. I. Muskhelishvili [19] and his pupils have obtained exact solutions in closed form for a very wide class of plane regions.

§ 1. The problem of eigenvalues for clamped plates of constant thickness

The problem of the determination of the natural frequencies of vibration of plates of constant thickness, clamped along the contour l of the region S of the plate, reduces to the determination of the eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$ of the boundary problem

$$\nabla^4 w - \lambda w = 0, \tag{1.1}$$

$$w\Big|_l = \frac{\partial w}{\partial \nu}\Big|_l = 0, \tag{1.2}$$

where $w = w(x, y)$ is the deflection of the plate, λ is a parameter and ν is the outward normal to the contour l.

The union of the equations (1.1)–(1.2) may be written in the form of an operator equation in Hilbert space

$$\begin{gathered} Aw - \lambda w = 0, \\ w \in D_A, \end{gathered} \tag{1.3}$$

where D_A denotes the region of definition of the operator, i.e., the union of the functions w, satisfying the boundary conditions (1.2).

It may be shown [9] that the operator A is positive definite on D_A, i.e.,

$$(Aw, w) = \iint_S \nabla^4 w \cdot w \, dS \geq \lambda_1 \|w\|^2 = \lambda_1 \iint_S w^2 \, dS, \tag{1.4}$$

where $\lambda_1 > 0$ and the scalar product and the norm in the Hilbert space are defined by the formulae

$$(w, w_1) = \iint_S w w_1 \, dS, \quad \|w\|^2 = \iint_S w^2 \, dS.$$

We will assume that the operator has a fully continuous inverse operator A^{-1} and that the problem of the determination of the eigenvalues of the boundary problem (1.1)–(1.2) is equivalent to the problem

of the determination of the eigenvalues of the equation

$$\mu w - A^{-1}w = \mu w(x, y) - \iint_S G(x, y, \xi, \eta)\, d\xi\, d\eta = 0, \tag{1.5}$$

where $G(x, y, \xi, \eta)$ is the Green function for the clamped plate, i.e.,

$$\begin{gathered} \nabla^4_{\xi,\eta} G(x, y, \xi, \eta) = \delta, \\ G\Big|_l = \frac{\partial G}{\partial \nu}\Big|_l = 0, \end{gathered} \tag{1.6}$$

and δ is the Dirac function.

Side by side with (1.6) we will consider the equation

$$\mu w - A_1^{-1}w = \mu w - \iint_S G_1(x, y, \xi, \eta) w(\xi, \eta)\, d\xi\, d\eta, \tag{1.7}$$

where $G(x, y, \xi, \eta)$ is the symmetric, fully continuous kernel of the equation (1.7) which in some sense is "similar" to the kernel $G(x, y, \xi, \eta)$.

Let $\mu_1 \geq \mu_2 \geq \mu_3 \geq \cdots$ and $\mu_{11} \geq \mu_{12} \geq \mu_{13} \geq \cdots$ be the eigenvalues of the equations (1.5) and (1.7) respectively, where each eigenvalue is assumed as many times as its inverse. Then we have immediately, on the basis of theorems of Courant and Weyl [10],

$$|\mu_n - \mu_{1n}| = \left| \frac{1}{\lambda_n} - \frac{1}{\lambda_{1n}} \right| \leq \|A^{-1} - A_1^{-1}\| \leq \alpha, \tag{1.8}$$

where α is a constant.

Hence, if the inequalities

$$\left| \frac{1}{\lambda_n} - \frac{1}{\lambda_{1n}} \right| < \alpha, \quad \left| \frac{1}{\lambda_{1,n\pm1}} - \frac{1}{\lambda_{1n}} \right| > 2\alpha \tag{1.9}$$

are fulfilled, it follows [18] that there is no other eigenvalue on the interval

$$\begin{gathered} \lambda_{1n} - \delta_1 < \lambda < \lambda_{1n} + \delta_2, \\ \delta_1 = \frac{\alpha\lambda_{1n}^2}{1 + \alpha\lambda_{1n}}, \quad \delta_2 = \frac{\alpha\lambda_{1n}^2}{1 - \alpha\lambda_{1n}} \end{gathered} \tag{1.10}$$

except λ_n. Further, we have

$$\|A^{-1} - A_1^{-1}\|^2 = \sup \frac{\iint_S dx\, dy \left\{ \iint_S u(x, y, \xi, \eta) w(\xi, \eta)\, d\xi\, d\eta \right\}^2}{\iint_S [w(\xi, \eta)]^2\, d\xi\, d\eta},$$

$$u = G - G_1,$$

whence, applying the inequality of Buniakovskii-Schwarz, we find

$$\|A^{-1} - A_1^{-1}\|^2 \leq \iint_S \left\{ \iint_S u^2(x, y, \xi, \eta)\, d\xi\, d\eta \right\} dx\, dy \leq \alpha^2. \tag{1.11}$$

In order to find estimates for the quantity α, determined by (1.11), different methods may be used. We will apply here the following procedure.

We will choose the kernel $G_1(x, y, \xi, \eta)$ in the form

$$G_1(x, y, \xi, \eta) = \sum_{r=1}^{m} \sum_{k=1}^{m} b_{kr} \varphi_r(x, y) \varphi_k(\xi, \eta), \tag{1.12}$$

where the functions $\varphi_k(x, y)$ satisfy the boundary conditions (1.2), i.e.,

$$\varphi_k \Big|_l = \frac{\partial \varphi_k}{\partial \nu} \Big|_l = 0, \tag{1.13}$$

and b_{kr} are constants ($b_{kr} = b_{rk}$).

For this choice of the functions $\varphi_k(x, y)$ the symmetric kernel $G_1(x, y, \xi, \eta)$ satisfies the same boundary conditions (1.2) as the function $G(x, y, \xi, \eta)$. It follows from this that the function $u = G - G_1$, considered as a function of (x, y), satisfies the boundary value problem

$$\begin{gathered} \nabla^4_{\xi,\eta} u = \nabla^4_{\xi,\eta}[G - G_1] = \delta - \nabla^4_{\xi,\eta} G_1 = f(x, y, \xi, \eta), \\ u \Big|_l = \frac{\partial u}{\partial \nu} \Big|_l = 0, \end{gathered} \tag{1.14}$$

where δ is the Dirac function.

Further, if u is a solution of the boundary problem (1.14), we have for $|u|$ the estimate [20]

$$|u|^2 \leq \iint_S [q(x, y, \xi, \eta)]^2\, d\xi\, d\eta \cdot \iint_S [H(x, y, \xi, \eta)]^2\, d\xi\, d\eta, \tag{1.15}$$

where

$$\frac{\partial^2 q}{\partial \xi \partial \eta} = \delta, \quad \frac{\partial^2 H}{\partial \xi \partial \eta} = f(x, y, \xi, \eta) = \delta - \nabla^4 G_1, \tag{1.16}$$

and one may choose for the function $q(x, y, \xi, \eta)$ a function of the form [20] (Fig. 1)

$$\begin{gathered} q(x, y, \xi, \eta) = q_0 + a_0 + a_{01} x + a_{02} y + \cdots, \\ q_0(x, y, \xi, \eta) = \begin{cases} +\frac{1}{4}, \text{ for } (\xi - x)(\eta - y) \geq 0 \\ -\frac{1}{4}, \text{ for } (\xi - x)(\eta - y) < 0. \end{cases} \end{gathered} \tag{1.17}$$

Here $a_0, a_{01}, \cdots$ are constants which must be chosen such that the first integral on the right hand side of (1.15) has a minimum value.

Instead of (1.17) one may also take q_0 to be equal to the constants C_1, C_2, C_3, C_4 in the corresponding four quadrants (Fig. 1) for the condition $C_1 + C_3 - (C_2 + C_4) = 1$. In particular, one may take $C_1 = C_3 = \frac{1}{4}$, $C_2 = C_4 = -\frac{1}{4}$. Further, in order to simplify the calculations, one may take $a_0 = a_{01} = a_{02} = \ldots = 0$.

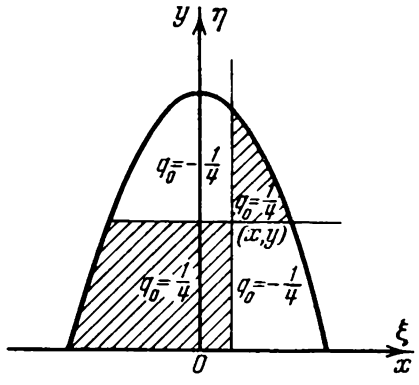

Fig. 1

Thus, we will assume on the basis of (1.16)

$$H = q_0(x, y, \xi, \eta) - H_0(x, y, \xi, \eta),$$

$$H_0 = \int_y^{\eta}\int_x^{\xi} \nabla^4_{\xi,\eta} G_1(x, y, \xi, \eta)\, d\xi\, d\eta, \tag{1.18}$$

and substitute (1.17)–(1.18) into (1.15). Further, substituting (1.15) into (1.11), we find an estimate α for the integral (1.11) as a function of the parameters b_{kr}.

In order that the quantity α will have a minimum one has to determine the parameters b_{kr} from the minimum conditions for α, i.e., from the system of algebraic equations

$$\frac{\partial \alpha}{\partial b_{kr}} = 0, \quad k, r = 1, 2, \cdots, m. \tag{1.19}$$

If the coefficients b_{kr} are determined in any other manner, for example by the Ritz-Galerkin method, the corresponding estimate for the integral (1.11) will be larger than $\alpha_{\min}$, determined from the conditions (1.19).

We will now dwell on the problem of the evaluation of the integrals in (1.11)–(1.18). For the utilization of the deduced estimates of the eigenvalues it is very important that the integrals in (1.11)–(1.18) should be evaluated in closed form. We will state here only one such case.

We assume that the equation of the contour may be written in the form

$$\omega(x, y) = [y - f_1(x)][y - f_2(x)] \ldots [y - f_n(x)] = 0, \tag{1.20}$$

where $f_1(x), \ldots, f_n(x)$ are polynomials of x and the function $\omega(x, y)$ does not vanish inside the region S. For example, the equation of the contour of the region bounded by the parabola $y = -x^2 + b$ and the straight

line $y = 0$ may be written in the form

$$\omega(x, y) = y(y + x^2 - b) = 0,$$

where $\omega(x, y) \neq 0$ inside the region (Fig. 1). It is seen from (1.20) that $\omega(x, y)$ is a polynomial in x and y.

We will now introduce into (1.12)

$$\varphi_k(x, y) = \omega^2(x, y)\psi_k(x, y),$$

where $\psi_k(x, y)$ are functions of the form $x^r y^s$ with r, s constants. For such a choice of the functions $\varphi_k(x, y)$ the function H_0, defined by (1.18), will also be a polynomial in x, y, ξ, η.

Further, if we take the region of integration S on the four quadrants $\xi - x \geq 0$, $\eta - y \geq 0$, etc., then the function $H = q_0 - H_0$ will be a polynomial in ξ, η in each quadrant and the double integrals in (1.15) may be evaluated with consideration of the equation of the boundary (1.20) in closed form, because

$$\iint_{S_1} H^2 \, d\xi \, d\eta = \int_x^{x_0} \left\{ \int_y^{\eta(\xi)} H^2(x, y, \xi, \eta) \, d\eta \right\} d\xi, \tag{1.21}$$

where $\eta(\xi)$ is a polynomial, S_1 is the first quadrant $\xi - x \geq 0, \eta - y \geq 0$. In analogy with (1.21), one may also write down expressions for the remaining quadrants S_2, S_3, S_4. Then it is obvious that integrals of the form (1.21) will be polynomials in x, y, while the integral on the right hand side of (1.11) may also be evaluated in closed form taking into consideration the contour (1.20).

We note that in the case when on the interval (x, x_0) the equation of the contour is determined by different functions $\eta(\xi)$ on separate parts of the interval (x, x_0), the integral (1.21) must be broken up into separate parts. However, the integral on the right hand side of (1.21) will again be a polynomial in x, y.

It will be of interest to give more detailed consideration to classes of regions which are simply as well as multiply connected and to systems of functions $\varphi_k(x, y)$ for which the integrals in (1.11)–(1.15) may be evaluated in closed form.

NOTE: If the region S lies completely inside a region S', then it is known [1,9] that the squares of the natural frequencies of vibration of the clamped plate S' will be smaller than the squares of the corresponding frequencies of oscillation of the plate S. Therefore, in order to obtain an estimate from below for the natural frequencies of vibration of a plate S, one may in many cases replace the region S by a region S' the contour of which possesses the above stated properties.

§ 2. Plates of variable thickness

The determination of the natural frequencies of vibration of plates of variable thickness, clamped along their contours, reduces to the determination of the eigenvalues λ_n of the equation

$$\frac{\partial^2}{\partial x^2}\left(D\frac{\partial^2 w}{\partial x^2}\right)+\frac{\partial^2}{\partial y^2}\left(D\frac{\partial^2 w}{\partial y^2}\right)+\frac{\partial^2}{\partial x^2}\left(\nu D\frac{\partial^2 w}{\partial y^2}\right)+$$

$$+\frac{\partial^2}{\partial y^2}\left(\nu D\frac{\partial^2 w}{\partial x^2}\right)+2\frac{\partial^2}{\partial x\partial y}\left((1-\nu)D\frac{\partial^2 w}{\partial x\partial y}\right)-\lambda g w=0, \tag{2.1}$$

$$w\Big|_l=\frac{\partial w}{\partial \nu}\Big|_l=0,\quad D=\frac{Eh^3}{12(1-\nu^2)},\quad g(x,y)=\rho h(x,y),$$

where D is the flexural rigidity, $h=h(x,y)$ is the thickness of the plate, w is the deflection of the plate and ρ is the density of its material. The boundary value problem (2.1) may be written in the form of the following operator equation, analogous to (1.3):

$$\begin{aligned} &Aw-\lambda B\dot{w}=0,\\ &w\in D_A,\ D_A\subset D_B. \end{aligned} \tag{2.2}$$

As above we assume that the operator A has a fully continuous inverse operator A^{-1}, and the problem of the determination of the eigenvalues of the equation (2.1) is equivalent to the problem of the determination of the eigenvalues of the equation

$$w-\lambda A^{-1}Bw=w-\lambda\iint_S G(x,y,\xi,\eta)g(\xi,\eta)w(\xi,\eta)\,d\xi\,d\eta=0, \tag{2.3}$$

where $G(x,y,\xi,\eta)$ is the Green function for the problem of bending of a plate of variable thickness, clamped along its contour. Taking into consideration that $g(\xi,\eta)\geq g_0>0$, the equation (2.3) may be symmetrized in the usual manner. We obtain

$$w'(x,y)-\lambda\iint G'(x,y,\xi,\eta)w'(\xi,\eta)\,d\xi\,d\eta=0, \tag{2.4}$$

where

$$w'(x,y)=w(x,y)\sqrt{g(x,y)},\quad G'=G\sqrt{g(x,y)}\,\sqrt{g(\xi,\eta)}.$$

Side by side with the equation (2.4) we consider the following equation with the fully continuous kernel G_1':

$$w_1(x,y)-\lambda\iint_S G_1'(x,y,\xi,\eta)w_1(\xi,\eta)\,d\xi\,d\eta=0, \tag{2.5}$$

where we will take $G_1'(x, y, \xi, \eta)$ in the form

$$G_1' = G_1(x, y, \xi, \eta)\sqrt{g(x, y)}\,\sqrt{g(\xi, \eta)},$$

$$G_1(x, y, \xi, \eta) = \sum_{r=1}^{m}\sum_{k=1}^{m} b_{kr}\varphi_r(x, y)\varphi_k(\xi, \eta), \quad b_{kr} = b_{rk}, \tag{2.6}$$

$$\varphi_k\Big|_l = \frac{\partial\varphi_k}{\partial\nu}\Big|_l = 0.$$

Then, in order to find estimates from below for the eigenvalues λ_n of the equation (2.2) by use of the formulae (1.9)–(1.10), we must find an estimate α^2 for the quantity

$$\iint_S dx\,dy \iint_S [u'(x, y, \xi, \eta)]^2\,d\xi\,d\eta, \quad u' = G' - G_1', \tag{2.7}$$

analogous to (1.11).

Taking into consideration (2.4) and (2.6), we find

$$\iint_S dx\,dy\left\{\iint_S [u']^2\,d\xi\,d\eta\right\} \le g_1^2 \iint_S dx\,dy\left\{\iint_S u^2(x, y, \xi, \eta)\,d\xi\,d\eta\right\} \le \alpha^2,$$

$$u = G - G_1, \quad g_1 = \max g(x, y).$$

Further, it is obvious that for our choice of the functions G_1 and G_1' the function u, considered as a function of ξ, η, satisfies the following boundary problem, analogous to (1.14):

$$Au = A[G - G_1] = \delta - AG_1 = \delta - \psi(x, y, \xi, \eta) = f(x, y, \xi, \eta),$$

$$u\Big|_l = \frac{\partial u}{\partial\nu}\Big|_l = 0, \tag{2.8}$$

where δ is the Dirac function.

Different methods may be applied in order to obtain estimates of the function u and the integral (2.7). We will consider here the following method which is analogous to that applied above to plates of constant thickness. If u is the solution of the boundary value problem (2.8), we have for the function u the estimate [21]

$$|u|^2 \le \iint_S \frac{[q(x, y, \xi, \eta)]^2}{(1 - \nu)D(\xi, \eta)}\,d\xi\,d\eta \iint_S \frac{H^2}{(1 - \nu)D(\xi, \eta)}\,d\xi\,d\eta, \tag{2.9}$$

where q and H have the same meaning as in § 1, if the right hand side of (1.16) is replaced by the right hand side of (2.8) and correspondingly the integrand $\nabla^4_{\xi,\eta}G_1$ in (1.18) is replaced by the function AG_1 in (2.8).

The estimate (2.9) may be improved by writing

$$|u|^2 < \frac{1}{(1-\nu)^2 D_0^2} \iint_S q^2\, d\xi\, d\eta \cdot \iint_S H^2\, d\xi\, d\eta, \tag{2.10}$$

where

$$D_0 = \min D = \frac{Eh_0^3}{12(1-\nu^2)}, \quad h_0 = \min h.$$

Substituting (2.10) into (2.7), we find the unknown estimate α in the formulae (1.9)–(1.10), where one must now understand by λ_n the eigenvalues of the equation (2.1) and λ_{1n} are the eigenvalues of the equation (2.5) with the generating kernel

$$G_1' = G_1 \sqrt{g(x,y)}\, \sqrt{g(\xi,\eta)}.$$

There still remains the problem of the evaluation of the integrals in (2.5). If the contour of the region S of the plate of variable thickness or the contour of a region S' completely containing the region S may be written in the form of the equation (1.20) and, besides, if $\sqrt{g(x,y)} = \sqrt{\rho h(x,y)}$ is a polynomial in x, y, then the integrals entering into (2.5)–(2.10) may be evaluated in closed form under the same conditions as for the plates of constant thickness. As in § 1, in order to find under these conditions the optimum estimate for the second integral in (2.7), one must determine the coefficients b_{kr} in (2.6) and (2.7) from the conditions for a minimum of this integral, i.e., from the condition $\alpha = \alpha_{\min}$.

NOTE: It is seen directly from the expression

$$\frac{(Aw, w)}{(Bw, w)} = \frac{\iint_S D\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\nu \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right] dx\, dy}{\iint_S g w^2\, dx\, dy}$$

and on the basis of the minimal properties of the eigenvalues that, if we replace $g(x,y)$ by a polynomial $g_1(x,y) \geq 0$, where $g_1(x,y) < g(x,y)$ in the region S, and the rigidity $D(x,y)$ by a polynomial $D_1(x,y) > 0$ in the region S with $D_1 \geq D$, the eigenvalues of the equation

$$A_1 w - \lambda B_1 w = 0$$

will not be larger than the eigenvalues of the equation (2.1).

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ON A CERTAIN CLASS OF ISOTROPIC CONGRUENCES

V. I. SMIRNOV
Leningrad, U.S.S.R.

§ 1. The aim of this paper is the study of one class of isotropic congruences in three-dimensional space in connection with the functionally invariant solutions of the corresponding partial differential equations. It represents a further development of **a** problem studied in the Author's paper "On conjugate functions, II" (Vestn. Leningr. un-ta, 1953, No. 11, 3–12).

We reproduce the basic definitions and results of this paper which we will use in what follows. Two functions $u(M)$, $v(M)$, defined in some region of three-dimensional space and satisfying the relations

$$\begin{aligned} (\operatorname{grad} u)^2 &= (\operatorname{grad} v)^2, \\ \operatorname{grad} u \cdot \operatorname{grad} v &= 0 \end{aligned} \tag{1.1}$$

will be called conjugate and the congruence (L) of the lines, defined by the equations

$$u(M) = C_1, \; v(M) = C_2, \tag{1.2}$$

where C_1 and C_2 are arbitrary constants, will be called an isotropic congruence. It will be assumed that in the above region $(\operatorname{grad} u)^2 \neq 0$.

Let **a** be the field of unit vectors tangential to the lines of the congruence and (α, β, γ) the components of **a** in some Cartesian coordinate system. The necessary and sufficient condition for the isotropy of congruences, expressed in terms of (α, β, γ), has the form

$$\begin{aligned} \beta\alpha_z - \gamma\alpha_y &= \tfrac{1}{2}(1 - \alpha^2)\mathbf{a} \cdot \operatorname{rot} \mathbf{a}, \\ \gamma\beta_x - \alpha\beta_z &= \tfrac{1}{2}(1 - \beta^2)\mathbf{a} \cdot \operatorname{rot} \mathbf{a}, \\ \alpha\gamma_y - \beta\gamma_x &= \tfrac{1}{2}(1 - \gamma^2)\mathbf{a} \cdot \operatorname{rot} \mathbf{a}, \end{aligned} \tag{1.3}$$

where one of these equalities is a consequence of the other two. The necessary and sufficient condition of isotropy may likewise be written in the form

$$\begin{aligned} \alpha_x - \alpha l &= \tfrac{1}{2}(1 - \alpha^2) \operatorname{div} \mathbf{a}, \\ \beta_y - \beta m &= \tfrac{1}{2}(1 - \beta^2) \operatorname{div} \mathbf{a}, \\ \gamma_z - \gamma n &= \tfrac{1}{2}(1 - \gamma^2) \operatorname{div} \mathbf{a}, \end{aligned} \tag{1.4}$$

where (l, m, n) are the components of the curvature vector **N** of the

congruence lines

$$l = \alpha\alpha_x + \beta\alpha_y + \gamma\alpha_z,$$
$$m = \alpha\beta_x + \beta\beta_y + \gamma\beta_z,$$
$$n = \alpha\gamma_x + \beta\gamma_y + \gamma\gamma_z.$$

One of the equalities (1.4) is a consequence of the other two.

We call a system of coordinates local at a point M, if the axis X is directed along $\mathbf{a}$ at the point M. In this coordinate system the necessary and sufficient condition of isotropy is expressed at the point M by the equalities

$$\beta_y = \gamma_z, \ \beta_z = -\gamma_y. \tag{1.5}$$

In what follows we will confine ourselves to the local point of view and we will assume that all functions are sufficiently smooth in the region under consideration. In addition, we assume that the quantities (α, β, γ) differ from zero.

If (L) is some isotropic congruence and $u(M)$ and $v(M)$ are some conjugate functions which generate it, then $\varphi = u + vi$ is a solution of the equation

$$\Delta\varphi + (\mathbf{N} + \sigma\mathbf{a})\cdot\operatorname{grad}\varphi = 0, \tag{1.6}$$

where σ is an arbitrary point function. The solution $u + vi$ is functionally invariant, i.e., any analytic function $F(u + vi)$ is also a solution of (1.6). Conversely, if some equation of the form

$$\Delta\varphi + \mathbf{b}\cdot\operatorname{grad}\varphi = 0,$$

where $\mathbf{b}$ is some real vector, has the functionally invariant solution $u + vi$, then $u(M)$ and $v(M)$ are conjugate functions and $\mathbf{b} = \mathbf{N} + \sigma\mathbf{a}$, where $\mathbf{N}$ and $\mathbf{a}$ correspond to the isotropic congruence generating the functions $u(M)$, $v(M)$ and σ is some point function. Thus the equation (1.6) represents the general form of linear equations with the principal term $\Delta\varphi$ which have functionally invariant solutions.

§ 2. In what follows we will consider mainly the case when for some choice of Cartesian coordinates the quantities (α, β, γ) do not depend on z. Then the conditions (1.4) take the form

$$\begin{aligned}(-1 + \alpha^2)\alpha_x + 2\alpha\beta\alpha_y + (1 - \alpha^2)\beta_y &= 0,\\ (1 - \beta^2)\alpha_x + 2\alpha\beta\beta_x + (-1 + \beta^2)\beta_y &= 0,\end{aligned} \tag{2.1}$$

and γ is determined by the obvious relation

$$\alpha^2 + \beta^2 + \gamma^2 = 1. \tag{2.2}$$

We will take α and β as independent variables and assume x and y to be known functions of these variables

$$x = \psi(\alpha, \beta), \quad y = \varphi(\alpha, \beta). \tag{2.3}$$

We obtain for ψ and φ the system of linear differential equations

$$\begin{aligned} (1 - \beta^2)\varphi_\beta + (1 - \alpha^2)\psi_\alpha &= 0, \\ 2\alpha\beta\psi_\alpha + (1 - \beta^2)(\varphi_\alpha - \psi_\beta) &= 0. \end{aligned} \tag{2.4}$$

Using known methods (cf., I. N. Vekua, Generalized analytic functions, Moscow, Fizmatgiz, 1959, p. 137), we reduce the system (2.4) to the canonical form. We obtain as a result the system of Cauchy-Riemann; the system (2.4) is thus integrated and its general solution contains an arbitrary analytic function. We will only produce the final result, omitting all wearisome manipulations. The variables (x, y) are expressed in terms of the variables (α, β, γ) in the following manner:

$$x = \alpha\lambda + \frac{\beta}{\gamma}\mu, \quad y = \beta\lambda - \frac{\alpha}{\gamma}\mu, \tag{2.5}$$

where $\lambda + \mu i$ is an arbitrary analytic function of the argument

$$w = \arctan\frac{\alpha}{\beta} + \frac{i}{2}\log\frac{1+\gamma}{1-\gamma}. \tag{2.6}$$

Instead of this argument we may take any analytic function in terms of it

$$\cos w = \frac{\beta - \alpha\gamma i}{\alpha^2 + \beta^2}, \quad \sin w = \frac{\alpha + \beta\gamma i}{\alpha^2 + \beta^2}, \quad e^{wi} = \frac{\beta + \alpha i}{1 + \gamma}, \text{ etc.} \tag{2.7}$$

The formulae (2.5) may be written in the form

$$x \sin w + y \cos w = \lambda + \mu i,$$

where the right hand side contains an arbitrary analytic function of the argument w or of any of the arguments (2.7). Using as argument $t = \sin w$, we obtain the equation

$$xt + y\sqrt{1 - t^2} = f(t). \tag{2.8}$$

This equation determines t as a complex function of (x, y). Its real part gives $\alpha/(\alpha^2 + \beta^2)$, its imaginary part $\beta\gamma/(\alpha^2 + \beta^2)$. We recall that the quantity γ is determined by the relations (2.2). One has to keep in mind the fact that (α, β) must satisfy the inequality $(\alpha^2 + \beta^2) < 1$.

We note that the quantities (α, β, γ) are expressed in terms of the

quantity t, defined by the equation (2.8), by the formulae

$$\alpha = \frac{1 + t\bar{t} - \sqrt{(t^2 - 1)(\bar{t}^2 - 1)}}{t + \bar{t}},$$

$$\beta = \frac{\bar{t}\sqrt{t^2 - 1} - t\sqrt{\bar{t}^2 - 1}}{i(t + \bar{t})}, \tag{2.9}$$

$$\gamma = \frac{\sqrt{t^2 - 1} + \sqrt{\bar{t}^2 - 1}}{t + \bar{t}}.$$

§ 3. We have stated a general method for finding all isotropic congruences on which the quantities (α, β, γ) do not depend on z. We will now demonstrate a method for the construction of pairs of conjugate functions, corresponding to such congruences. We define the functions $\sigma(x, y)$ and $\tau(x, y)$ by the following two relations:

$$\sigma_x = \frac{\beta}{\alpha^2 + \beta^2}, \quad \sigma_y = -\frac{\alpha}{\alpha^2 + \beta^2},$$

$$\tau_x = \frac{\alpha\gamma}{\alpha^2 + \beta^2}, \quad \tau_y = \frac{\beta\gamma}{\alpha^2 + \beta^2}, \tag{3.1}$$

where (α, β, γ) do not depend on z and correspond to some isotropic congruence. We must verify the two conditions of integrability

$$\left(\frac{\beta}{\alpha^2 + \beta^2}\right)_y = -\left(\frac{\alpha}{\alpha^2 + \beta^2}\right)_x, \quad \left(\frac{\alpha\gamma}{\alpha^2 + \beta^2}\right)_y = \left(\frac{\beta\gamma}{\alpha^2 + \beta^2}\right)_x.$$

The first of these reduces to the form

$$(\beta^2 - \alpha^2)\alpha_x - 2\alpha\beta(\alpha_y + \beta_x) + (\alpha^2 - \beta^2)\beta_y = 0,$$

and this equation is a direct consequence of the equations of the system (2.1). It is easily verified that also the second condition of integrability

$$\gamma(1 - \gamma^2)(\beta_x - \alpha_y) + (1 + \gamma^2)(\beta\gamma_x - \alpha\gamma_y) = 0$$

is a consequence of the system (2.1). It is sufficient to multiply it by γ and to take into consideration the obvious relations

$$\alpha\alpha_x + \beta\beta_x + \gamma\gamma_x = 0,$$

$$\alpha\alpha_y + \beta\beta_y + \gamma\gamma_y = 0. \tag{3.2}$$

Thus, the relations (3.1) afford the possibility to determine $\sigma(x, y)$ and $\tau(x, y)$. We will now construct the functions

$$u(x, y, z) = \tau(x, y) - z,$$

$$v(x, y) = \sigma(x, y). \tag{3.3}$$

By virtue of (3.1) the functions $u(M)$, $v(M)$ satisfy the relations (1.1), i.e., they are conjugate functions. In addition, it follows from (3.1) that

$$\alpha u_x + \beta u_y + \gamma u_z = 0,$$

$$\alpha v_x + \beta v_y = 0,$$

i.e., $u(M)$ and $v(M)$ assume constant values along the lines of the isotropic congruences, determined by the quantities (α, β, γ).

The real and imaginary parts of any analytic function $F(u + vi)$ represent a pair of conjugate functions determining such an isotropic congruence. The functions (3.3) have a special structure: the function $v(M)$ does not depend on z and $u(M)$ contains z linearly. We note that the formulae (3.1) may be written in the form

$$\sigma_x - \tau_x i = \cos w,$$

$$\sigma_y - \tau_y i = -\sin w.$$

§ 4. We have seen above that an isotropic congruence, on which the quantities (α, β, γ) do not depend on z, may be determined by the conjugate functions (3.3), one of which does not depend on z, while the second depends on z in a special manner. Conversely, we will postulate that conjugate functions are to have the form (3.3). It is not difficult to see that the quantities (α, β, γ) on the isotropic congruences, corresponding to them, do not depend on z. This follows directly from the equalities

$$\alpha\sigma_x + \beta\sigma_y = 0, \quad \alpha\tau_x + \beta\tau_y - \gamma = 0$$

and the relations (2.2). We will now prove a more general proposition:

THEOREM. *If one of the two conjugate functions $u(x, y, z)$ and $v(x, y)$ does not depend on z and $\mathbf{a} \cdot rot\, \mathbf{a} \neq 0$, then the quantities (α, β, γ) do not depend on z on the isotropic congruences corresponding to them.*

The following equalities are true:

$$\alpha v_x + \beta v_y = 0, \quad \alpha u_x + \beta u_y + \gamma u_z = 0, \tag{4.1}$$

$$u_x v_x + u_y v_y = 0, \quad u_x^2 + u_y^2 + u_z^2 = v_x^2 + v_y^2. \tag{4.2}$$

It follows from the first of the equations (4.1) that the ratio β/α does not depend on z, and hence

$$\beta\alpha_z - \alpha\beta_z = 0. \tag{4.3}$$

By virtue of this, adding the first two of the relations (1.3), we obtain

the relation

$$\beta_x - \alpha_y = \frac{1+\gamma^2}{2\gamma}\,\mathbf{a}\cdot\mathrm{rot}\,\mathbf{a}, \tag{4.4}$$

which will be used below.

We readily obtain from the relations (4.1) and (4.2) the equalities

$$v_x = \beta\omega,\ v_y = -\alpha\omega, \tag{4.5}$$

$$u_x = \alpha\gamma\omega,\ u_y = \beta\gamma\omega,\ u_z = -(\alpha^2+\beta^2)\omega, \tag{4.6}$$

where ω is some point function. The conditions $u_{xz} = u_{zx}$, $u_{yz} = u_{zy}$ and $(\alpha\omega)_z = 0$ lead to the three equalities

$$\begin{gathered}(\alpha^2+\beta^2)\omega_x + (\alpha\gamma_z - 2\gamma\gamma_x)\omega = 0,\\ (\alpha^2+\beta^2)\omega_y + (\beta\gamma_z - 2\gamma\gamma_y)\omega = 0,\\ \alpha\omega_z + \omega\alpha_z = 0.\end{gathered} \tag{4.7}$$

Hence it follows that the expressions

$$\frac{\alpha\gamma_z - 2\gamma\gamma_x}{\alpha^2+\beta^2} - \frac{\alpha_x}{\alpha} \text{ and } \frac{\beta\gamma_z - 2\gamma\gamma_y}{\alpha^2+\beta^2} - \frac{\alpha_y}{\alpha}$$

are partial derivatives of some function which does not depend on z. Multiplying the first of these by $-\beta/\alpha$ and adding it to the second, we obtain the following expression which does not depend on z:

$$H(x,y) = 2\gamma\,\frac{\beta\gamma_x - \alpha\gamma_y}{\alpha(\alpha^2+\beta^2)} + \frac{\beta\alpha_x}{\alpha^2} - \frac{\alpha_y}{\alpha},$$

or, using (1.4),

$$H(x,y) = -\frac{\gamma}{\alpha}\,\mathbf{a}\cdot\mathrm{rot}\,\mathbf{a} + \frac{\beta\alpha_x}{\alpha^2} - \frac{\alpha_y}{\alpha}.$$

But the difference

$$\frac{\beta\alpha_x}{\alpha^2} - \frac{\beta_x}{\alpha} = -\left(\frac{\beta}{\alpha}\right)_x$$

does not depend on z, and, consequently, also the expression

$$G(x,y) = -\frac{\gamma}{\alpha}\,\mathbf{a}\cdot\mathrm{rot}\,\mathbf{a} + \frac{\beta_x - \alpha_y}{\alpha}$$

is independent of z. Using the formulae (4.4) and (1.3) we obtain

$$G(x,y) = \frac{1-\gamma^2}{2\alpha\gamma}\,\mathbf{a}\cdot\mathrm{rot}\,\mathbf{a} = \frac{\alpha\gamma_y - \beta\gamma_x}{\alpha\gamma}. \tag{4.8}$$

Introducing the function $\rho = \beta/\alpha$, which does not depend on z, we obtain, using the formulae (3.2) and (4.4),

$$\mathbf{a}\cdot\operatorname{rot}\mathbf{a} = \frac{\alpha^2\gamma(\alpha\rho_x + \beta\rho_y) + \alpha\gamma_y - \beta\gamma_x}{\alpha^2 + \beta^2},$$

while the third of the conditions (1.3) leads to the equality

$$\alpha\gamma_y - \beta\gamma_x = \alpha^2\gamma(\alpha\rho_x + \beta\rho_y),$$

i.e., by (4.8), to

$$\alpha^2(\rho_x + \rho\rho_y) = G(x, y). \tag{4.9}$$

By assumption $\mathbf{a}\cdot\operatorname{rot}\mathbf{a} \neq 0$, i.e., $G(x, y) \neq 0$, and it follows from (4.9) that α is independent of z; hence it follows directly that also β and γ are independent of z. It is known that for $\mathbf{a}\cdot\operatorname{rot}\mathbf{a} = 0$ an isotropic congruence is a congruence of orthogonal trajectories of a family of planes or a family of spheres, depending on one parameter. Conversely, each congruence of orthogonal trajectories of planes or spheres is an isotropic congruence.

§ 5. We consider now the case when on an isotropic congruence $\mathbf{a}\cdot\operatorname{rot}\mathbf{a} = 0$ and one of the conjugate functions, $v(x, y)$, does not depend on z. Then the formulae (4.1)–(4.4) apply and from the last formula follows the equality $\beta_x = \alpha_y$.

Taking into consideration the condition (1.3), we may verify that the case under consideration is determined by the independence of $v(x, y)$ on z and by the equalities

$$\beta_x = \alpha_y,\ \gamma\beta_x = \alpha\beta_z = \beta\alpha_z,\ \alpha\gamma_y = \beta\gamma_x. \tag{5.1}$$

The first two of the formulae (4.7) may be written for this purpose in the form

$$\begin{aligned}(\alpha^2 + \beta^2)\omega_x + \alpha(\alpha_x + \beta_y)\omega = 0,\\ (\alpha^2 + \beta^2)\omega_y + \beta(\alpha_x + \beta_y)\omega = 0.\end{aligned} \tag{5.2}$$

From the formulae (4.5) follows

$$v_{xx} = \omega\beta_x + \omega_x\beta,\ v_{yy} = -\omega\alpha_y - \omega_y\alpha,$$

and, taking (5.1) and (5.2) into consideration, it may be seen that $v(x, y)$ must be a harmonic function. We introduce now the harmonic function $w(x, y)$, conjugate to $v(x, y)$, and we will seek the general form of the function $u(x, y, z)$, conjugate to $v(x, y)$, in the form

$$u(x, y, z) = w(x, y) + \varepsilon(x, y, z). \tag{5.3}$$

We obtain for the function ε the two equations

$$v_x\varepsilon_x + v_y\varepsilon_y = 0,\ \varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2 + 2(w_x\varepsilon_x + w_y\varepsilon_y) = 0. \tag{5.4}$$

Taking into consideration the equality $v_xw_x + v_yw_y = 0$, it may be verified that ε is a function of w and z and that the equations (5.4) reduce for this to the single equation

$$(\varepsilon_w^2 + 2\varepsilon_w)(w_x^2 + w_y^2) + \varepsilon_z^2 = 0. \tag{5.5}$$

It follows from this equation that the sum $w_x^2 + w_y^2$ is a function of w only. But it is not difficult to see that this is only possible in the following two cases:

$$w = Ax + By + C,\ w = A\log\rho + C, \tag{5.6}$$

where A, B, C are real constants and $\rho = \sqrt{(x - x_0)^2 + (y - y_0)^2}$. Without limiting generality one may assume $C = x_0 = y_0 = 0$.

Before proceeding to the consideration of these two cases, we note that (α, β, γ), corresponding to the considered isotropic congruence, are obtained from the formulae

$$\alpha v_x + \beta v_y = 0,\ \alpha w_x(1 + \varepsilon_w) + \beta w_y(1 + \varepsilon_w) + \gamma\varepsilon_z = 0, \tag{5.7}$$

where ε must satisfy the equation (5.5) and we assume, naturally, that $\varepsilon_z \neq 0$. Taking into consideration (5.7) and the conjugacy of the functions $v(x, y)$ and $w(x, y)$ we obtain

$$\alpha = \pm\frac{w_x}{w_x^2 + w_y^2}\varepsilon_z,$$

$$\beta = \pm\frac{w_y}{w_x^2 + w_y^2}\varepsilon_z,$$

$$\gamma = \pm(1 + \varepsilon_w),$$

where one must take the same sign in the first two formulae and, without limiting generality, one may take the plus sign. The second of the equalities (5.7) shows that for this one must take in the expression for γ the minus sign

$$\alpha = \frac{w_x}{w_x^2 + w_y^2}\varepsilon_z,\ \beta = \frac{w_y}{w_x^2 + w_y^2}\varepsilon_z,\ \gamma = -(1 + \varepsilon_w). \tag{5.8}$$

It is not difficult to verify that then all the conditions (5.1) are satisfied.

The equation of the family of surfaces (planes or spheres), orthogonal

to the considered isotropic congruence, is obtained from the equation

$$\frac{\varepsilon_z}{w_x^2 + w_y^2}(w_x dx + w_y dy) - (1 + \varepsilon_w)dz = 0$$

or

$$\varepsilon_z dw - (w_x^2 + w_y^2)(1 + \varepsilon_w)dz = 0,$$

where $w_x^2 + w_y^2$ is a function of w only. Multiplying both parts of the equation by ε_{zw} and taking (5.5) into consideration, it may be verified that the equation of the unknown family of surfaces has the form

$$\varepsilon_z = C, \tag{5.9}$$

where ε is some solution of the equation (5.5). We will now proceed to the utilization of the formulae (5.6).

In the first case, $w_x^2 + w_y^2 = A^2 + B^2$, and the equation (5.5) has the form

$$(\varepsilon_w^2 + 2\varepsilon_w)(A^2 + B^2) + \varepsilon_z^2 = 0. \tag{5.10}$$

Its complete integral will be

$$\varepsilon = kz\sqrt{-b^2 - 2b} + bw + c,\ k^2 = A^2 + B^2. \tag{5.11}$$

For the construction of the general integral we form the two equations

$$\begin{aligned} &\varepsilon = kz\sqrt{-b^2 - 2b} + bw + \varphi(b), \\ &-k\frac{b+1}{\sqrt{-b^2-2b}}z + w + \varphi'(b) = 0. \end{aligned} \tag{5.12}$$

The second equation determines b as a function of (z, w); substituting this value in the first equation, we obtain an expression for ε in terms of (z, w). The family (5.9) is obtained by differentiation of this expression with respect to z. It is not difficult to see that this leads to the fact that b is a constant. In other words, the family of surfaces (5.9) is a family of planes the equation of which is the second of the equations (5.12) for constant value of b and $w = Ax + By$. The equation (5.8) has the form

$$\alpha = \frac{A}{A^2 + B^2}\varepsilon_z,\ \beta = \frac{B}{A^2 + B^2}\varepsilon_z,\ \gamma = -(1 + \varepsilon_w),$$

where $\varepsilon_z = k\sqrt{-b^2 - 2b}$ and b are determined by the second of the equalities (5.12).

We proceed now to the study of the second of the cases (5.6). Without

limiting generality we may assume $A = 1$, so that $w = \log \rho$, where $\rho = \sqrt{x^2 + y^2}$. Then

$$w_x^2 + w_y^2 = \frac{1}{\rho^2} = e^{-2w},$$

and the equation (5.5) has the form

$$(\varepsilon_w^2 + 2\varepsilon_w)e^{-2w} + \varepsilon_z^2 = 0.$$

The complete integral of this equation is

$$\varepsilon = bz + \sqrt{1 - b^2\rho^2} - \log\left(1 + \sqrt{1 - b^2\rho^2}\right) + c.$$

For the construction of the general integral one must set $c = \varphi(a)$ and add to the equation obtained the following:

$$z + \frac{\sqrt{1 - b^2\rho^2}}{b} - \frac{1}{b} + \varphi'(b) = 0.$$

For constant b this equation represents a family of spheres, orthogonal to the corresponding isotropic congruence.

§ 6. We now proceed to equations of the form

$$\Delta\varphi + \mathbf{b}\cdot\operatorname{grad}\varphi = 0 \tag{6.1}$$

which have functionally invariant solutions, generated by the isotropic congruences studied above. We write the equation (6.1) in terms of those Cartesian coordinates for which the quantities (α, β, γ) do not depend on z

$$\varphi_{xx} + \varphi_{yy} + \varphi_{zz} + p\varphi_x + q\varphi_y + r\varphi_z = 0, \tag{6.2}$$

where

$$\begin{aligned} p &= \alpha\alpha_x + \beta\alpha_y + \sigma\alpha, \\ q &= \alpha\beta_x + \beta\beta_y + \sigma\beta, \\ r &= \alpha\gamma_x + \beta\gamma_y + \sigma\gamma, \end{aligned} \tag{6.3}$$

and σ is an arbitrary function of (x, y, z). Given an isotropic congruence, we may construct the differential equation (6.2) which has functionally invariant solutions $u + vi$, corresponding to that congruence. We have determined these solutions above in terms of the quantities (α, β, γ).

Let us now give certain considerations concerning the inverse problem. It has to be made clear for what conditions functions p, q, and r of the independent variables (x, y, z) may be represented by the formulae (6.3), where (α, β, γ) do not depend on z and correspond to some isotropic congruence.

Eliminating from the relations (6.3) the function σ which is not subject to any conditions we obtain

$$\begin{aligned} \gamma p - \alpha r &= \gamma(\alpha\alpha_x + \beta\alpha_y) - \alpha(\alpha\gamma_x + \beta\gamma_y), \\ \gamma q - \beta r &= \gamma(\alpha\beta_x + \beta\beta_y) - \beta(\alpha\gamma_x + \beta\gamma_y). \end{aligned} \tag{6.4}$$

These relations and one of the equations (6.3) (for the determination of σ) replace the system (6.3).

Using the equations (3.2), we may eliminate from the equations (6.4) the quantities γ_x and γ_y. Taking also the equation (2.1) into consideration, which expresses the condition of isotropy of the congruence, we obtain the four equations

$$\begin{aligned} \beta\alpha_x - \alpha\alpha_y &= \gamma(\gamma q - \beta r), \\ \beta\beta_x - \alpha\beta_y &= \gamma(\alpha r - \gamma p), \\ (1 - \alpha^2)(-\alpha_x + \beta_y) + 2\alpha\beta\alpha_y &= 0, \\ (1 - \beta^2)(\alpha_x - \beta_y) + 2\alpha\beta\beta_x &= 0. \end{aligned}$$

Solving them for the derivatives we obtain

$$\begin{aligned} \alpha_x &= \frac{p\alpha(1 - \alpha^2) + q\beta(1 - \beta^2 - 2\alpha^2) - r\gamma(\alpha^2 + \beta^2)}{\alpha^2 + \beta^2}, \\ \beta_y &= \frac{p\alpha(1 - \alpha^2 - 2\beta^2) + q\beta(1 - \beta^2) - r\gamma(\alpha^2 + \beta^2)}{\alpha^2 + \beta^2}, \\ \alpha_y &= \frac{p\beta(1 - \alpha^2) - q\alpha(1 - \alpha^2)}{\alpha^2 + \beta^2}, \\ \beta_x &= \frac{-p\beta(1 - \beta^2) + q\alpha(1 - \beta^2)}{\alpha^2 + \beta^2}. \end{aligned} \tag{6.5}$$

We apply to this Pfaff system the ordinary method and write down the conditions $\alpha_{xy} = \alpha_{yx}$ and $\beta_{xy} = \beta_{yx}$, using the equations of the system. This leads us to the following two relations:

$$\begin{aligned} &P\alpha(1 - \alpha^2) + Q\beta(1 - \alpha^2) - R\beta(1 - 2\alpha^2 - \beta^2) - \\ &\quad - r^2\beta(\alpha^2 + \beta^2) + 2qr\gamma(\alpha^2 + \beta^2) - r_y\gamma(\alpha^2 + \beta^2) = 0, \\ &P\beta(1 - \beta^2) - Q\alpha(1 - \alpha^2 - 2\beta^2) + R\alpha(1 - \beta^2) - \\ &\quad - r^2\alpha(\alpha^2 + \beta^2) + 2pr\gamma(\alpha^2 + \beta^2) - r_x\gamma(\alpha^2 + \beta^2) = 0, \end{aligned} \tag{6.6}$$

where

$$P = p_y + q_x - 2pq, \quad Q = p^2 - p_x, \quad R = q^2 - q_y. \tag{6.7}$$

After constructing a linear combination of the relations (6.6) we find

$$\begin{aligned}&\gamma[P(\alpha^2-\beta^2)+2(Q-R)\alpha\beta]+\\&\qquad+(\alpha^2+\beta^2)(2qr\alpha-2pr\beta+r_x\beta-r_y\alpha)=0,\\&\gamma[(1-\alpha^2)Q-(1-\beta^2)R]+r^2(\alpha^2-\beta^2)\gamma+2qr\beta(1-\beta^2)-\\&\qquad-2pr\alpha(1-\alpha^2)+r_x\alpha(1-\alpha^2)-r_y\beta(1-\beta^2)=0.\end{aligned}\tag{6.8}$$

§ 7. We will now consider some examples. Let $p=q=0$. We proceed directly to the system (6.5)

$$\alpha_x=-r\gamma,\ \alpha_y=0,\ \beta_x=0,\ \beta_y=-r\gamma.$$

It follows from this system that

$$\alpha=Cx+C_1,\ \beta=Cy+C_2,$$

$$\gamma=\sqrt{1-(Cx+C_1)^2-(Cy+C_2)^2},\ r=-\frac{C}{\gamma},$$

where C, C_1 and C_2 are arbitrary constants.

We take $C_1=C_2=0$ and $C=1$, i.e., $\alpha=x$, $\beta=y$, $\gamma=\sqrt{1-x^2-y^2}$ and

$$r=-\frac{1}{\sqrt{1-x^2-y^2}}.\tag{7.1}$$

Using the formulae (3.1), we may determine the functionally invariant solution

$$u=\sqrt{1-x^2-y^2}-\tfrac{1}{2}\log\frac{1-\sqrt{1-x^2-y^2}}{1+\sqrt{1-x^2-y^2}}-z,$$

$$v=-\arctan\frac{y}{z}.$$

The curvature vector of the congruence has the components

$$l=x,\ m=y,\ n=-\frac{x^2+y^2}{\sqrt{1-x^2-y^2}},$$

and one obtains for the coefficients (p, q, r) the formulae

$$p=x(1+\sigma),\ q=y(1+\sigma),$$

$$r=-\frac{x^2+y^2}{\sqrt{1-x^2-y^2}}+\sigma\sqrt{1-x^2-y^2}.$$

For $\sigma=-1$, we obtain $p=q=0$ and r follows from the formula (7.1).

Let now $q = r = 0$. The relations (6.8) take the form

$$\begin{aligned} p_y(\alpha^2 - \beta^2) + (p^2 - p_x)2\alpha\beta &= 0, \\ (1 - \alpha^2)(p^2 - p_x) &= 0. \end{aligned} \tag{7.2}$$

They lead to the equations $p^2 - p_x = 0$ and $p_y = 0$, whence $p = 1/(C - x)$ or, setting $C = 0$, we obtain $p = -1/x$. For this one obtains the complete integrating system of Pfaff

$$\begin{aligned} \alpha_x &= -\frac{\alpha(1 - \alpha^2)}{x(\alpha^2 + \beta^2)}, \quad \beta_x = \frac{\beta(1 - \beta^2)}{x(\alpha^2 + \beta^2)}, \\ \alpha_y &= -\frac{\beta(1 - \alpha^2)}{x(\alpha^2 + \beta^2)}, \quad \beta_y = -\frac{\alpha(1 - \alpha^2 - 2\beta^2)}{x(\alpha^2 + \beta^2)}. \end{aligned} \tag{7.3}$$

For the determination of the quantities (α, β, γ) we obtain the system of equations

$$\frac{\gamma}{\alpha\beta} = \frac{2C}{y}, \quad \frac{\gamma}{1 - \alpha^2} = \frac{C}{x}, \quad \alpha^2 + \beta^2 + \gamma^2 = 1, \tag{7.4}$$

where one must add to x and y arbitrary constant terms. Using (7.3), we obtain the components of the curvature vector

$$l = -\frac{1 - \alpha^2}{x}, \quad m = \frac{\alpha\beta}{x}, \quad n = \frac{\alpha\gamma}{x}.$$

Possible coefficients (p, q, r) of the equation (6.2), corresponding to the quantities (α, β, γ) which are determined by the formulae (7.4), will have the form

$$p = -\frac{1 - \alpha^2}{x} + \sigma\alpha, \quad q = \frac{\alpha\beta}{x} + \sigma\beta, \quad r = \frac{\alpha\gamma}{x} + \sigma\gamma,$$

and the coefficients $(-1/x, 0, 0)$ are obtained for $\sigma = -\alpha/x$.

We will now assume that $r = 0$. The relations (6.8) reduce to the form

$$\begin{aligned} P(\alpha^2 - \beta^2) + 2(Q - R)\alpha\beta &= 0, \\ (1 - \alpha^2)Q - (1 - \beta^2)R &= 0. \end{aligned} \tag{7.5}$$

Solving these equations with respect to (α, β) and substituting them into the system (6.5) for $r = 0$, we obtain the following four relations for (p, q) and their derivatives:

$$\begin{aligned} L_x &= -2Lp - Mq, \quad M_x = -Mp - 2q, \\ L_y &= LMp + 2Lq, \quad M_y = (M^2 - 2L)p + Mq \end{aligned} \tag{7.6}$$

where

$$L = \frac{R}{Q}, \quad M = \frac{P}{Q}. \tag{7.7}$$

If we introduce the quantities

$$M = \frac{P}{Q}, \quad N = \frac{P}{R}, \tag{7.8}$$

we obtain instead of (7.6)

$$\begin{aligned} &M_x = -Mp - 2q, \quad N_x = Np + \left(N^2 - 2\frac{N}{M}\right)q, \\ &M_y = \left(M^2 - 2\frac{M}{N}\right)p + Mq, \quad N_y = -2p - Nq. \end{aligned} \tag{7.9}$$

These relations express the necessary and sufficient conditions for

$$\Delta\varphi + p\varphi_x + q\varphi_y = 0 \tag{7.10}$$

to have functionally invariant solutions.

We note that it follows from the system (6.6) for $r = 0$ that p and q must not depend on z. If p and q satisfy the system (7.6), we obtain from the system (7.5) the quantities α and β, and thereby also γ, and we have the isotropic congruence, corresponding to the obtained (α, β, γ). This congruence will correspond to the equation

$$\Delta\varphi + p_1\varphi_x + q_1\varphi_y + r_1\varphi_z = 0,$$

where

$$p_1 = p + \sigma\alpha, \quad q_1 = q + \sigma\beta, \quad r_1 = \sigma\gamma,$$

and $\sigma(x, y, z)$ is an arbitrary function. Obviously the vector (p_1, q_1, r_1) may be represented by the formula

$$(p_1, q_1, r_1) = \mathbf{N} + \sigma_0\mathbf{a},$$

where σ_0 is some function of (x, y).

Many years ago at one of the meetings of the Academy of Sciences of the USSR consideration was given to a plan of work in mathematics and N. I. Muskhelishvili, in connection with the application of functionally invariant solutions of the wave equation to problems of the theory of elasticity, commented on the advantage of finding solutions of a general type for partial differential equations. The recollection of his comment gave me the courage to present this paper in the collection dedicated to the highly esteemed Nikolai Ivanovich. I would like to do this not on the basis of mathematical considerations only. For me many bright memories of the distant past are connected with the name of Nikolai Ivanovich Muskhelishvili.

ON LUR'E's SOLUTION OF THE EQUATIONS OF THERMOELASTIC EQUILIBRIUM

I. N. SNEDDON and R. J. TAIT
Glasgow, Scotland

§ 1. Introduction

In recent years there has been a revival of interest in the calculation of the stress field in an elastic body in which there is a steady distribution of temperature. Early work in the subject is summarized in the book by Melan and Parkus [1]. More recent developments are described fully in the excellent monograph by Nowacki [2] and some of the more important technical applications are dealt with by Gatewood in his introductory text [3].

In his book on three-dimensional problems in the theory of elasticity Lur'e [4] derives a very simple and elegant solution of the equations of thermoelastic equilibrium and illustrates the method by considering a typical problem, namely the determination of the stress distribution in a semi-infinite solid when a point source of heat is placed on its boundary. An interesting consequence of Lur'e's solution is that it shows that the stress field induced in a thick plate (or semi-infinite solid) by an arbitrary distribution of temperature on the boundaries is plane and parallel to the boundaries. This result was discovered independently by Sternberg and McDowell [5, 6] using the method of the Green's function. Another proof of it, this time in cylindrical coordinates, was given by Muki [7, 8] using the theory of Hankel transforms and Fourier series. An attempt to prove the result in Cartesian coordinates by means of multiple Fourier transforms led Sneddon and others to the discussion of several special problems [9–12]. None of these other methods is as simple as Lur'e's.

In this paper an attempt is made to exploit more fully the possibilities presented by Lur'e's solution. In § 3 a slightly modified form of Lur'e's solution is given for the dimensionless equations presented in § 2. The next section (§ 4) shows how simple harmonic functions may be used to obtain solutions of problems of physical interest. In § 5 Fourier techniques are used to derive solutions of problems concerning semi-infinite solids and thick plates. The two last sections illustrate how the method may be used to calculate the thermal stresses in an elastic parallelepiped or finite cylinder which is embedded in a rigid plate which does not conduct heat.

In all the problems considered the solutions are presented in a form suitable for numerical computation. Such computations are being carried out and will be presented elsewhere.

§ 2. The equations of thermoelastic equilibrium

To facilitate the subsequent calculations we shall state the equations of thermoelastic equilibrium in dimensionless form [13]. If we take a typical length l as our unit of length, the reference temperature T as the unit of temperature, and the rigidity modulus μ as unit of stress we find that the equations of thermoelastic equilibrium in the absence of body forces and heat sources can be written in the dimensionless forms

$$\sigma_{ij,j} = 0, \tag{2.1}$$

$$\sigma_{ij} = [(\beta^2 - 2)\Delta - b\theta)]\delta_{ij} + 2\gamma_{ij}, \tag{2.2}$$

$$\nabla^2\theta = 0, \tag{2.3}$$

where, in terms of the Lamé constants λ, μ and the coefficient of linear expansion α,

$$\beta^2 = \frac{\lambda + 2\mu}{\mu}, \quad b = \frac{(3\lambda + 2\mu)\alpha T}{\mu}; \tag{2.4}$$

σ_{ij} denotes the ij-th component of the stress tensor, $f_{,j}$ means $\partial f/\partial x_j$ and the summation convention is used. The components γ_{ij} of the strain tensor are defined by the equation

$$\gamma_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}). \tag{2.5}$$

In cylindrical coordinates (ρ, ϕ, z) the equations of equilibrium (2.1) are replaced by

$$\begin{aligned}
&\frac{\partial \sigma_\rho}{\partial \rho} + \frac{1}{\rho}\frac{\partial \tau_{\rho\phi}}{\partial \phi} + \frac{\partial \tau_{\rho z}}{\partial z} + \frac{\sigma_\rho - \sigma_\phi}{\rho} = 0,\\
&\frac{\partial \tau_{\rho\phi}}{\partial \rho} + \frac{1}{\rho}\frac{\partial \sigma_\phi}{\partial \phi} + \frac{\partial \tau_{\phi z}}{\partial z} + \frac{2\tau_{\rho\phi}}{\rho} = 0,\\
&\frac{\partial \tau_{\rho z}}{\partial \rho} + \frac{1}{\rho}\frac{\partial \tau_{\phi z}}{\partial \phi} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{\rho z}}{\rho} = 0,
\end{aligned} \tag{2.6}$$

and the stress-strain relations (2.2) are replaced by

$$(\sigma_\rho, \sigma_\phi, \sigma_z) = (\beta^2 - 2)\Delta - b\theta + 2\left(\frac{\partial u_\rho}{\partial \rho}, \frac{u_\rho}{\rho} + \frac{1}{\rho}\frac{\partial u_\phi}{\partial \phi}, \frac{\partial u_z}{\partial z}\right), \tag{2.7}$$

$$\tau_{\phi z} = \frac{\partial u_\phi}{\partial z} + \frac{1}{\rho}\frac{\partial u_z}{\partial \phi}, \quad \tau_{z\rho} = \frac{\partial u_z}{\partial \rho} + \frac{\partial u_\rho}{\partial z}, \tag{2.8}$$

$$\tau_{\rho\phi} = \frac{\partial u_\phi}{\partial \rho} + \frac{1}{\rho}\frac{\partial u_\rho}{\partial \phi} - \frac{u_\phi}{\rho}.$$

In these equations the dilatation Δ is given by the equation

$$\Delta = \frac{\partial u_\rho}{\partial \rho} + \frac{u_\rho}{\rho} + \frac{1}{\rho}\frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_z}{\partial z}. \tag{2.9}$$

In problems in which there is axial symmetry (about the z-axis) we have $u_\phi = 0$, $\partial/\partial\phi \equiv 0$.

§ 3. The basic solutions

We shall now state a slightly modified form of Lur'e's solution [4].

If the displacement vector has components (u, v, w) referred to Cartesian coordinates (x, y, z) and if we write

$$u = \frac{\partial \psi}{\partial x}, \quad v = \frac{\partial \psi}{\partial y}, \quad w = -\frac{\partial \psi}{\partial z} + \chi(z), \tag{3.1}$$

$$b\theta = -2(\beta^2 - 1)\frac{\partial^2 \psi}{\partial z^2} + \beta^2 \chi'(z), \tag{3.2}$$

where ψ is a harmonic function so that

$$\nabla^2 \psi = 0, \tag{3.3}$$

and $\chi(z)$ is a quadratic function of z so that

$$\chi(z) = fz^2 + gz + h, \tag{3.4}$$

where f, g, h are constants, then we can readily show that

$$\tau_{xz} = \tau_{yz} = \sigma_z = 0 \tag{3.5}$$

at every point and that

$$\sigma_x = -2\left(\frac{\partial^2 \psi}{\partial y^2} + \chi'(z)\right), \quad \sigma_y = -2\left(\frac{\partial^2 \psi}{\partial x^2} + \chi'(z)\right), \tag{3.6}$$

$$\tau_{xy} = 2\frac{\partial^2 \psi}{\partial x \partial y}. \tag{3.7}$$

The solution expressed by equations (3.1), (3.2) is therefore appropriate to any problem in which the conditions (3.5) are satisfied at a boundary.

Similarly, in problems in which we use cylindrical coordinates, we may write

$$u_\rho = \frac{\partial \psi}{\partial \rho}, \quad u_\phi = \frac{1}{\rho}\frac{\partial \psi}{\partial \phi}, \quad u_z = -\frac{\partial \psi}{\partial z} + \chi(z), \tag{3.8}$$

where ψ, χ are given by equations (3.3), (3.4) respectively. The temperature fluctuation θ is again given by equation (3.2) and we also have

$$\sigma_z = \tau_{\phi z} = \tau_{z\rho} = 0, \tag{3.9}$$

and

$$\begin{gathered} \sigma_\rho = 2\left(\frac{\partial^2 \psi}{\partial \rho^2} + \frac{\partial^2 \psi}{\partial z^2}\right) - 2\chi'(z), \quad \tau_{\rho\phi} = 2\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial \psi}{\partial \rho}\right), \\ \sigma_\phi = -2\frac{\partial^2 \psi}{\partial \rho^2} - 2\chi'(z), \end{gathered} \tag{3.10}$$

so that a solution of the type given by equations (3.8), (3.2) is appropriate to any problem in which the conditions (3.9) are satisfied on a boundary.

In the case of axial symmetry we have $\tau_{\rho\phi} = 0$ everywhere and

$$\sigma_\rho - \sigma_\phi = 2\rho\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial \psi}{\partial \rho}\right). \tag{3.11}$$

§ 4. Elementary solutions

In this section we shall show by assuming a simple form of harmonic function $\psi(\rho, z)$ we can obtain solutions which are physically interesting. The method is then extended by a simple integration procedure to cover more complicated cases.

Consider the harmonic function

$$\psi(\rho, z) = -\frac{b}{4(\beta^2 - 1)}\sum_n \theta_n \log\frac{r_n - (z + a_n)}{r_n + (z + a_n)}, \tag{4.1}$$

where θ_n, a_n are constants and

$$r_n^2 = \rho^2 + (z + a_n)^2.$$

If, in addition, we choose $\chi(z) \equiv 0$, we find that

$$\theta(\rho, z) = \sum_n \theta_n \frac{z + a_n}{r_n^3}, \tag{4.2}$$

so that on the surface $z = 0$ we have

$$\theta(\rho, 0) = \sum_n \frac{a_n\theta_n}{(\rho^2 + a_n^2)^{\frac{1}{2}}}. \tag{4.3}$$

For this solution we find from equation (3.11) that

$$\sigma_\rho - \sigma_\phi = \frac{b}{\beta^2 - 1} \sum_n (z + a_n)\theta_n \left(\frac{1}{r_n^3} + \frac{2}{\rho^2 r_n}\right). \tag{4.4}$$

We shall illustrate by a simple example. If we take $a_1 = k + 1$, $a_2 = -(k - 1)$, $(k > 2)$, $\theta_1 = \theta_2 = -(k^2 - 4)^2\theta_0/(8k)$, we find from equation (4.2) that for this solution $\theta(\rho, -1) = 0$ and that $\theta(\rho, 1) = f(\rho)$, where

$$f(\rho) = \frac{(k^2 - 4)^2\theta_0}{8k} \left\{ \frac{k - 2}{[\rho^2 + (k - 2)^2]^{\frac{1}{2}}} - \frac{k + 2}{[\rho^2 + (k + 2)^2]^{\frac{1}{2}}} \right\}. \tag{4.5}$$

Hence the potential function

$$\psi(\rho, z) = \frac{(k^2 - 4)^2 b\theta_0}{32k(\beta^2 - 1)} \left\{ \log \frac{\sqrt{\rho^2 + (k + 1 + z)^2} - (k + 1 + z)}{\sqrt{\rho^2 + (k + 1 + z)^2} + (k + 1 + z)} - \right. \tag{4.6}$$

$$\left. - \log \frac{\sqrt{\rho^2 + (k - 1 - z)^2} + (k - 1 - z)}{\sqrt{\rho^2 + (k - 1 - z)^2} - (k - 1 - z)} \right\}, \quad (k > 2),$$

gives the solution of the problem in which the plane surfaces of the plate $-1 \leq z \leq 1$ are free from applied stress, the plate being deformed by the temperature field arising from the boundary conditions $\theta = 0$ on $z = -1$, $\theta = f(\rho)$ on $z = 1$, $f(\rho)$ being given by equation (4.5). This is the solution obtained by other means by Lockett and Sneddon [9].

If, instead of applying a simple summation process to the basic solution as in (4.1), we integrate over a parameter α we obtain the potential function

$$\psi(\rho, z) = -\frac{b}{4(\beta^2 - 1)} \int \Psi(\alpha) \log \left\{ \frac{r(\alpha) - (z + \alpha)}{r(\alpha) + (z + \alpha)} \right\} d\alpha, \tag{4.7}$$

where $r(\alpha) = [\rho^2 + (z + \alpha)^2]^{\frac{1}{2}}$ and the function $\Psi(\alpha)$ is arbitrary. For this potential function

$$\theta(\rho, z) = \int \Psi(\alpha) \frac{z + \alpha}{[r(\alpha)]^3} d\alpha \tag{4.8}$$

and

$$\sigma_\rho - \sigma_\phi = \frac{b}{\beta^2 - 1} \int (z + \alpha)\Psi(\alpha) \left\{ \frac{1}{[r(\alpha)]^3} + \frac{2}{\rho^2 r(\alpha)} \right\} d\alpha. \tag{4.9}$$

If we take $\Psi(\alpha)$ to have the (constant) value $-\frac{1}{2}(k^2-1)\theta_0$ when $-(k+1)<\alpha<k-1$ and to be zero otherwise, then it follows from equation (4.8) that

$$\theta(\rho,z)=-\tfrac{1}{2}(k^2-1)\theta_0\left\{\frac{1}{[\rho^2+(k+1-z)^2]^{\frac{1}{2}}}-\frac{1}{[\rho^2+(k-1+z)^2]^{\frac{1}{2}}}\right\}.$$

Hence, for this solution $\theta(\rho,1)=0$ and $\theta(\rho,0)=g(\rho)$ where

$$g(\rho)=\tfrac{1}{2}(k^2-1)\theta_0\left\{\frac{1}{[\rho^2+(k-1)^2]^{\frac{1}{2}}}-\frac{1}{[\rho^2+(k+1)^2]^{\frac{1}{2}}}\right\}. \tag{4.10}$$

Substituting this value for $\chi(\alpha)$ into equation (4.9) we see that the difference of the principal stresses in a plate $0\leq z\leq 1$ whose surfaces are free from applied stress and whose surface temperature distributions are defined by the equation $\theta(\rho,1)=0$, $\theta(\rho,0)=g(\rho)$ is given by the equation

$$\sigma_\rho-\sigma_\phi=\frac{b(k^2-1)\theta_0}{2(\beta^2-1)}(\rho_2-\rho_1)\left(\frac{1}{\rho_1\rho_2}+\frac{2}{\rho^2}\right), \tag{4.11}$$

where $\rho_1^2=\rho^2+(k-1+z)^2$, $\rho_2^2=\rho^2+(k+1-z)^2$.

If we take $\Psi(\alpha)=(Q/8\pi\kappa)e^{-h(\alpha-\delta)}$ in equation (4.7) and take (δ,∞) with $\delta>0$ as limits of integration, we obtain the potential function

$$\psi(\rho,z)=-\frac{bQ}{32(\beta^2-1)\pi\kappa}\int_\delta^\infty e^{-(\alpha-\delta)h}\log\left\{\frac{r(\alpha)-(z+\alpha)}{r(\alpha)+(z+\alpha)}\right\}d\alpha \tag{4.12}$$

which leads to the temperature field

$$\theta(\rho,z)=\frac{Qe^{h\delta}}{8\pi\kappa}\int_\delta^\infty\frac{(z+\alpha)e^{-h\alpha}\,d\alpha}{[(z+\alpha)^2+\rho^2]^{\frac{1}{2}}}$$

whose surface value is

$$\theta(\rho,0)=\frac{Qe^{h\delta}}{8\pi\kappa}\int_\delta^\infty\frac{\alpha e^{-h\alpha}\,d\alpha}{(\alpha^2+\rho^2)^{\frac{1}{2}}}.$$

Now it has been shown by Thomas [14] that this is the surface temperature distribution produced by a heat source of strength Q placed at the point $(0,0,-\delta)$ above the semi-infinite solid $z\geq 0$ with boundary condition

$$h\theta-\frac{\partial\theta}{\partial z}=\frac{Q\delta}{4\pi\kappa(\delta^2+\rho^2)^{\frac{1}{2}}} \tag{4.13}$$

on $z=0$.

§ 5. Application of Fourier techniques

In this section we shall make use of the techniques of Fourier analysis to derive solutions of certain problems for the semi-infinite solid and a thick plate. Put

$$(5.1)\qquad \psi(x, y, z) = -\frac{b}{4\pi(\beta^2 - 1)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\bar{f}(\xi, \eta)}{\zeta^2} e^{-i(\xi x+\eta y)-\zeta z}\, d\xi d\eta$$

with $\zeta^2 = \xi^2 + \eta^2$. For this potential function

$$\theta(x, y, z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{f}(\xi, \eta) e^{-i(\xi x+\eta y)-\zeta z}\, d\xi d\eta,$$

so that, if we assume $\chi(z) \equiv 0$, $\theta \to 0$ as $z \to \infty$ and $\theta(x, y, 0) = f(x, y)$, where $\bar{f}(\xi, \eta)$ is the Fourier transform of $f(x, y)$. The potential function (5.1) therefore gives the stress field in the semi-infinite solid $z \geq 0$ when the temperature distribution on its traction-free surface $z = 0$ is given by the equation $\theta = f(x, y)$.

Similarly, the functions $\chi(z) = 0$ and

$$(5.2)\qquad \psi(x, y, z) =$$

$$= -\frac{b}{4\pi(\beta^2-1)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\bar{f}(\xi, \eta)\sinh(1+z)\zeta + \bar{g}(\xi, \eta)\sinh(1-z)\zeta}{\zeta^2 \sinh 2\zeta} e^{-i(\xi x+\eta y)} d\xi d\eta$$

give the stress field in the thick plate $-1 \leqslant z \leqslant 1$, when the temperature distributions on its traction-free surfaces are given by the equations $\theta(x, y, 1) = f(x, y)$, $\theta(x, y, -1) = g(x, y)$. This solution was obtained by other methods in [9].

On the other hand, the potential function

$$(5.3)\qquad \psi(x, y, z) = -\frac{b}{4\pi(\beta^2 - 1)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\bar{f}(\xi, \eta)\cosh \zeta z}{\zeta^2 \cosh \zeta} e^{-i(\xi x+\eta y)}\, d\xi\, d\eta$$

(taken with $\chi(z) = 0$) has the property that $\theta(x, y, 1) = f(x, y)$ and $\partial\theta/\partial z = w = 0$ on $z = 0$. It therefore gives the stress field in an elastic layer $0 \leq z \leq 1$ which is resting on a rigid foundation whose boundary $z = 0$ is impervious to the flow of heat; the layer is deformed by the application of a temperature distribution $\theta(x, y, 1) = f(x, y)$. The properties of this solution are discussed in [10] where it was derived in another way.

For problems with axial symmetry we use a Hankel transform solution instead of a double Fourier one. For example,

$$\psi(\rho, z) = -\frac{b}{2(\beta^2 - 1)} \int_0^\infty \frac{\bar{f}(\xi)}{\xi} J_0(\xi\rho) e^{-\xi z}\, d\xi, \tag{5.4}$$

together with $\chi(z) = 0$ is the solution of the problem for a semi-infinte solid $z \geq 0$ whose stress-free boundary is deformed by the temperature distribution $\theta(\rho, 0) = f(\rho)$, $\bar{f}(\xi)$ denoting the zero-order Hankel transform of $f(\rho)$. It is assumed, in addition, that $\theta \to 0$ as $z \to \infty$. If we substitute this potential function into equations (3.8), we get the solution derived by Lockett and Sneddon [9].

Similarly, the potential function

$$\psi(\rho, z) = \tag{5.5}$$

$$= -\frac{b}{2(\beta^2 - 1)} \int_0^\infty \frac{\bar{f}(\xi)\sinh \xi(1 + z) + \bar{g}(\xi)\sinh \xi(1 - z)}{\xi \sinh 2\xi} J_0(\rho\xi)\, d\xi$$

provides the solution of the problem for a thick plate $-1 \leq z \leq 1$ to whose stress free boundaries are applied the temperature distributions $\theta(\rho, 1) = f(\rho)$, $\theta(\rho, -1) = g(\rho)$. The solution so obtained is in agreement with that previously found by Lockett and Sneddon [9].

$$\psi(\rho, z) = -\frac{b}{2(\beta^2 - 1)} \int_0^\infty \frac{\bar{Q}(\xi)\sinh \xi(1 + z)}{\xi^2 \cosh 2\xi} J_0(\rho\xi)\, d\xi \tag{5.6}$$

provides the solution of the problem of a thick plate $-1 \leq z \leq 1$ whose surface $z = -1$ is maintained at zero temperature while the flux $\partial\theta/\partial z$ on $z = 1$ is $q(\rho)$; the function $\bar{Q}(\xi)$ is the zero order Hankel transform of $q(\rho)$. This is the solution derived by Lur'e ([4], p. 196) who considers in some detail the special case in which $q(\rho) = (2\pi\rho)^{-1} Q_0 \delta(\rho)$. [See also Nowacki, [2], p. 97].

For an elastic layer $0 \leq z \leq 1$ resting on a rigid foundation $z \leq 0$ whose boundary $z = 0$ is impervious to the flow of heat we have the potential function

$$\psi(\rho, z) = -\frac{b}{2(\beta^2 - 1)} \int_0^\infty \bar{f}(\xi) \frac{\cosh \xi z}{\xi \cosh \xi} J_0(\rho\xi)\, d\xi \tag{5.7}$$

corresponding to a temperature distribution $\theta(\rho, 1) = f(\rho)$ on the free surface of the layer. The details of this solution are discussed in [10].

Cylindrical coordinates (ρ, ϕ, z) can be used profitably even in cases in which there is no axial symmetry.

If we take

$$\psi(\rho, \phi, z) = -\frac{b}{2(\beta^2 - 1)} \sum_{m=0}^{\infty} \cos m\phi \int_0^{\infty} \frac{a_m(\xi) J_m(\xi\rho)}{\xi} e^{-\xi z}\, d\xi, \tag{5.8}$$

then it follows from equation (3.2) — with $\chi(z) = 0$ — that the corresponding temperature field is given by the equation

$$\theta(\rho, \phi, z) = \sum_{m=0}^{\infty} \cos m\phi \int_0^{\infty} \xi a_m(\xi) J_m(\xi\rho) e^{-\xi z}\, d\xi$$

from which it follows that $\theta \to 0$ as $z \to \infty$ and

$$\theta(\rho, \phi, 0) = \sum_{m=0}^{\infty} \cos m\phi \int_0^{\infty} \xi a_m(\xi) J_m(\xi\rho) d\xi. \tag{5.9}$$

The potential function (5.8) is therefore appropriate to a semi-infinite solid $z \geq 0$ whose boundary $z = 0$ is kept free from applied stress. If the prescribed distribution of temperature on the surface is

$$\theta(\rho, \phi, 0) = \sum_{m=0}^{\infty} \theta_m(\rho) \cos m\phi, \tag{5.10}$$

then it follows from a comparison of equations (5.9) and (5.10) and the use of the Hankel inversion theorem that the a_m must be chosen so that

$$a_m(\xi) = \int_0^{\infty} \rho \theta_m(\rho) J_m(\rho\xi)\, d\rho \equiv \bar{\theta}_m(\xi). \tag{5.11}$$

If we substitute from equation (5.8) into equations (3.8) and write

$$I_{m+q}(\rho, z) = \int_0^{\infty} \bar{\theta}_m(\xi) J_{m+q}(\rho\xi) e^{-\xi z}\, d\xi,$$

we obtain Muki's solution [7]

$$u_\rho = \frac{b}{4(\beta^2 - 1)} \sum_{m=0}^{\infty} \cos m\phi \{I_{m+1} - I_{m-1}\}, \tag{5.12}$$

$$u_\phi = \frac{b}{4(\beta^2 - 1)} \sum_{m=0}^{\infty} \sin m\phi \{I_{m+1} + I_{m-1}\}, \tag{5.13}$$

$$u_z = -\frac{b}{4(\beta^2 - 1)} \sum_{m=0}^{\infty} \cos m\phi I_m. \tag{5.14}$$

We turn our attention now to the thick plate $-1 \leq z \leq 1$. If we take $\chi(z) = 0$ and

$$(5.15)\quad \psi(\rho, \phi, z) = $$

$$= -\frac{b}{2(\beta^2-1)} \sum_{m=0}^{\infty} \cos m\phi \int_0^{\infty} \frac{\bar{\theta}_{1,m}(\xi)\sinh \xi(1+z)+\bar{\theta}_{2,m}(\xi)\sinh \xi(1-z)}{\xi \sinh 2\xi} \cdot$$

$$\cdot J_m(\rho\xi)\, d\xi,$$

where $\bar{\theta}_{1,m}(\xi)$, $\bar{\theta}_{2,m}(\xi)$ denote the Hankel transforms of order m of the functions $\theta_{1,m}(\rho)$, $\theta_{2,m}(\rho)$, then, since by equation (3.2)

$$(5.16)\quad \theta(\rho, \phi, z) = \sum_{m=0}^{\infty} \cos m\phi \int_0^{\infty} [\bar{\theta}_{1,m}(\xi)\sinh \xi(1+z)+\bar{\theta}_{2,m}(\xi)\sinh \xi(1-z)] \cdot$$

$$\cdot \frac{\xi J_m(\rho\xi)\, d\xi}{\sinh 2\xi},$$

it follows that the surface temperatures are given by the equations

$$(5.17)\quad \theta(\rho, \phi, 1) = \sum_{m=0}^{\infty} \theta_{1,m}(\rho)\cos m\phi, \quad \theta(\rho, \phi, -1) = \sum_{m=0}^{\infty} \theta_{2,m}(\rho)\cos m\phi.$$

From equations (3.8) we find that the components u_ρ, u_ϕ of the displacement vector are given by equations (5.12), (5.13), but now with

$$(5.18)\quad I_{m+q} = \int_0^{\infty} [\bar{\theta}_{1,m}(\xi)\sinh \xi(1+z) + \bar{\theta}_{2,m}(\xi)\sinh \xi(1-z)] \frac{J_m(\rho\xi)\, d\xi}{\sinh 2\xi},$$

while u_z is given by the equation

$$(5.19)\quad u_z = -\frac{b}{4(\beta^2-1)} \sum_{m=0}^{\infty} \cos m\phi \cdot$$

$$\cdot \int_0^{\infty} \frac{\bar{\theta}_{1,m}(\xi)\cosh \xi(1+z) - \bar{\theta}_{2,m}(\xi)\cosh \xi(1-z)}{\sinh 2\xi} J_m(\rho\xi)\, d\xi.$$

Again these expressions agree with Muki's results.

If we have an elastic layer $0 \leq z \leq 1$ resting on an impervious rigid foundation $z \leq 0$ and deformed by the application of a temperature

$$\theta(\rho, \phi, 1) = \sum_{m=0}^{\infty} \theta_m(\rho)\cos m\phi$$

to its free surface, then we obtain the desired solution by putting $\bar{\theta}_{1,m}=\bar{\theta}_m$ in equation (5.15). Then

$$\psi(\rho, \phi, z) = -\frac{b}{2(\beta^2-1)} \sum_{m=0}^{\infty} \cos m\phi \int_0^{\infty} \bar{\theta}_m(\xi) \frac{\cosh \xi z}{\xi \sinh \xi} J_m(\xi\rho)\, d\xi.$$

It follows from equations (3.8) that the components u_ρ, u_ϕ of the displacement vector are given by equations (5.12), (5.13), but now with

$$I_{m+q} = \int_0^{\infty} \bar{\theta}_m(\xi)(\cosh \xi z/\cosh \xi) J_{m+q}(\xi\rho)\, d\xi, \tag{5.20}$$

while u_z is given by the equation

$$u_z = -\frac{b}{4(\beta^2-1)} \sum_{m=0}^{\infty} \cos m\phi \int_0^{\infty} \bar{\theta}_m(\xi) \frac{\sinh \xi z}{\cosh \xi} J_m(\xi\rho)\, d\xi. \tag{5.21}$$

§ 6. Solutions for rectangular parallelepipeds

If we use double Fourier series instead of double Fourier integrals, we get solutions appropriate to rectangular parallelepipeds. To keep the formal calculations as simple as possible we shall assume that the parallelepiped has one pair of faces in the form of a square, the length of whose edge is taken to be twice our unit of length. We shall assume that the faces of the parallelepiped perpendicular to these faces are impervious to heat and that the normal displacements on these faces vanish. The solutions we derive therefore give the stresses in an elastic plug contained in a rigid plate (see Figs. 1 and 2). For instance, if we take

$$\psi(x, y, z) = -\frac{b}{2(\beta^2-1)} \cdot$$

$$\cdot \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a_{mn} \sinh(\delta+z)p_{mn} + b_{mn} \sinh(\delta-z)p_{mn}}{p_{mn}^2 \sinh 2p_{mn}\delta} \cos m\pi x \cos n\pi y \tag{6.1}$$

with $(m, n) \neq (0, 0)$, $p_{mn}^2 = \pi^2(m^2+n^2)$, and

$$\chi(z) = \frac{ba_0}{4\beta^2\delta}(\delta+z)^2 - \frac{bb_0}{4\beta^2\delta}(\delta-z)^2, \tag{6.2}$$

we find that the temperature field is given by the equation

$$\theta(x, y, z) = \tfrac{1}{2}a_0\left(1 + \frac{z}{\delta}\right) + \tfrac{1}{2}b_0\left(1 - \frac{z}{\delta}\right) + \tag{6.3}$$

$$+ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{a_{mn}\sinh(\delta + z)p_{mn} + b_{mn}\sinh(\delta - z)p_{mn}}{\sinh 2p_{mn}\delta} \cos m\pi x \cos n\pi y.$$

It follows therefore that for this solution $\partial\theta/\partial n$ vanishes on the faces $x = \pm 1$, $y = \pm 1$, while $\theta(x, y, \delta) = f(x, y)$, $\theta(x, y, -\delta) = g(x, y)$ provided that the constants a_{mn}, b_{mn}, a_0, b_0 are chosen so that

$$f(x, y) = a_0 + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}\cos m\pi x \cos n\pi y,$$

$$g(x, y) = b_0 + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} b_{mn}\cos m\pi x \cos n\pi y,$$

it being remembered that in the double sums the term $m = n = 0$ is excluded.

From the theory of Fourier series

$$a_0 = \tfrac{1}{4}\iint f\,dx\,dy, \quad a_{m0} = \tfrac{1}{2}\iint f\cos m\pi x\,dx\,dy,$$

$$a_{0n} = \tfrac{1}{2}\iint f\cos n\pi y\,dx\,dy, \tag{6.4}$$

$$a_{mn} = \iint f\cos m\pi x\cos n\pi y\,dx\,dy,$$

the integrations being taken over the square $-1 \leq x \leq 1$, $-1 \leq y \leq 1$. The constants b_0, b_{mn} are given by similar equations with f replaced by g.

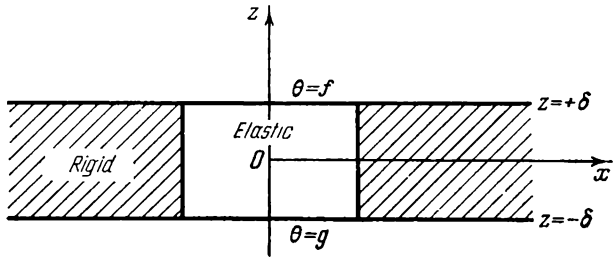

Fig. 1

Similarly, if we choose

$$\psi(x, y, z) = -\frac{b}{2(\beta^2 - 1)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_{mn}\cosh p_{mn}z}{p_{mn}^2\cosh p_{mn}\delta}\cos m\pi x\cos n\pi y \tag{6.5}$$

with $(m, n) \neq (0, 0)$, $p_{mn}^2 = \pi^2(m^2 + n^2)$ and

$$\chi(z) = ba_0 z/\beta^2, \tag{6.6}$$

then it follows that

$$\theta(x, y, z) = a_0 + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \frac{\cosh p_{mn} z}{\cosh p_{mn} \delta} \cos m\pi x \cos n\pi y.$$

For this solution $\partial\theta/\partial n$ vanishes on the faces $x = \pm 1, y = \pm 1, z = 0$, while $\theta(x, y, \delta) = f(x, y)$ where the constants a_0, a_{mn} are given by the

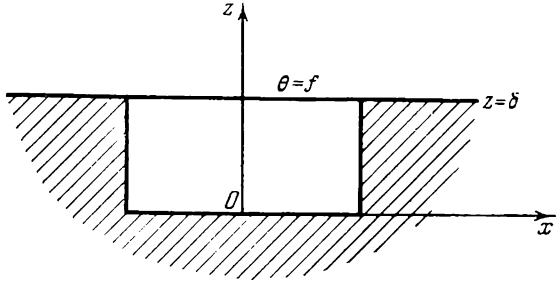

Fig. 2

equations (6.4) above. This solution therefore refers to an elastic parallelepiped inserted in a rigid block in the manner shown in Fig. 2.

§ 7. Solutions for finite cylinders

We can carry out the analysis of the previous section, but with the double Fourier series replaced by a Dini series. In this way we obtain solutions for cylinders of finite height analogous to those obtained above for parallelepipeds. We take the radius of the cylinder to be our unit of length.

Consider the potential function

$$(7.1)\quad \psi(\rho, z) = -\frac{b}{2(\beta^2 - 1)} \sum_{i=1}^{\infty} \frac{a_i \sinh(\delta + z)\lambda_i + b_i \sinh(\delta - z)\lambda_i}{\lambda_i^2 \sinh(2\lambda_i \delta)} J_0(\lambda_i \rho),$$

where the sum is taken over the positive zeros $\lambda_1, \lambda_2, \cdots$ of the function $J_1(\lambda)$, and the quadratic function

$$(7.2)\qquad \chi(z) = \frac{b}{4\beta^2 \delta} [a_0(\delta + z)^2 - b_0(\delta - z)^2].$$

From equation (3.2) we find that $u_\rho = 0$ on the surface $\rho = 1$ and that

$$\theta(\rho, z) = \tfrac{1}{2}a_0\left(1 + \frac{z}{\delta}\right) + \tfrac{1}{2}b_0\left(1 - \frac{z}{\delta}\right) + \\ + \sum_{i=1}^{\infty} [a_i \sinh(\delta + z)\lambda_i + b_i \sinh(\delta - z)\lambda_i] J_0(\lambda_i \rho) \operatorname{cosech}(2\lambda_i \delta),$$

so that $\partial\theta/\partial\rho = 0$ and $\rho = 1$ and

$$\theta(\rho, \delta) = a_0 + \sum_{i=1}^{\infty} a_i J_0(\lambda_i \rho). \tag{7.3}$$

From the theory of Dini series ([15], p. 597) we know that if we define the constants $a_0, a_1, a_2, \cdots$ by the equations

$$a_0 = 2\int_0^1 tf(t)\,dt, \quad a_i = \frac{2}{J_0^2(\lambda_i)}\int_0^1 tf(t)J_0(\lambda_i t)\,dt, \quad (i \geq 1), \tag{7.4}$$

then the series on the right hand side of equation (7.2) converges to the sum $f(\rho)$ for $0 \leq \rho \leq 1$. Similarly, if we define constants $b_0, b_1, b_2, \cdots$ by the equations

$$b_0 = 2\int_0^1 \tan(t)\,dt, \quad b_i = \frac{2}{J_0^2(\lambda_i)}\int_0^1 \tan(t)J_0(\lambda_i t)\,dt, \tag{7.5}$$

we find that $\theta(\rho, -\delta) = g(\rho)$.

If we insert the constants (7.4), (7.5) into equations (7.1) and (7.2), we obtain the solution of the equations of thermoelastic equilibrium satisfying the boundary conditions:

$$\sigma_z = \tau_{\rho z} = 0, \quad \theta = f(\rho), \quad \text{on } z = \delta,$$

$$\sigma_z = \tau_{\rho z} = 0, \quad \theta = g(\rho), \quad \text{on } z = -\delta,$$

$$u_\rho = 0, \quad \partial\theta/\partial\rho = 0, \text{ on } \rho = 1, \quad -\delta \leq z \leq \delta.$$

This is the cylindrical analogue of the situation shown in Fig. 1 (with x replaced by ρ).

The corresponding expressions for the quantities of physical interest are

$$u_\rho = \frac{b}{2(\beta^2 - 1)} \sum_{i=1}^{\infty} \frac{a_i \sinh(\delta + z)\lambda_i + b_i \sinh(\delta - z)\lambda_i}{\lambda_i \sinh(2\lambda_i\delta)} J_1(\lambda_i\rho),$$

$$u_z = \frac{b}{4\beta^2\delta}[a_0(\delta + z)^2 - b_0(\delta - z)^2] +$$

$$+ \frac{b}{2(\beta^2 - 1)} \sum_{i=1}^{\infty} \frac{a_i \cosh(\delta + z)\lambda_i - b_i \cosh(\delta - z)\lambda_i}{\lambda_i \sinh(2\lambda_i\delta)} J_0(\lambda_i\rho),$$

$$\sigma_\phi - \sigma_\rho = \frac{b}{\beta^2 - 1} \sum_{i=1}^{\infty} \frac{a_i \sinh(\delta + z)\lambda_i + b_i \sinh(\delta - z)\lambda_i}{\sinh(2\lambda_i\delta)} J_2(\lambda_i\rho).$$

Similarly, the boundary value problem

$$\sigma_z = \tau_{\rho z} = 0, \quad \theta = f(\rho), \text{ on } z = \delta,$$

$$u_z = 0, \quad \frac{\partial \theta}{\partial z} = 0, \text{ on } z = 0,$$

$$u_\rho = 0 \quad \frac{\partial \theta}{\partial \rho} = 0, \text{ on } \rho = 1,$$

which is the analogue of that shown in Fig. 2 has the solution given by the functions

$$(7.6)\quad \psi(\rho, z) = -\frac{b}{2(\beta^2 - 1)} \sum_{i=1}^{\infty} \frac{a_i \cosh \lambda_i z}{\lambda_i^2 \cosh \lambda_i \delta} J_0(\lambda_i \rho), \quad \chi(z) = \frac{a_0 z}{\beta^2 \delta},$$

where $a_0, a_1, a_2, \cdots$ are defined by the equations (7.4). The quantities of physical interest are then given by the expressions.

$$u_\rho = \frac{b}{2(\beta^2 - 1)} \sum_{i=1}^{\infty} \frac{a_i \cosh \lambda z}{\lambda_i \cosh \lambda_i \delta} J_1(\lambda_i \rho),$$

$$u_z = \frac{a_0 z}{\beta^2 \delta} + \frac{b}{2(\beta^2 - 1)} \sum_{i=1}^{\infty} \frac{a_i \sinh \lambda_i z}{\lambda_i \cosh \lambda_i \delta} J_0(\lambda_i \rho),$$

$$\sigma_\phi - \sigma_\rho = \frac{b}{\beta^2 - 1} \sum_{i=1}^{\infty} \frac{a_i \cosh \lambda_i z}{\cosh \lambda_i \delta} J_2(\lambda_i \rho).$$

The situation is not quite so simple, if we prescribe the flux of heat over the plane faces of the cylinder. If we take

$$\psi(\rho, z) = -\frac{b}{2(\beta^2 - 1)} \sum_{i=1}^{\infty} \frac{a_i \cosh \lambda_i(\delta + z) - b_i \cosh \lambda_i(\delta - z)}{\lambda_i^3 \sinh 2\lambda_i \delta} J_0(\lambda_i \rho)$$

with $\chi(z) = ba_0 z^2 / 2\beta^2$, we find that

$$\frac{\partial \theta}{\partial z} = a_0 + \sum_{i=1}^{\infty} \frac{a_i \sinh \lambda_i(\delta + z) + b_i \sinh \lambda_i(\delta - z)}{\sinh 2\lambda_i \delta} J_0(\lambda_i \rho),$$

so that on $z = \delta$,

$$\frac{\partial \theta}{\partial z} = a_0 + \sum_{i=1}^{\infty} a_i J_0(\lambda_i \rho),$$

while on $z = -\delta$,

$$\frac{\partial \theta}{\partial z} = a_0 + \sum_{i=1}^{\infty} b_i J_0(\lambda_i \rho).$$

This solution is therefore applicable only to problems in which

$$\int_0^1 \rho \left(\frac{\partial \theta}{\partial z}\right)_{z=\delta} d\rho = \int_0^1 \rho \left(\frac{\partial \theta}{\partial z}\right)_{z=-\delta} d\rho.$$

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A PROBLEM OF PLASTICITY THEORY

V. V. SOKOLOVSKII
Moscow, U.S.S.R.

The present paper is devoted to the problem of drawing of a plastic strip between rigid walls. The problem has been solved in simple closed form on the basis of a yield condition of a special type [1].

As usual, the plane plastic equilibrium is determined by the differential equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \tag{1}$$

and by a yield condition of the special type

$$\tfrac{1}{4}(\sigma_x - \sigma_y)^2 + \tau_{xy}^2 = k^2 \sin^2\left(\frac{\sigma_x + \sigma_y}{2k} + \frac{m}{k}\right) \tag{2}$$

for

$$\frac{\pi k}{2} \leq \tfrac{1}{2}(\sigma_x + \sigma_y) + m \leq \pi k.$$

This system of equations is easily transformed by transition from the components σ_x, σ_y and τ_{xy} to new variables ψ and φ by use of the formulae

$$\left.\begin{matrix}\sigma_x \\ \sigma_y\end{matrix}\right\} = k(2\psi \pm \sin 2\psi \cos 2\varphi) - m, \quad \tau_{xy} = k \sin 2\psi \sin 2\varphi. \tag{3}$$

Introducing the expressions (3) into the differential equations (1), we obtain a system of two first order equations for the new variables ψ and φ. The system is hyperbolic and by introduction of the variables

$$\xi = \psi + \varphi, \ \eta = \psi - \varphi \text{ or } 2\psi = \xi + \eta, \ 2\varphi = \xi - \eta \tag{4}$$

it can be reduced to two differential equations of the form

$$\cos\eta \frac{\partial \eta}{\partial x} - \sin\eta \frac{\partial \eta}{\partial y} = 0, \quad \cos\xi \frac{\partial \xi}{\partial x} + \sin\xi \frac{\partial \xi}{\partial y} = 0. \tag{5}$$

The equations of the characteristics

$$x \sin\eta + y \cos\eta = \text{const.}, \quad \eta = \text{const.},$$
$$x \sin\xi - y \cos\xi = \text{const.}, \quad \xi = \text{const.},$$

determine two families of straight lines in the xy-plane. These straight lines are inclined to the Ox axis at angles $\varphi \mp \psi$ and intersect at variable angles 2ψ.

The basic system of equations (5) has the simple integrals

$$x \sin \eta + y \cos \eta = f(\eta), \quad x \sin \xi - y \cos \xi = g(\xi), \tag{6}$$

containing the arbitrary functions $f(\eta)$ and $g(\xi)$.

Besides, there exist particular integrals corresponding to constant ξ and η.

Some attention must be paid now to the particular integrals

$$\xi = \xi_0, \quad x \sin \eta + y \cos \eta = f(\eta) \tag{7}$$

which have a net of characteristics, composed of the non-parallel straight lines $\eta = \text{const}$, and a family of parallel straight lines.

Further, some consideration should be given to the particular integrals

$$\eta = \eta_0, \quad x \sin \xi - y \cos \xi = g(\xi) \tag{8}$$

with a net of characteristics, formed by a family of parallel straight lines and the non-parallel straight lines $\xi = \text{const}$.

Finally, the particular integrals $\xi = \xi_0$, $\eta = \eta_0$ should be noted with a net of characteristics, consisting of two families of parallel straight lines.

It is easy to see that along characteristics $\eta = \text{const.}$, inclined to the Ox axis at angles $\varphi - \psi$, the normal stress components σ_n and the tangential stress component τ have the forms

$$\sigma_n = k(2\psi - \sin 2\psi \cos 2\psi) - m, \quad \tau = k \sin^2 2\psi.$$

Similarly, it is easy to show that along the characteristics $\xi = \text{const.}$ forming with the Ox axis the angles $\varphi + \psi$ the normal stress component σ_n and the tangential stress component τ will be

$$\sigma_n = k(2\psi - \sin 2\psi \cos 2\psi) - m, \quad \tau = -k \sin^2 2\psi.$$

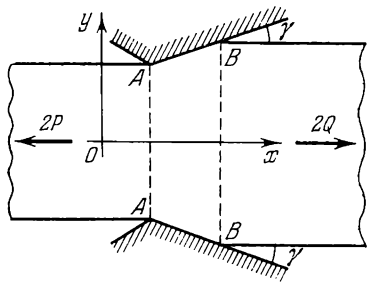

Fig. 1

Let us now study the stress distribution in a plastic strip, while it is being drawn between the rigid smooth walls of a die [2] forming an angle 2γ (Fig. 1).

It is assumed that uniformly distributed normal pressure n is acting along the segment of contact AB, while forward and backward forces $2P$ and $2Q$ are applied to the strip.

Owing to the symmetry of the problem with respect to the Ox axis, one may consider only the upper half-strip $y \geq 0$, taking into account that along the axis the shear stress $\tau_{xy} = 0$ and the angle $\varphi = 0$ (Fig. 2).

The coordinates of the point A will be denoted by x_1, y_1, and the coordinates of the point B by x_2, y_2.

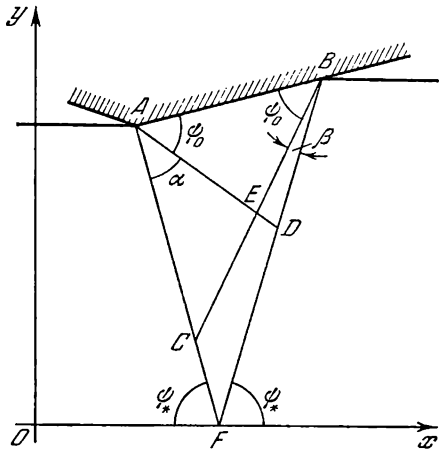

Fig. 2

The horizontal component of the uniform pressure acting along the segment AB must be balanced by the difference $P - Q$. Hence it follows that

$$P - Q = n(y_2 - y_1), \quad P = py_1, \quad Q = qy_2.$$

Now it is not difficult to construct the quantities ξ and η in the regions AEB, CAE, EBD and $CEDF$ using the particular integrals (7) and (8). The quantities ξ and η must satisfy the equations (5) and be continuous on the straight lines AED and BEC.

A relative decrease of the thickness of the strip, i.e., a reduction of the strip r is determined by

$$r = 1 - \frac{y_1}{y_2} = 1 - \frac{\sin(\psi_* - \gamma)}{\sin(\psi_* + \gamma)},$$

while the angles α and β are

$$\alpha = \psi_* - \psi_0 + \gamma, \quad \beta = \psi_* - \psi_0 - \gamma.$$

The above equality can be transformed quite easily into the form

$$(p - q)\tan\psi_* = (2n + p + q)\tan\gamma. \tag{9}$$

In the region AEB, one has for the angles $\psi = \psi_0$, $\varphi = \gamma$, so that the quantities ξ and η are constant,

$$\xi = \psi_0 + \gamma, \quad \eta = \psi_0 - \gamma,$$

and the net of characteristics is formed by two families of parallel straight lines, inclined to the Ox axis at angles $\psi_0 \mp \gamma$. The angle ψ_0 is related to the normal pressure n in the following manner:

$$m - n = k(2\psi_0 - \sin 2\psi_0). \tag{10}$$

In the region CAE, the quantities ξ and η are determined by

$$\xi = \psi_0 + \gamma, \quad \tan\eta = -\frac{y - y_1}{x - x_1},$$

and the net of characteristics consists of a pencil of straight lines passing through the point A and of straight lines parallel to BC.

In the region EBD, the quantities ξ and η are determined by

$$\tan\xi = \frac{y - y_2}{x - x_2}, \quad \eta = \psi_0 - \gamma,$$

and the net of characteristics consists of a pencil of straight lines passing through the point B and of straight lines parallel to AD.

In the region $CEDF$, the quantities ξ and η are given by

$$\tan\xi = \frac{y - y_2}{x - x_2}, \quad \tan\eta = -\frac{y - y_1}{x - x_1},$$

and the net of characteristics is formed by two pencils of straight lines passing through the points A and B.

The horizontal component of the total force applied to ACF must be equal to the forward force P. Hence it follows that

$$P = \int_0^{y_1} [\sigma_n + \tau \operatorname{cotan} \psi_*]dy, \quad P = py_1,$$

or

$$(p + m)y_1 = \int_0^{y_1} [(\sigma_n + m) + \tau \operatorname{cotan} \psi_*]dy.$$

It is obvious that along the segment AF: $\eta = \psi_*$, while the stress components σ_n and τ are expressed by

$$\sigma_n + m = k[(\xi + \psi_*) - \sin(\xi + \psi_*)\cos(\xi + \psi_*)], \quad \tau = k\sin^2(\xi + \psi_*).$$

It is also not difficult to show that on the segment AC: $\xi = \psi_0 + \gamma$, at the point F: $\xi = \psi_*$, while one has along CF the relation

$$y - y_1 = -y_1 \frac{\sin 2\psi_* \sin(\xi - \gamma)}{\sin(\psi_* - \gamma)\sin(\xi + \psi_*)}.$$

After computation of the preceding integral and some transformations we obtain

$$(11) \quad p + m = 2k\left\{\psi_* + \frac{\cos\psi_*}{\sin(\psi_* - \gamma)}[\sin\psi_0 \sin(\psi_0 + \gamma) - (\psi_0 + \gamma - \psi_*)\sin\gamma]\right\}.$$

The horizontal component of the total force applied to BDF must be equal to the backward force Q. Hence it follows that

$$Q = \int_0^{y_2} [\sigma_n - \tau \operatorname{cotan} \psi_*] dy, \quad Q = q y_2,$$

or

$$(q + m) y_2 = \int_0^{y_2} [(\sigma_n + m) - \tau \operatorname{cotan} \psi_*] dy.$$

It is clear that along the segment BF: $\xi = \psi_*$, while the stress components σ_n and τ are given by

$$\sigma_n + m = k[(\psi_* + \eta) - \sin(\psi_* + \eta)\cos(\psi_* + \eta)], \quad \tau = -k \sin^2(\psi_* + \eta).$$

Besides, it is easily seen that on the segment BD: $\eta = \psi_0 - \gamma$, as before, at the point F: $\eta = \psi_*$, while one has along DF the relation

$$y - y_2 = -y_2 \frac{\sin 2\psi_* \sin(\gamma + \eta)}{\sin(\psi_* + \gamma) \sin(\psi_* + \eta)}.$$

Computing the integral in the same manner as before we find

$$(12) \quad q + m = 2k \left\{ \psi_* + \frac{\cos \psi_*}{\sin(\psi_* + \gamma)} [\sin \psi_0 \sin(\psi_0 - \gamma) + (\psi_0 - \gamma - \psi_*) \sin \gamma] \right\}.$$

One can easily see that (11) is a consequence of (9) and (12), while (12) follows from (9) and (11). Therefore one has for the determination of n, ψ_0 and p or q the equations (9), (10) and (11) or (12).

It should be noted that the above solution is valid as long as the angles satisfy the conditions

$$\beta \geq 0, \quad \psi_0 \leq \psi_* - \gamma.$$

The particular case when the side BEC coincides with the side BDF and the region $CEDF$ reduces to the segment DF is given by

$$\beta = 0, \quad \psi_0 = \psi_* - \gamma.$$

Figs 3 and 4 show the graphs of the changes of the quantities $\bar{p} = (p + m)/k$ and $\bar{q} = (q + m)/k$ as functions of r for the angle $\gamma = 10°$ and for various values of the angle ψ_0 from 45° to 80° at steps of 5°. The dotted curves refer to $\beta = 0$ or $\psi_0 = \psi_* - \gamma$ and they connect the points

$$r = 0, \quad \bar{p} = \bar{q} = \pi$$

with the points

$$r = 0.22, \quad \bar{p} = 2.86, \quad \bar{q} = 2.36.$$

In conclusion, it should be noted that the problem considered in this paper becomes for $p = q = 0$ and $\gamma = 0$ the problem of the equilibrium

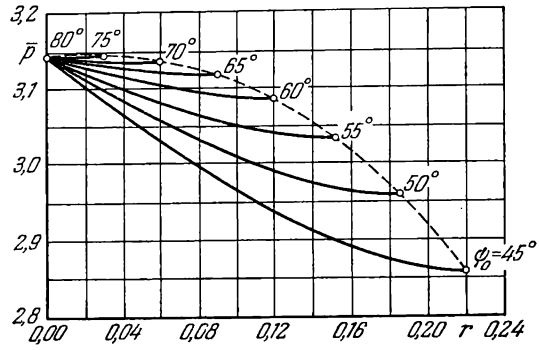

Fig. 3

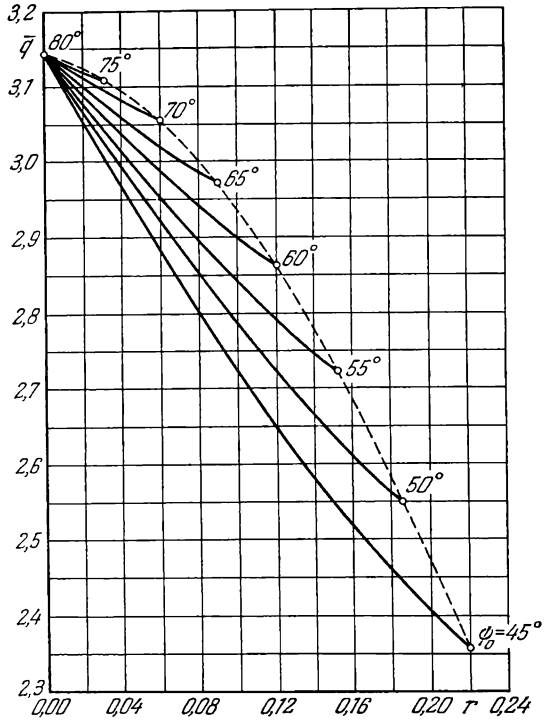

Fig. 4

of a plastic strip subject to normal pressure uniformly distributed along opposing segments, which has been studied before.

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ELASTIC WAVES ARISING FROM NORMAL STRESSES APPLIED TO THE SURFACE OF A HALF-SPACE

L. N. SRETENSKII
Moscow, U.S.S.R.

§ 1. We will consider an elastic half-space and assume that to the points of the rectangle $|x| < a$, $|y| < b$ of its boundary – the XOY plane – normal stresses are applied which change with time in accordance with the law

$$Z_z = P\, e^{i\omega t} \quad \text{for } |x| < a,\ |y| < b,$$

$$Z_z = 0 \qquad \text{for all other values of } x, y.$$

Here ω is the frequency of oscillation of the stresses, P is a constant quantity. In addition, we assume that the tangential stresses are zero at all points of the plane $z = 0$. It will be our objective to determine for these conditions the displacements of the points of the plane $z = 0$.

The elastic displacements u, v, w may be represented in the form

$$u = u' - \frac{\partial \varphi}{\partial x},\ v = v' - \frac{\partial \varphi}{\partial y},\ w = w' - \frac{\partial \varphi}{\partial z},$$

where we have for the determination of u', v', w' – the displacements due to the passage of transverse waves – and of the potential φ of the longitudinal displacements the equations

$$h^2 \frac{\partial^2 \varphi}{\partial t^2} = \Delta \varphi,$$

$$\kappa^2 \frac{\partial^2 u'}{\partial t^2} = \Delta u',\ \kappa^2 \frac{\partial^2 v'}{\partial t^2} = \Delta v',\ \kappa^2 \frac{\partial^2 w'}{\partial t^2} = \Delta w',$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0.$$

The quantities κ and h are expressed in terms of the Lamé constants λ and μ and of the density ρ by the formulae

$$\kappa^2 = \frac{\rho}{\mu},\ h^2 = \frac{\rho}{\lambda + 2\mu}.$$

We will find first for the solution of the problem posed the displacements accompanying normal stresses of one particular form.

If the surface of the elastic half-space is subject to loading determined for all values x, y by the formulae

$$X_z = 0,\ Y_z = 0,\ Z_z = P \cos mx \cos ny\, e^{i\omega t},$$

the corresponding displacements will be given by the formulae

$$u' = \frac{2s\gamma}{\mu} \cdot \frac{mP}{S} \sin mx \cos ny\, e^{sz+i\omega t},$$

$$v' = \frac{2s\gamma}{\mu} \cdot \frac{nP}{S} \cos mx \sin ny\, e^{sz+i\omega t},$$

$$w' = -\frac{2(m^2+n^2)}{\mu} \cdot \frac{\gamma P}{S} \cos mx \cos ny\, e^{sz+i\omega t},$$

$$\varphi = -\frac{s^2+m^2+n^2}{\mu} \frac{P}{S} \cos mx \cos ny\, e^{\gamma z+i\omega t}.$$

The following notation has been used here:

$$s^2 = m^2 + n^2 - \kappa^2\omega^2,\ \gamma^2 = m^2 + n^2 - h^2\omega^2,$$

$$S = (2s^2 + \kappa^2\omega^2)^2 - 4s\gamma(s^2 + \kappa^2\omega^2).$$

Accepting these formulae, we will now turn to our problem. We will assume that we have the function $f(x, y)$, determined by the conditions

$$f(x, y) = P \quad \text{for } |x| < a,\ |y| < b,$$

$$f(x, y) = 0 \quad \text{for all other values of } x, y.$$

We may represent this function in the form of the double integral

$$f(x, y) = \frac{P}{\pi^2} \int_{-\infty}^{\infty}\int \frac{\sin ma}{m} \cdot \frac{\sin nb}{n} \cos mx \cos ny\, dm\, dn,$$

whence it follows that the function φ and the displacements u', v', w' will be determined in our problem by the formulae

$$(1.1) \quad \varphi = -\frac{P\, e^{i\omega t}}{\pi^2\mu} \int_{-\infty}^{\infty}\int \frac{s^2+m^2+n^2}{S} \frac{\sin ma}{m} \frac{\sin nb}{n} \cos mx \cos ny\, e^{\gamma z}\, dm\, dn,$$

$$(1.2) \quad u' = \frac{2P\, e^{i\omega t}}{\pi^2\mu} \int_{-\infty}^{\infty}\int \frac{s\gamma}{S} \sin ma \frac{\sin nb}{n} \sin mx \cos ny\, e^{sz}\, dm\, dn,$$

$$(1.3)\quad v' = \frac{2P\,e^{i\omega t}}{\pi^2\mu}\int_{-\infty}^{\infty}\int \frac{s\gamma}{S}\,\frac{\sin ma}{m}\sin nb\cos mx\sin ny\,e^{sz}\,dm\,dn,$$

$$(1.4)\quad w' = -\frac{2P\,e^{i\omega t}}{\pi^2\mu}\int_{-\infty}^{\infty}\int \frac{\gamma(m^2+n^2)}{S}\,\frac{\sin ma}{m}\,\frac{\sin nb}{n}\cos mx\cos ny\,e^{sz}\,dm\,dn.$$

We note that the complex numbers s and γ must have positive real parts, because in the elastic body under consideration $z < 0$.

§ 2. For the study of the formulae obtained we must find the zeros of the function S. If we set in the formula determined by this function

$$s^2 = \kappa^2\omega^2\xi,$$

we will find a new expression for the function, written with the help of a variable ξ

$$S = \kappa^4\omega^4\left[(2\xi+1)^2 - 4(\xi+1)\sqrt{\xi}\sqrt{\xi+\frac{\kappa^2-h^2}{\kappa^2}}\right].$$

We will denote the expression in square brackets by $E(\xi)$. Since the real parts of the quantities s and γ are positive, the equation $E(\xi) = 0$ is satisfied by only one number ξ' which for the hypothesis due to Poisson, i.e., $\lambda = \mu$, is approximately equal to 0.183. This number is related to the velocity of propagation of Rayleigh waves V by the formula

$$V = \frac{1}{\kappa\sqrt{\xi'+1}}.$$

The value m' of the variable m, corresponding to the number ξ', will be

$$m' = \sqrt{\alpha^2 - n^2},$$

where α denotes the quantity $\kappa\omega\sqrt{1+\xi'}$. For values $|n| > \alpha$ the number m' will be purely imaginary and we will take the coefficient of $\sqrt{-1}$ in this number to be negative.

§ 3. We now turn our attention on the formulae of § 1 and consider first the magnitudes of the displacements u', v', w'; we will study these displacements for $z = 0$.

The double integral of (1.2) contains the integral

$$(3.1)\qquad J = \int_{-\infty}^{\infty}\frac{s\gamma}{S}\sin ma\sin mx\,dm.$$

This integral may be represented in the form

$$J = \tfrac{1}{4}\int_{-\infty}^{\infty} \frac{s\gamma}{S} e^{(x-a)mi}\, dm + \tfrac{1}{4}\int_{-\infty}^{\infty} \frac{s\gamma}{S} e^{-(x-a)mi}\, dm - \tag{3.2}$$

$$- \tfrac{1}{4}\int_{-\infty}^{\infty} \frac{s\gamma}{S} e^{(x+a)mi} - \tfrac{1}{4}\int_{-\infty}^{\infty} \frac{s\gamma}{S} e^{-(x+a)mi}\, dm.$$

A study of the integrals in (3.2) may be reduced to the study of the integral

$$J = \int_{-\infty}^{\infty} \frac{s\gamma}{S} e^{\sigma mi}\, dm, \tag{3.3}$$

in which σ is a real, positive or negative number. The multi-valued function $s\gamma$ of the variable m

$$s\gamma = \sqrt{m^2 - (\kappa^2\omega^2 - n^2)}\ \sqrt{m^2 - (h^2\omega^2 - n^2)}$$

enters into the expression for the integral J.

For large positive values of m each of these roots is to be taken with a plus sign.

The multi-valued function $s\gamma$ under consideration has the four branch points

$$m_1 = \sqrt{\kappa^2\omega^2 - n^2},\ -m_1,\ m_2 = \sqrt{h^2\omega^2 - n^2},\ -m_2.$$

The position of these points depends on the magnitude of the number n. We will introduce in the plane of the complex variable m cuts which join these points and make the function under consideration single-valued.

We assume first that the following inequality applies:

$$-h\omega < n < h\omega. \tag{3.4}$$

We will join by the cut l_1 the points $-m_1$, $-m_2$, lying on the real axis and by the cut l_2 the points m_2, m_1, lying on the real axis to the right of the point O. The integrand in (3.3) will be single-valued in the entire plane of the complex variable m with the cuts l_1 and l_2.

The dotted line in Fig. 1 shows the initial path of integration.

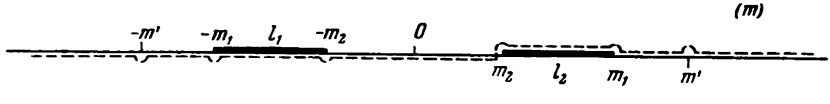

Fig. 1.

It must be noted that the points $-m'$ and m', arranged in this case in order to fulfill the inequality (3.4) on the real axis, must be by-passed along semi-circles, the point $-m'$ along a semi-circle in the lower half-plane and the point m' along a semi-circle in the upper half-plane. For such a passage around the Rayleigh points the additional condition for the study of elastic waves at infinity will be observed and, thereby, waves will be absent which might originate from infinity. If the number σ is larger than zero, the stated path of integration may be transformed into a path of integration embracing the cut l_1; if $\sigma < 0$, the original path may be transformed into a path containing the cut l_2; it is necessary for these transformations to take into consideration the poles $-m'$ and m' and their residues.

Completing the calculations, we obtain for $\sigma > 0$

$$J = i \int_{m_2}^{m_1} M(m)\, e^{-\sigma m i}\, dm - \frac{2\pi i}{m'} G\, e^{-\sigma m' i},$$

and for $\sigma < 0$

$$J = i \int_{m_2}^{m_1} M(m)\, e^{\sigma m i}\, dm - \frac{2\pi i}{m'} G\, e^{\sigma m' i},$$

where

$$M(m) = \frac{2(2s^2 + \kappa^2\omega^2)^2 \sqrt{m_1^2 - m^2}\, \sqrt{m^2 - m_2^2}}{(2s^2 + \kappa^2\omega^2)^4 + 16(m_1^2 - m^2)(m^2 - m_2^2)(s^2 + \kappa^2\omega^2)^2},$$

$$G = \frac{\sqrt{\xi'}\sqrt{\xi' + \dfrac{\kappa^2 - h^2}{\kappa^2}}}{2\left(\dfrac{dE}{d\xi}\right)_{\xi=\xi'}}.$$

On the basis of these calculations, the formula (3.2) assumes the form

$$(3.5)\quad J = \frac{i}{2} \int_{m_2}^{m_1} M(m)\, [e^{-(x-a)mi} - e^{-(x+a)mi}]\, dm - \\ - \frac{\pi i}{m'} G\, [e^{-(x-a)m'i} - e^{-(x+a)m'i}].$$

It should be noted that for this definition of the multi-valued function $s\gamma$ the function S vanishes only at the point m'; the remaining two roots of the equation $S = 0$ belong to other branches of the function under consideration.

§ 4. We will now assume that the variable n satisfies the inequality

$$h\omega < |n| < \kappa\omega. \tag{4.1}$$

In this case the numbers m_2 and $-m_2$ will be purely imaginary and the numbers m_1 and $-m_1$ remain real. We will understand by m_2 a purely imaginary number with a negative coefficient of $\sqrt{-1}$.

We will cut the plane of the complex variable m along the two segments l_1 and l_2: the segment l_1 connects the points $-m_1$ and $-m_2$ and the segment l_2 the points m_1 and m_2 (Fig. 2).

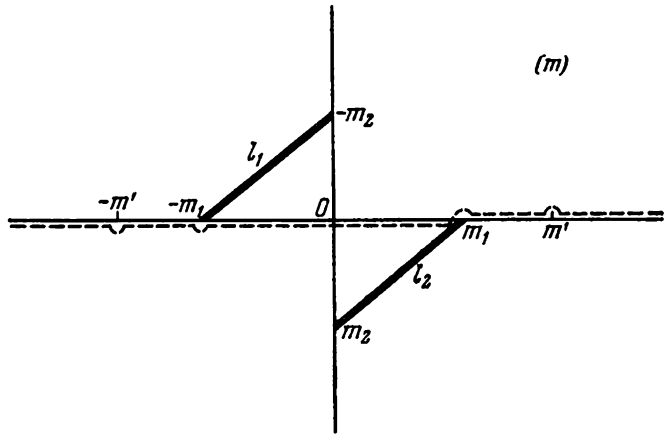

Fig. 2.

Completing the same transformations as in the preceding section we find for J the expression

$$J = \frac{i}{2}\int_{m_2}^{m_1} M(m)\,[e^{-(x-a)mi} - e^{-(x+a)mi}]\,dm - \frac{\pi i}{m'}\,G\,[e^{-(x-a)m'i} - e^{-(x+a)m'i}]. \tag{4.2}$$

This expression in its above form coincides with the expression (3.5), but differs essentially from it in that the integral for the variable m is taken along a straight path in the plane of the complex variable and, owing this to the integrand involves imaginary symbols.

§ 5. We will now assume that the following inequality is observed:

$$\kappa\omega < |n| < \kappa\omega\sqrt{\xi' + 1}. \tag{5.1}$$

When these inequalities are fulfilled, the numbers m_1 and m_2 will be purely imaginary and the cuts in the plane of the complex variable m, connecting the branch points $-m_1$, $-m_2$ and m_1, m_2, will lie along the

imaginary axis (Fig. 3). We note that we assume the quantities $\mu_1 = -m_1/i$, $\mu_2 = -m_2/i$ to be positive. In the presence of the inequalities (5.1) the Rayleigh points $-m'$ and m' lie on the real axis; as above, these points are by-passed along semi-circles.

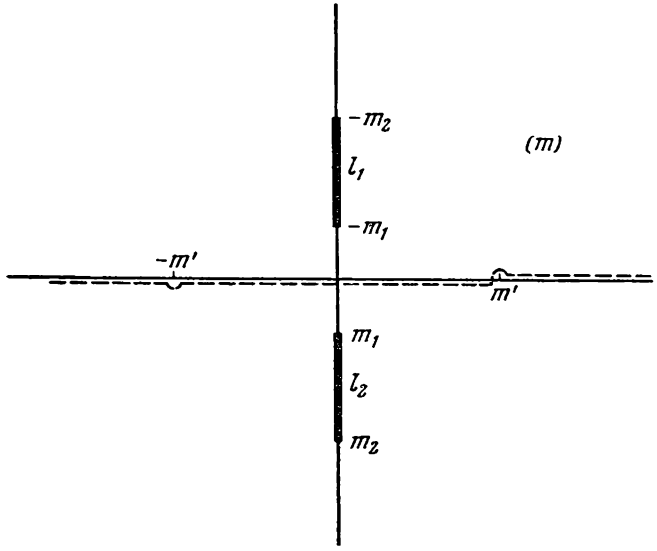

Fig. 3.

After a number of transformations the integral (3.2) will be written in this case in the form

$$J = -\tfrac{1}{2}\int_{\mu_1}^{\mu_2} N(\mu)\,[e^{-(x-a)\mu} - e^{-(x+a)\mu}]\,d\mu - \qquad (5.2)$$

$$- \frac{\pi i}{m'} G\,[e^{-(x-a)m'i} - e^{-(x+a)m'i}],$$

where

$$\mu_1 = \sqrt{n^2 - \kappa^2\omega^2},\ \mu_2 = \sqrt{n^2 - h^2\omega^2},$$

$$s^2 = n^2 - \kappa^2\omega^2 - \mu^2,\ \gamma^2 = n^2 - h^2\omega^2 - \mu^2,$$

$$N(\mu) = \frac{2(2s^2 + \kappa^2\omega^2)^2\sqrt{\mu^2 - \mu_1^2}\,\sqrt{\mu_2^2 - \mu^2}}{(2s^2 + \kappa^2\omega^2)^4 + 16(\mu^2 - \mu_1^2)(\mu_2^2 - \mu^2)(s^2 + \kappa^2\omega^2)^2}.$$

§ 6. We must now consider the last case: $|n| > \alpha$. In this case the number $m' = \sqrt{\alpha^2 - n^2}$ will be purely imaginary and, consequently, we will have no singular points on the path of integration of the integral (3.2). Hence there is no necessity to transform the path of integration and the expression of the integral J remains unchanged.

§ 7. We will now transform the expression (1.2) on the basis of the formulae for the integral J deduced in the preceding sections.

We find

$$(7.1)\quad u' : \frac{2P\, e^{i\omega t}}{\pi^2 \mu} = D_1 + D_2 + D_3 + D_4 + \\ + 2\pi i G \int_0^{\alpha} \frac{\sin nb}{n} \cos ny \left[e^{-(x+a)m'i} - e^{-(x-a)m'i}\right] \frac{dn}{m'},$$

where D_1, D_2, D_3, D_4 are double integrals of the form

$$D_1 = i \int_0^{h\omega} \frac{\sin nb}{n} \cos ny\, dn \int_{m_2}^{m_1} M(m) \left[e^{-(x-a)mi} - e^{-(x+a)mi}\right] dm,$$

$$D_2 = i \int_{h\omega}^{\kappa\omega} \frac{\sin nb}{n} \cos ny\, dn \int_{m_2}^{m_1} M(m) \left[e^{-(x-a)mi} - e^{-(x+a)mi}\right] dm,$$

$$D_3 = \int_{\kappa\omega}^{\alpha} \frac{\sin nb}{n} \cos ny\, dn \int_{\mu_1}^{\mu_2} N(\mu) \left[e^{-(x-a)\mu} - e^{-(x+a)\mu}\right] d\mu,$$

$$D_4 = 2 \int_{\alpha}^{\infty} \frac{\sin nb}{n} \cos ny\, dn \int_{-\infty}^{\infty} \frac{s\gamma}{S} \sin ma \sin mx\, dm.$$

§ 8. We now turn to the transformation of the formulae (1.3). We consider for this purpose first the integral with the variable m

$$J' = \int_{-\infty}^{\infty} \frac{s\gamma}{S} \frac{\sin ma}{m} \cos mx\, dm.$$

This integral may be written in the form

$$(8.1)\quad J' = \frac{1}{4i} \int_{-\infty}^{\infty} \frac{s\gamma}{S} e^{im(x+a)} \frac{dm}{m} - \frac{1}{4i} \int_{-\infty}^{\infty} \frac{s\gamma}{S} e^{-im(x+a)} \frac{dm}{m} - \\ - \frac{1}{4i} \int_{-\infty}^{\infty} \frac{s\gamma}{S} e^{im(x-a)} \frac{dm}{m} + \frac{1}{4i} \int_{-\infty}^{\infty} \frac{s\gamma}{S} e^{-im(x-a)} \frac{dm}{m}.$$

We will take here as integration path the same path as in the preceding sections, however, with one supplement: the point $m = 0$ is to be bypassed along a semi-circle from below.

We consider the integral

$$J' = \int_{-\infty}^{\infty} \frac{s\gamma}{S} e^{i\sigma m} \frac{dm}{m}.$$

We may find for this integral the following expression. If $|n| < h\omega$, then

for $\sigma > 0$

$$J' = i \int_{m_2}^{m_1} M(m)\, e^{-\sigma m i} \frac{dm}{m} + \frac{2\pi i}{m'^2} G\, e^{-\sigma m' i} - \\ - \frac{2\pi i \sqrt{\kappa^2\omega^2 - n^2}\, \sqrt{h^2\omega^2 - n^2}}{(2n^2 - \kappa^2\omega^2)^2 + 4n^2 \sqrt{\kappa^2\omega^2 - n^2}\, \sqrt{h^2\omega^2 - n^2}},$$

for $\sigma < 0$

$$J' = -i \int_{m_2}^{m_1} M(m)\, e^{\sigma m i} \frac{dm}{m} - \frac{2\pi i}{m'^2} G\, e^{\sigma m' i}.$$

If $h\omega < |n| < \kappa\omega$, then

for $\sigma > 0$

$$J' = i \int_{m_2}^{m_1} M(m)\, e^{-\sigma m i} \frac{dm}{m} + \frac{2\pi i}{m'^2} G\, e^{-\sigma m' i} - \\ - \frac{2\pi \sqrt{\kappa^2\omega^2 - n^2}\, \sqrt{n^2 - h^2\omega^2}}{(2n^2 - \kappa^2\omega^2)^2 - 4in^2 \sqrt{\kappa^2\omega^2 - n^2}\, \sqrt{n^2 - h^2\omega^2}},$$

for $\sigma < 0$

$$J' = i \int_{m_2}^{m_1} M(m)\, e^{\sigma m i} \frac{dm}{m} - \frac{2\pi i}{m'^2} G\, e^{\sigma m' i}.$$

If $\kappa\omega < |n| < \alpha$, then

for $\sigma > 0$

$$J' = i \int_{\mu_1}^{\mu_2} N(\mu)\, e^{-\sigma\mu} \frac{d\mu}{\mu} + \frac{2\pi i}{m'^2} G\, e^{-\sigma m' i} + \\ + \frac{2\pi i \sqrt{n^2 - \kappa^2\omega^2}\, \sqrt{n^2 - h^2\omega^2}}{(2n^2 - \kappa^2\omega^2)^2 - 4n^2 \sqrt{n^2 - \kappa^2\omega^2}\, \sqrt{n^2 - h^2\omega^2}},$$

for $\sigma < 0$

$$J' = i\int_{\mu_1}^{\mu_2} N(\mu)\, e^{\sigma\mu}\,\frac{d\mu}{\mu} - \frac{2\pi i}{m'^2}\, G\, e^{\sigma m' i}.$$

For values $|n| > \alpha$, there is no need to transform the integral.

Having established these formulae, we will apply them to the equality (8.1); we find for all values of $|n|$ from 0 to α

$$J' = \frac{\pi G}{m'^2}\,[e^{-(x+a)m'i} - e^{-(x-a)m'i}] + \text{an integral.}$$

We will now find the expression for the displacement v'. Using the formula (1.3), we obtain for $z = 0$

$$(8.2)\quad v' : \frac{2P\, e^{i\omega t}}{\pi^2\mu} = 2\pi G\int_0^{\alpha} \sin nb \sin ny\,[e^{-(x+a)m'i} - e^{-(x-a)m'i}]\,\frac{dn}{m'^2} +$$

$$+ \text{double integrals.}$$

§ 9. We turn now to the formula (1.4) for the displacement w' and transform it to a new form by contour integration.

Modifying somewhat the calculations of the preceding section, we find for w' the expression

$$(9.1)\quad w' : \frac{2P\, e^{i\omega t}}{\pi^2\mu} =$$

$$= -\frac{2\pi\kappa\omega(1+\xi')}{\sqrt{\xi'}}\, G\int_0^{\alpha} \frac{\sin nb}{n} \cos ny\,[e^{-(x+a)m'i} - e^{-(x-a)m'i}]\,\frac{dn}{m'^2} +$$

$$+ \text{double integrals.}$$

The modifications stated concern the path of integration. We have for the study of the displacements u' and v' one non-single-valued function, formed by the product $s\gamma$; for the study of w' we have besides this function still the function γ with the branch points $-m_2$, m_2. Thanks to this we have to consider simultaneously with the integrals along the cuts l_1 and l_2 also integrals taken along the vertical straight lines from the point $-m_2$ to the point $-m_2 + \infty i$ and from the point m_2 to the point $m_2 - \infty i$. These integrals will enter in the final result into the double integrals of the formula (9.1).

§ 10. Finally, we turn our attention to the transformation of the formula (1.1), determining the potential φ. For this transformation one

may use all the methods of the preceding sections and we obtain for the integral

$$J'' = \int_{-\infty}^{\infty} \frac{2m^2 + 2n^2 - \kappa^2\omega^2}{S} \frac{\sin ma}{m} \cos mx \, dm$$

for $|n| < \alpha$

$$J'' = \pi H \left[e^{-im'(x+a)} - e^{-im'(x-a)}\right] \frac{1}{m'^2} + \text{an integral},$$

where

$$H = \frac{2\xi' + 1}{2\left(\frac{dE}{d\xi}\right)_{\xi=\xi'}}.$$

The formula (1.1) may now be rewritten

$$\varphi : \frac{P e^{i\omega t}}{\pi^2\mu} = -\pi H \int_0^{\alpha} \frac{\sin nb}{n} \cos ny \left[e^{-im'(x+a)} - e^{-im'(x-a)}\right] \frac{dn}{m'^2} +$$

$$+ \text{double integrals.}$$

Hence we find for $z = 0$

$$\frac{\partial\varphi}{\partial x} : \frac{P e^{i\omega t}}{\pi^2\mu} =$$

$$= \pi i H \int_0^{\alpha} \frac{\sin nb}{n} \cos ny \left[e^{-im'(x+a)} - e^{-im'(x-a)}\right] \frac{dn}{m'} + \cdots,$$

(10.1)

$$\frac{\partial\varphi}{\partial y} : \frac{P e^{i\omega t}}{\pi^2\mu} =$$

$$= \pi H \int_0^{\alpha} \sin nb \sin ny \left[e^{-im'(x+a)} - e^{-im'(x-a)}\right] \frac{dn}{m'^2} + \cdots.$$

With the help of the formulae (1.1) we find further for $z = 0$

$$\frac{\partial\varphi}{\partial z} : \frac{P e^{i\omega t}}{\pi^2\mu} =$$

$$= -\int_{-\infty}^{\infty}\int \frac{\gamma(2m^2 + 2n^2 - \kappa^2\omega^2)}{S} \frac{\sin ma}{m} \frac{\sin nb}{n} \cos mx \cos ny \, dm \, dn.$$

Considering first in this integral the integration with respect to m and

using for this the transformations of § 9 for the contour integrals, we find for the partial derivative of φ with respect to z for $z = 0$ the expression

$$(10.2)\quad \frac{\partial \varphi}{\partial z} : \frac{P e^{i\omega t}}{\pi^2 \mu} = -\pi H \kappa \omega \sqrt{\xi' + \frac{\kappa^2 - h^2}{\kappa^2}} \cdot$$

$$\cdot \int_0^{\alpha} \frac{\sin nb}{n} \cos ny \left[e^{-im'(x+a)} - e^{-im'(x-a)}\right] \frac{dn}{m'^2} + \cdots .$$

The double integrals on the right hand sides of the formulae (10.1) and (10.2) have not been written down explicitly.

§ 11. The given transformations of the formulae (1.1)–(1.4) permit us to establish certain asymptotic formulae for the evaluation of the elastic displacements in regions which are sufficiently far away from the region of application of the pressure.

We will assume initially that, leaving y unchanged, we move away from this region increasing x beyond all limits. We find from the formulae (7.1), (8.2), (9.1), (10.1) and (10.2) the asymptotic values of the elastic displacements. In the last section we shall obtain values of the double integrals entering into the above formulae and we shall show that for large values of x these integrals will be significantly less than the simple integrals on the right hand sides of the formulae under consideration. We will introduce the following notation for these simple integrals:

$$T_1 = \int_0^{\infty} \frac{\sin nb}{n} \cos ny \left[e^{-(x+a)m'i} - e^{-(x-a)m'i}\right] \frac{dn}{m'^2},$$

$$T_2 = \int_0^{\infty} \sin nb \sin ny \left[e^{-(x+a)m'i} - e^{-(x-a)m'i}\right] \frac{dn}{m'^2},$$

$$T_3 = \int_0^{\infty} \frac{\sin nb}{n} \cos ny \left[e^{-(x+a)m'i} - e^{-(x-a)m'i}\right] \frac{dn}{m'^2}.$$

The study of these three integrals can be effected by the same method. First, we shall introduce instead of n the new variable ν by the formula

$$\nu^2 = \alpha^2 - n^2, \quad dn = -\frac{\nu\, d\nu}{\sqrt{\alpha^2 - \nu^2}}.$$

Hence we obtain for our integrals

$$T_1 = \int_0^\alpha \frac{\sin(\sqrt{\alpha^2-\nu^2}\,b)}{\alpha^2-\nu^2}\cos(\sqrt{\alpha^2-\nu^2}\,y)\,[e^{-(x+a)\nu i} - e^{-(x-a)\nu i}]\,d\nu,$$

$$T_2 = \int_0^\alpha \frac{\sin(\sqrt{\alpha^2-\nu^2}\,b)}{\sqrt{\alpha^2-\nu^2}}\sin(\sqrt{\alpha^2-\nu^2}\,y)\,[e^{-(x+a)\nu i} - e^{-(x-a)\nu i}]\,\frac{d\nu}{\nu},$$

$$T_3 = \int_0^\alpha \frac{\sin(\sqrt{\alpha^2-\nu^2}\,b)}{\alpha^2-\nu^2}\cos(\sqrt{\alpha^2-\nu^2}\,y)\,[e^{-(x+a)\nu i} - e^{-(x-a)\nu i}]\,\frac{d\nu}{\nu}.$$

We will transform these integrals into double integrals by use of the formula

$$e^{-(x+a)\nu i} - e^{-(x-a)\nu i} = \frac{\nu}{i}\int_{x-a}^{x+a} e^{-i\nu\xi}\,d\xi,$$

as the result of which we find the expressions

$$(11.1)\quad T_1 = -i\int_{x-a}^{x+a} d\xi \int_0^\alpha \frac{\sin(\sqrt{\alpha^2-\nu^2}\,b)}{\alpha^2-\nu^2}\cos(\sqrt{\alpha^2-\nu^2}\,y)\nu\, e^{-i\nu\xi}\,d\nu,$$

$$(11.2)\quad T_2 = -i\int_{x-a}^{x+a} d\xi \int_0^\alpha \frac{\sin(\sqrt{\alpha^2-\nu^2}\,b)}{\sqrt{\alpha^2-\nu^2}}\sin(\sqrt{\alpha^2-\nu^2}\,y)\, e^{-i\nu\xi}\,d\nu,$$

$$(11.3)\quad T_3 = -i\int_{x-a}^{x+a} d\xi \int_0^\alpha \frac{\sin(\sqrt{\alpha^2-\nu^2}\,b)}{\alpha^2-\nu^2}\cos(\sqrt{\alpha^2-\nu^2}\,y)\, e^{-i\nu\xi}\,d\nu.$$

Since the magnitude of x is large, one may take for the inner integrals asymptotic formulae for large ξ. We have

$$\int_0^\alpha \frac{\sin(\sqrt{\alpha^2-\nu^2}\,b)}{\alpha^2-\nu^2}\cos(\sqrt{\alpha^2-\nu^2}\,y)\nu\, e^{-i\nu\xi}\,d\nu = \frac{b\sqrt{\pi\alpha}}{\sqrt{2}}\,\frac{e^{-i(\alpha\xi-\pi/4)}}{\sqrt{\xi}} + \cdots,$$

$$\int_0^\alpha \frac{\sin(\sqrt{\alpha^2-\nu^2}\,b)}{\sqrt{\alpha^2-\nu^2}}\sin(\sqrt{\alpha^2-\nu^2}\,y)\, e^{-i\nu\xi}\,d\nu = -\frac{i}{\alpha\xi}\sin\alpha b\sin\alpha y + \cdots,$$

$$\int_0^\alpha \frac{\sin(\sqrt{\alpha^2-\nu^2}\,b)}{\alpha^2-\nu^2}\cos(\sqrt{\alpha^2-\nu^2}\,y)\, e^{-i\nu\xi}\,d\nu = \frac{b\sqrt{\pi}}{\sqrt{2\alpha}}\cdot\frac{e^{-i(\alpha\xi-\pi/4)}}{\sqrt{\xi}} + \cdots.$$

Hence

$$T_1 = b\sqrt{\frac{\pi\alpha}{2}}\, e^{-\frac{1}{4}\pi i}\int_{x-a}^{x+a}\frac{e^{-i\alpha\xi}}{\sqrt{\xi}}\,d\xi + \cdots = b\sqrt{\frac{2\pi}{\alpha}}\sin a\alpha\,\frac{e^{-i(\alpha x+\pi/4)}}{\sqrt{x}} + \cdots,$$

$$T_2 = -\sin\alpha b\sin\alpha y\int_{x-a}^{x+a}\frac{d\xi}{\alpha\xi} + \cdots = -\frac{2a\sin\alpha b\sin\alpha y}{\alpha x} + \cdots,$$

$$T_3 = b\sqrt{\frac{\pi}{2\alpha}}\, e^{-\frac{1}{4}\pi i}\int_{x-a}^{x+a}\frac{e^{-i\alpha\xi}}{\sqrt{\xi}}\,d\xi + \cdots = b\sqrt{\frac{2\pi}{\alpha^3}}\sin a\alpha\,\frac{e^{-i(\alpha x+\pi/4)}}{\sqrt{x}} + \cdots.$$

§ 12. Having obtained these formulae we will now construct the complete values of the elastic displacements, limiting ourselves, however, to the consideration of terms of order $x^{-\frac{1}{2}}$ only.

We have from (7.1) and (9.2)

$$(12.1)\qquad u = \frac{\xi'(2\xi'+1)}{(\xi'+1)^{5/4}E'(\xi')}\,\frac{Pb}{\mu}\,\frac{\sin a\alpha}{\sqrt{2\pi\kappa\omega}}\,\frac{e^{-i(\alpha x-\omega t-\pi/4)}}{\sqrt{x}} + \cdots.$$

From (8.2) and (10.1) we find to an accuracy of $x^{-\frac{1}{2}}$ that $v = 0$. From (9.1) and (10.2) we obtain

$$(12.2)\qquad w = -\frac{(2\xi'+3)\sqrt{\xi' + \dfrac{\kappa^2 - h^2}{\kappa^2}}}{(\xi'+1)^{3/4}E'(\xi')}\,\frac{Pb}{\mu}\cdot$$

$$\cdot\,\frac{\sin a\alpha}{\sqrt{2\pi\kappa\omega}}\,\frac{e^{-i(\alpha x-\omega t+\pi/4)}}{\sqrt{x}} + \cdots.$$

Thus, in a region bounded by two straight lines parallel to the OX axis (for the sake of definiteness we may choose as these straight lines the lines $y = b$, $y = -b$) waves will propagate (for an unlimited increase of x) with a velocity which is equal to the velocity of Rayleigh waves

$$\frac{\omega}{\alpha} = \frac{1}{\kappa\sqrt{\xi'+1}} = V.$$

In their motion these displacement-waves will be damped inversely proportionally to $x^{\frac{1}{2}}$. The displacements u and w may vanish completely for definite dimensions of the region of pressure. This will be the case when

$$a\kappa\omega\sqrt{\xi'+1} = \pi N,$$

where N is any integer. These waves are also propagated in the direction of the negative OX axis between the straight lines $y = \pm b$.

We may likewise consider the propagation of waves between the straight lines $x = \pm a$, giving $|y|$ large values. For such waves u will be zero and the values of v and w are found from the formulae (12.1) and (12.2) with the corresponding replacement of quantities.

It may be seen from the formulae (12.1) and (12.2) that the end of the displacement vector of a given point describes an ellipse about a mean position of the point; the axes of this ellipse are parallel to the coordinate axes OX and OY, their lengths increase inversely proportionally to $x^{\frac{1}{2}}$.

§ 13. We will now consider the displacements u, v, w when the place of observation is at a large distance from the region of application of pressure in a direction which makes an arbitrary angle with the sides of the rectangle.

For the completion of this study we must consider the integrals T_1, T_2, T_3 of § 11.

We will take the integral T_1 in the form (11.1). We let

$$x = R\cos\theta,\quad y = R\sin\theta$$

and introduce for ξ the new integration variable σ by use of the formula

$$\xi = a\sigma + R\cos\theta,\quad -1 < \sigma < 1.$$

After a minor transformation we obtain

$$(13.1)\quad T_1 = -\tfrac{1}{2}ai\int_{-1}^{1} d\sigma \int_{0}^{\alpha} \nu\, e^{-ia\sigma\nu}\,\frac{\sin\left(b\sqrt{\alpha^2-\nu^2}\right)}{\alpha^2-\nu^2}\, e^{iRN_1(\nu)}\,d\nu -$$

$$-\tfrac{1}{2}ai\int_{-1}^{1} d\sigma \int_{0}^{\alpha} \nu\, e^{-ia\sigma\nu}\,\frac{\sin\left(b\sqrt{\alpha^2-\nu^2}\right)}{\alpha^2-\nu^2}\, e^{-iRN_2(\nu)}\,d\nu.$$

The exponents of the last factors contain here the functions

$$N_1(\nu) = \sin\theta\sqrt{\alpha^2-\nu^2} - \nu\cos\theta,$$

$$N_2(\nu) = \sin\theta\sqrt{\alpha^2-\nu^2} + \nu\cos\theta.$$

We will find the values of ν for which the derivatives of these functions vanish.

The derivative

$$\frac{dN_1}{d\nu} = -\left(\cos\theta + \frac{\nu\sin\theta}{\sqrt{\alpha^2-\nu^2}}\right)$$

vanishes for $\nu = -\alpha \cos\theta$; however, this value of ν is found to be outside the integration limits for θ, if we consider the propagation of waves in the first or in the fourth quadrant; however, such a study will be sufficient. It follows therefore that the first integral of the formula (13.1) for large values of R does not contribute a principal term to the asymptotic expansion of the integral T_1 for large R.

The derivative

$$\frac{dN_2}{d\nu} = \cos\theta - \frac{\nu \sin\theta}{\sqrt{\alpha^2 - \nu^2}}$$

vanishes for $\nu = \nu_0 = \alpha \cos\theta$. For this value of ν we have

$$N_2(\nu_0) = \alpha, \quad \left(\frac{d^2N_2}{d\nu^2}\right)_{\nu_0} = -\frac{1}{\alpha \sin^2\theta}.$$

Using these results, we may find the first term of the asymptotic expansion of the inner integral of the second term of (13.1). Using the method of steepest descent, we obtain for large R

$$\int_0^{\pi} \alpha \cos\theta\, e^{-ia\alpha\sigma\cos\theta} \frac{\sin(\alpha b \sin\theta)}{\alpha^2 \sin^2\theta}\, e^{-i\alpha R} e^{\frac{1}{2}iR\frac{(\nu-\alpha\cos\theta)^2}{\alpha\sin^2\theta}}\, d\nu =$$

$$= \frac{\cos\theta}{\alpha \sin^2\theta} \sin(\alpha b \sin\theta)\, e^{-ia\alpha\sigma\cos\theta} e^{-i\alpha R} \int_{-\alpha\cos\theta}^{\alpha(1-\cos\theta)} e^{\frac{ix^2}{2\alpha\sin^2\theta}R}\, dx.$$

The asymptotic value of the integral on the right hand side of this formula is

$$\sqrt{\frac{2\pi\alpha}{R}}\, e^{\frac{1}{4}\pi i} \sin\theta.$$

Therefore the right hand side of the preceding formula assumes the form

$$\sqrt{\frac{2\pi}{\alpha}} \operatorname{cotan}\theta \sin(\alpha b \sin\theta)\, e^{-ia\alpha\sigma\cos\theta} \frac{e^{-i(\alpha R - \pi/4)}}{\sqrt{R}},$$

and we obtain the asymptotic expression for the integral

$$T_1 = -\tfrac{1}{2}ai \int_{-1}^{1} \sqrt{\frac{2\pi}{\alpha}} \operatorname{cotan}\theta \sin(\alpha b \sin\theta) \frac{e^{-i(\alpha R - \pi/4)}}{\sqrt{R}}\, e^{-ia\alpha\sigma\cos\theta}\, d\sigma + \cdots.$$

Evaluating the integral in this formula we obtain

$$T_1 = \frac{1}{\alpha}\sqrt{\frac{2\pi}{\alpha}} \cdot \frac{\sin(a\alpha\cos\theta)\sin(\alpha b\sin\theta)}{\sin\theta} \frac{e^{-i(\alpha R + \pi/4)}}{\sqrt{R}} + \cdots.$$

Using exactly the same computations we find the asymptotic formulae for the integrals T_2 and T_3

$$T_2 = \frac{1}{\alpha}\sqrt{\frac{2\pi}{\alpha}} \cdot \frac{\sin(a\alpha\cos\theta)\sin(\alpha b\sin\theta)}{\cos\theta} \cdot \frac{e^{-i(\alpha R-\pi/4)}}{\sqrt{R}} + \cdots,$$

$$T_3 = \frac{1}{\alpha^2}\sqrt{\frac{2\pi}{\alpha}} \cdot \frac{\sin(a\alpha\cos\theta)\sin(\alpha b\sin\theta)}{\sin\theta\cos\theta} \cdot \frac{e^{-i(\alpha R+\pi/4)}}{\sqrt{R}} + \cdots.$$

With the help of these formulae we may determine the elastic displacements u, v, w for large values of R

$$u = \frac{\xi'(2\xi'+1)}{(\xi'+1)^{5/4}E'(\xi')} \cdot \frac{P}{\mu\sqrt{2\pi}\,(\kappa\omega)^{3/2}} \cdot \frac{\sin(a\alpha\cos\theta)\sin(\alpha b\sin\theta)}{\sin\theta} \cdot \frac{e^{-i(\alpha R-\omega t-\pi/4)}}{\sqrt{R}} + \cdots,$$

$$v = \frac{\xi'(2\xi'+1)}{(\xi'+1)^{5/4}E'(\xi')} \cdot \frac{P}{\mu\sqrt{2\pi}\,(\kappa\omega)^{3/2}} \cdot \frac{\sin(a\alpha\cos\theta)\sin(\alpha b\sin\theta)}{\cos\theta} \cdot \frac{e^{-i(\alpha R-\omega t+\pi/4)}}{\sqrt{R}} + \cdots,$$

$$w = \frac{(2\xi'+3)\sqrt{\xi' + \dfrac{\kappa^2-h^2}{\kappa^2}}}{(\xi'+1)^{5/4}E'(\xi')} \cdot \frac{P}{\mu\sqrt{2\pi}(\omega\kappa)^{3/2}} \cdot \frac{\sin(a\alpha\cos\theta)\sin(\alpha b\sin\theta)}{\sin\theta\cos\theta} \cdot \frac{e^{-i(\alpha R-\omega t-\frac{3}{4}\pi)}}{\sqrt{R}} + \cdots. \tag{13.2}$$

We will now denote by u_R and u_θ the displacements in the directions of the radius vector and perpendicular to it respectively. We readily find from the preceding formulae that $u_\theta = 0$ and

$$u_R = \frac{\xi'(2\xi'+1)}{(\xi'+1)^{5/4}E'(\xi')} \cdot \frac{P}{\mu\sqrt{2\pi}\,(\kappa\omega)^{3/2}} \cdot \frac{\sin(a\alpha\cos\theta)\sin(\alpha b\sin\theta)}{\sin\theta\cos\theta} \cdot \frac{e^{-i(\alpha R-\omega t-\pi/4)}}{\sqrt{R}} + \cdots. \tag{13.3}$$

Thus, for large values of R, the entire process of propagation of elastic waves on the surface of a half-space is described by the formulae (13.2) and (13.3).

We see that the lines of equal phases of displacements u_R or w themselves represent circles propagating to infinity with the radial velocity of Rayleigh waves. The amplitude of these displacements decreases without limits as the radius of the line of equal phase increases. Along every line of equal phase the amplitude of the displacements changes in dependence on the angle θ, and, if the numbers $a\alpha$ and $b\alpha$ are sufficiently large, these amplitudes will vanish for a number of angles θ. In that case the entire map of the expansion of elastic waves on the surface may be likened to a uniformly expanding sectorial chess board consisting of fields which are unequal between them.

If the numbers $a\alpha$ and $b\alpha$ are very small, the magnitude of the displacements along the lines of equal phase does not depend substantially on the angle θ, and the formulae (13.2) and (13.3) go over into the usual formulae for the propagation of circular Rayleigh waves. Thus, the dependence of the displacements u_R and w on the angle θ, established by the formulae (13.2) and (13.3), is completely explained by the finite dimensions of the region which is subject to loading.

§ 14. In conclusion, a few remarks must still be made regarding the double integrals in the formulae for the displacements which were left out of consideration above. For example, we will take the double integral D_1 of § 7. If we study this integral for large values of $|x|$, the order of smallness of D_1 will be x^{-1}, since the quantity m does not have an analytic extreme value either on the boundary or inside the region of integration. For the study of this integral for large R one may note that the functions $\pm n \cos\theta \pm m \sin\theta$ do not have analytic extremes on the boundary of the region of integration or inside this region; hence it follows that D_1 will have the order of smallness R^{-2}. Such estimates may also be established for all the other double integrals. This gave us the opportunity to limit our consideration in the corresponding formulae to the simple integrals, closely connected with Rayleigh waves.

COMPRESSION OF ELASTIC BODIES TOUCHING ALONG A STRAIGHT SEGMENT

I. IA. SHTAERMAN (E. STEUERMANN)
Moscow, U.S.S.R.

The three-dimensional contact problem of the theory of elasticity has been studied for those cases where the region of contact of the compressed bodies in their unstressed state (i.e., before compression) is either a point [1] or some two-dimensional region; in the second case, i.e., in the so-called problem of pressure of a stamp on an elastic body, solution is known for elliptic and several other regions of contact [2].

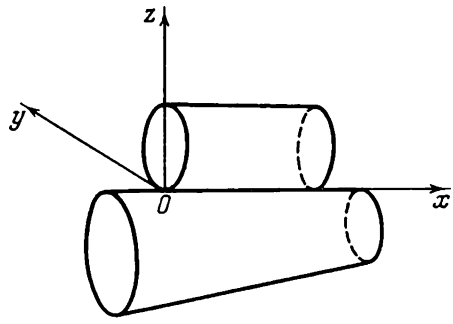

Fig. 1

In this paper we consider the case when the original region of contact of the compressed elastic bodies is a rectilinear segment. As we will show below, this problem reduces to the solution of a system of two one-dimensional integral equations.

We will direct the axis Ox along the line of original contact of the compressed bodies, we will place the axis Oy in the general tangential plane and we will direct the axis Oz along the normal to the plane of contact (cf. Fig. 1).

Let

$$z = f_1(x, y)$$

be the equation of the surface of the body, lying in the upper half space, and

$$z = -f_2(x, y)$$

be the equation of the surface of the body, lying in the lower half space. As the result of compression, the region of contact changes from a straight segment to a narrow strip. We will denote by $f(x)$ the half width of the contact strip at the section with the abscissa x and by l the length of this strip. We will denote by $p(x, y)$ the pressure at the point of the contact strip with coordinates x, y. As is known from the general theory of three-dimensional contact problems, one has to fulfill in the region of

contact the condition

$$(1)\quad (\vartheta_1+\vartheta_2)\int_0^l\int_{-f(\xi)}^{f(\xi)}\frac{p(\xi,\eta)\,d\xi\,d\eta}{\sqrt{(x-\xi)^2+(y-\eta)^2}}=\alpha+\beta x-f_1(x,y)-f_2(x,y),$$

where

$$\vartheta_1=\frac{1-\sigma_1{}^2}{\pi E_1},\quad \vartheta_2=\frac{1-\sigma_2{}^2}{\pi E_2},$$

E_1 and σ_1 are the modulus of elasticity and Poisson's ratio for one of the compressed bodies, E_2 and σ_2 are those for the other body, and $\alpha+\beta x$ is the "closing in" of the compressed bodies at the section with the abscissa x (α is the change in the distance of the bodies in the plane $x=0$, β is the angle of relative rotation of the compressed bodies).

In addition, one must satisfy the equilibrium conditions

$$(2)\quad \int_0^l\int_{-f(\xi)}^{f(\xi)}p(\xi,\eta)\,d\xi\,d\eta=P,\quad \int_0^l\int_{-f(\xi)}^{f(\xi)}p(\xi,\eta)\xi\,d\xi\,d\eta=M,$$

where P is the resultant of the compressive forces applied to one of the bodies and M is the moment of these forces about the axis Oy.

For the selected position of the coordinate axes the functions $f_1(x,y)$ and $f_2(x,y)$ will fulfill for $y=0$ the conditions

$$(3)\quad f_1=f_2=\frac{\partial f_1}{\partial y}=\frac{\partial f_2}{\partial y}=0.$$

Setting $y=0$ in (1), we obtain the equation

$$(4)\quad (\vartheta_1+\vartheta_2)\int_0^l\int_{-f(\xi)}^{f(\xi)}\frac{p(\xi,\eta)\,d\eta}{\sqrt{(x-\xi)^2+\eta^2}}=\alpha+\beta x,\quad 0<x<l.$$

By (1) and (4), one must have in the contact region the equality

$$(5)\quad (\vartheta_1+\vartheta_2)\int_{-l}^{l}\int_{-f(\xi)}^{f(\xi)}p(\xi,\eta)\left[\frac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2}}-\right.$$

$$\left.-\frac{1}{\sqrt{(x-\xi)^2+\eta^2}}\right]d\xi\,d\eta=-f_1(x,y)-f_2(x,y).$$

Executing in (5) the integration with respect to x, we obtain the

equation

$$(6)\quad \int_0^l \int_{-f(\xi)}^{f(\xi)} p(\xi,\eta)\log\left[\frac{x-\xi+\sqrt{(x-\xi)^2+(y-\eta)^2}}{x-\xi+\sqrt{(x-\xi)^2+\eta^2}}\cdot\right.$$

$$\left.\cdot\frac{x_0-\xi+\sqrt{(x_0-\xi)^2+\eta^2}}{x_0-\xi+\sqrt{(x_0-\xi)^2+(y-\eta)^2}}\right]d\xi\,d\eta =$$

$$= -\frac{1}{\vartheta_1+\vartheta_2}\int_{x_0}^{x}[f_1(\xi,y)+f_2(\xi,y)]\,d\xi.$$

By (3) there exists the limit

$$(7)\quad F(x)=\lim_{y\to 0}\left\{-\frac{1}{(\vartheta_1+\vartheta_2)y^2}\int_{x_0}^{x}[f_1(\xi,y)+f_2(\xi,y)]\,d\xi=\right.$$

$$\left.=-\frac{1}{2(\vartheta_1+\vartheta_2)}\int_{x_0}^{x}\left(\frac{\partial^2 f_1}{\partial y^2}+\frac{\partial^2 f_2}{\partial y^2}\right)\right\}_{x=\xi,\,y=0}d\xi.$$

Thus there must apply in the limit the equality

$$(8)\quad \lim_{y\to 0}\frac{1}{y^2}\int_0^l\int_{-f(\xi)}^{f(\xi)} p(\xi,\eta)\log\left[\frac{(x-\xi)+\sqrt{(x-\xi)^2+(y-\eta)^2}}{x-\xi+\sqrt{(x-\xi)^2+\eta^2}}\cdot\right.$$

$$\left.\cdot\frac{x_0-\xi+\sqrt{(x_0-\xi)^2+\eta^2}}{x_0-\xi+\sqrt{(x_0-\xi)^2+(y-\eta)^2}}\right]d\xi\,d\eta=F(x),\quad 0<x<l.$$

For $\xi\neq x$ and $\xi\neq x_0$ the limit

$$(9)\quad \lim_{y\to 0}\frac{1}{y^2}\int_{-f(\xi)}^{f(\xi)} p(\xi,\eta)\log\left[\frac{x-\xi+\sqrt{(x-\xi)^2+(y-\eta)^2}}{x-\xi+\sqrt{(x-\xi)^2+\eta^2}}\cdot\right.$$

$$\left.\cdot\frac{x_0-\xi+\sqrt{(x_0-\xi)^2+\eta^2}}{x_0-\xi+\sqrt{(x_0-\xi)^2+(y-\eta)^2}}\right]d\eta$$

exists, since in this case the integral, occurring under the limit sign, together with its first two derivatives with respect to y is continuous at $y=0$, and the same integral and its first derivative with respect to y vanishes at $y=0$. The derivative of this integral with respect to y vanishes for $y=0$, since the function $p(\xi,\eta)$ must be an even function

of η, and in this case the integral under consideration is an even function of y.

In order for the limit (9) to exist for $\xi = x$ $(x \neq x_0)$, there must exist the limit

$$\varphi(x) = \lim_{y \to 0} \frac{1}{y^2} \int_{-f(x)}^{f(x)} p(x, \eta) \log \left| \frac{y - \eta}{\eta} \right| d\eta;$$

in other words, apart from small higher order quantities compared with y^2, there must be fulfilled the equality

$$\int_{-f(x)}^{f(x)} p(x, \eta) \log \left| \frac{y - \eta}{\eta} \right| d\eta = \varphi(x) y^2. \tag{10}$$

Thus, for a fixed value of the displacement x, we arrive at the integral equation of the plane contact problem of the theory of elasticity, the solution of which is known to have the form [3]

$$p(x, y) = q(x) \sqrt{f^2(x) - y^2}. \tag{11}$$

Thus, the solution of the contact problem considered by us in this paper is reduced to finding the functions $q(x)$ and $f(x)$ determining the region of contact and the distribution of pressure in this region, and the constants α and β, determining the "closing in" of the compressed bodies.

Substituting $p(x, y)$ from (11) in (8), we obtain in the limit the equality

$$\lim_{y \to 0} \frac{1}{y^2} \int_0^l \int_{-f(\xi)}^{f(\xi)} q(\xi) \sqrt{f^2(\xi) - \eta^2} \log \left[\frac{x - \xi + \sqrt{(x - \xi)^2 + (y - \eta)^2}}{x - \xi + \sqrt{(x - \xi)^2 + \eta^2}} \cdot \right. \tag{12}$$

$$\left. \cdot \frac{x_0 - \xi + \sqrt{(x_0 - \xi)^2 + \eta^2}}{x_0 - \xi + \sqrt{(x_0 - \xi)^2 + (y - \eta)^2}} \right] d\xi \, d\eta = F(x), \quad 0 < x < l.$$

We will now introduce into the consideration the function

$$\Phi(s, t) = \int_{-1}^{1} \sqrt{1 - \tau^2} \log \frac{[s + \sqrt{s^2 + (t - \tau)^2}] \, |\tau|}{(s + \sqrt{s^2 + \tau^2}) |t - \tau|} \, d\tau. \tag{13}$$

Setting

$$s = \frac{x - \xi}{f(\xi)}, \quad t = \frac{y}{f(\xi)}, \quad \tau = \frac{\eta}{f(\xi)},$$

in (13), we find

$$\Phi\left(\frac{x-\xi}{f(\xi)},\ \frac{y}{f(\xi)}\right)=$$

$$=\frac{1}{f^2(\xi)}\int_{-f(\xi)}^{f(\xi)}\sqrt{f^2(\xi)-\eta^2}\log\left[\frac{x-\xi+\sqrt{(x-\xi)^2+(y-\eta)^2}}{x-\xi+\sqrt{(x-\xi)^2+\eta^2}}\right]\frac{|\eta|}{|y-\eta|}\,d\eta.$$

Hence there follows

$$\int_{-f(\xi)}^{f(\xi)}\sqrt{f^2(\xi)-\eta^2}\log\left[\frac{x-\xi+\sqrt{(x-\xi)^2+(y-\eta)^2}}{x-\xi+\sqrt{(x-\xi)^2+\eta^2}}\cdot\right.$$

$$\left.\cdot\frac{x_0-\xi+\sqrt{(x_0-\xi)^2+\eta^2}}{x_0-\xi+\sqrt{(x_0-\xi)^2+(y-\eta)^2}}\right]d\eta= \tag{14}$$

$$=f^2(\xi)\left[\Phi\left(\frac{x-\xi}{f(\xi)},\ \frac{y}{f(\xi)}\right)-\Phi\left(\frac{x_0-\xi}{f(\xi)},\ \frac{y}{f(\xi)}\right)\right].$$

By (14), the limiting equality (12) may be given the form

$$\lim_{y\to 0}\frac{1}{y^2}\int_0^l q(\xi)f^2(\xi)\left[\Phi\left(\frac{x-\xi}{f(\xi)},\ \frac{y}{f(\xi)}\right)-\Phi\left(\frac{x_0-\xi}{f(\xi)},\ \frac{y}{f(\xi)}\right)\right]d\xi= \tag{15}$$

$$=F(x),\quad 0<x<l.$$

Differentiating the formula (13) with respect to t we find

$$\frac{\partial\Phi}{\partial t}=\int_{-1}^{1}\frac{s\sqrt{1-\tau^2}}{(\tau-t)\sqrt{s^2+(t-\tau)^2}}\,d\tau, \tag{16}$$

where the integral must be understood in the sense of its Cauchy principal value. Introducing into this integral the transformation $\tau=t-\sigma$, we obtain for the derivative $\partial\Phi/\partial t$ the formula

$$\frac{\partial\Phi}{\partial t}=-\int_{-1+t}^{1+t}\frac{s\sqrt{1-(t-\sigma)^2}}{\sigma\sqrt{s^2+\sigma^2}}\,d\sigma. \tag{17}$$

Completing the differentiation in (17), we find

$$\frac{\partial^2\Phi}{\partial t^2}=\int_{-1+t}^{1+t}\frac{s(t-\sigma)}{\sigma\sqrt{(s^2+\sigma^2)[1-(t-\sigma)^2]}}\,d\sigma. \tag{18}$$

For $t = 0$ the derivative $\partial\Phi/\partial t$ vanishes, as it is an odd function of t as shown by the formula (16). Thus

$$\lim_{t\to 0} \frac{1}{t^2}\,\Phi(s,t) = \tfrac{1}{2}\left(\frac{\partial^2\Phi}{\partial t^2}\right)_{t=0},$$

and $\Phi(s, 0) = 0$ in accordance with (13), or, using (18), one finds

$$\lim_{t\to 0} \frac{1}{t^2}\,\Phi(s,t) = -\frac{s}{2}\int_{-1}^{1} \frac{d\sigma}{\sqrt{(1-\sigma^2)(s^2+\sigma^2)}} = -\,s\int_{0}^{1} \frac{d\sigma}{\sqrt{(1-\sigma^2)(s^2+\sigma^2)}}\,.$$

By means of the change of variable $\sigma = \cos\zeta$ one may give this limiting equality the form

$$\lim_{t\to 0} \frac{1}{t^2}\,\Phi(s,t) = \Psi(s), \tag{19}$$

where

$$\Psi(s) = -\,s\int_{0}^{\pi/2} \frac{d\varphi}{\sqrt{\cos^2\varphi + s^2}}\,. \tag{20}$$

By (19)

$$\lim_{y\to 0} \frac{f^2(\xi)}{y^2}\,\Phi\left(\frac{x-\xi}{f(\xi)},\ \frac{y}{f(\xi)}\right) = \Psi\left(\frac{x-\xi}{f(\xi)}\right),$$

and thus the limiting equality (15) may be given the form

$$\int_{0}^{l} q(\xi)\left[\Psi\left(\frac{x-\xi}{f(\xi)}\right) - \Psi\left(\frac{x_0-\xi}{f(\xi)}\right)\right] d\xi = F(x), \quad 0 < x < l. \tag{21}$$

Differentiating with respect to s in (20), we find

$$\Psi'(s) = -\int_{0}^{\pi/2} \frac{\cos^2\varphi\, d\varphi}{(\cos^2\varphi + s^2)^{\frac{3}{2}}}\,. \tag{22}$$

Using the identity

$$\frac{1}{1+s^2}\,\frac{d}{d\varphi}\left(\frac{\sin\varphi\cos\varphi}{\sqrt{\cos^2\varphi+s^2}}\right) = \frac{\cos^2\varphi}{(\cos^2\varphi+s^2)^{\frac{3}{2}}} - \frac{1}{1+s^2}\,\frac{\sin^2\varphi}{\sqrt{\cos^2\varphi+s^2}},$$

one may rewrite (22) in the form

$$\Psi^1(s) = -\frac{1}{1+s^2}\int_0^{\pi/2}\frac{\sin^2\varphi\,d\varphi}{\sqrt{\cos^2\varphi+s^2}} =$$

$$= -\frac{1}{\sqrt{1+s^2}}\int_0^{\pi/2}\frac{d\varphi}{\sqrt{1-\frac{\sin^2\varphi}{1+s^2}}} - \int_0^{\pi/2}\sqrt{1-\frac{\sin^2\varphi}{1+s^2}}\,d\varphi$$

or

$$\Psi'(s) = -\frac{1}{\sqrt{1+s^2}}\left[K\left(\frac{1}{\sqrt{1+s^2}}\right) - E\left(\frac{1}{\sqrt{1+s^2}}\right)\right], \tag{23}$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind,

$$K(k) = \int_0^{\pi/2}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}},\quad E(k) = \int_0^{\pi/2}\sqrt{1-k^2\sin^2\varphi}. \tag{24}$$

Differentiating (21) with respect to x and taking into consideration (7) and (23), we obtain the equation

$$\int_0^l \frac{q(\xi)}{\sqrt{(x-\xi)^2+f^2(\xi)}}\left[K\left(\frac{f(\xi)}{\sqrt{(x-\xi)^2+f^2(\xi)}}\right) - \right. \tag{25}$$

$$\left. - E\left(\frac{f(\xi)}{\sqrt{(x-\xi^2)+f^2(\xi)}}\right)\right]d\xi = \frac{1}{2(\vartheta_1+\vartheta_2)}\left(\frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_2}{\partial y^2}\right)_{y=0}.$$

Substituting $p(x, y)$ from (11) in (4), we obtain the equation

$$(\vartheta_1+\vartheta_2)\int_0^l q(\xi)\int_{-f(\xi)}^{f(\xi)}\frac{\sqrt{f^2(\xi)-\eta^2}}{\sqrt{(x-\xi)^2+\eta^2}}\,d\eta\,d\xi = \alpha+\beta x,\quad 0<x<l. \tag{26}$$

We will now introduce the function

$$\Theta(s) = \int_{-1}^{1}\frac{\sqrt{1-\tau^2}}{\sqrt{s^2+\tau^2}}\,d\tau. \tag{27}$$

Setting

$$s = \frac{x-\xi}{f(\xi)},\quad \tau = \frac{\eta}{f(\xi)}$$

in (27), we find

$$\Theta\left(\frac{x-\xi}{f(\xi)}\right)=\frac{1}{f(\xi)}\int_{-f(\xi)}^{f(\xi)}\frac{\sqrt{f^2(\xi)-\eta^2}}{\sqrt{(x-\xi^2)+\eta^2}}\,d\eta.$$

Thus, the equation (26) may be given the form

$$(28)\qquad (\vartheta_1+\vartheta_2)\int_0^l q(\xi)f(\xi)\Theta\left(\frac{x-\xi}{f(\xi)}\right)d\xi=\alpha+\beta x,\quad 0<x<l.$$

Setting $\tau=\cos\zeta$ in (27), we find

$$(29)\quad \Theta(s)=2\int_0^{\pi/2}\frac{\sin^2\varphi\,d\varphi}{\sqrt{\cos^2\varphi+s^2}}=2\sqrt{1+s^2}\left(\int_0^{\pi/2}\frac{d\varphi}{\sqrt{1-\frac{\sin^2\varphi}{1+s^2}}}-\right.$$

$$\left.-\int_0^{\pi/2}\sqrt{1-\frac{\sin^2\varphi}{1+s^2}}\,d\varphi\right)=2\sqrt{1+s^2}\left[K\left(\frac{1}{\sqrt{1+s^2}}\right)-E\left(\frac{1}{\sqrt{1+s^2}}\right)\right].$$

Substituting $\Theta(s)$ from (29) in (28), we obtain the equation

$$(30)\quad \int_0^l q(\xi)\sqrt{(x-\xi)^2+f^2(\xi)}\left[K\left(\frac{f(\xi)}{\sqrt{(x-\xi)^2+f^2(\xi)}}\right)-\right.$$

$$\left.-E\left(\frac{f(\xi)}{\sqrt{(x-\xi)^2+f^2(\xi)}}\right)\right]d\xi=\frac{\alpha+\beta x}{2(\vartheta_1+\vartheta_2)},\quad 0<x<l.$$

Substituting $p(x,y)$ from (11) in (2), we obtain

$$(31)\qquad \int_0^l q(\xi)f^2(\xi)\,d\xi=\frac{2P}{\pi},$$

$$\int_0^l q(\xi)f^2(\xi)\xi\,d\xi=\frac{2M}{\pi}.$$

Thus, we have obtained for the unknown functions $q(x)$ and $f(x)$ the two integral equations (25) and (30). The unknown constants α and β are determined by the supplementary conditions (31).

Introducing the notation

$$(32)\qquad F_1(k)=k[K(k)-E(k)],\ F_2(k)=\frac{1}{k}[K(k)-E(k)],$$

one may represent the system of integral equations (25) and (30) in the form

$$\int_0^l \frac{q(\xi)}{f(\xi)} F_1\left(\frac{f(\xi)}{\sqrt{(x-\xi)^2 + f^2(\xi)}}\right) d\xi = \frac{1}{2(\vartheta_1 + \vartheta_2)} \left(\frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_2}{\partial y^2}\right)_{y=0},$$

(33)

$$\int_0^l q(\xi) f(\xi) F_2\left(\frac{f(\xi)}{\sqrt{(x-\xi)^2 + f^2(\xi)}}\right) d\xi = \frac{\alpha + \beta x}{2(\vartheta_1 + \vartheta_2)}, \quad 0 < x < l.$$

Thanks to the presence of the logarithmic singularity of the function $K(k)$ at the point $k = 1$ the integrands in the equations (33) have logarithmic singularities at the point $\xi = x$. At the same time, due to the smallness of the ratio $f(x)/l$, the integrands in the equations (33) decrease rapidly as $|x - \xi|$ increases; for $k \to 0$ we have $F_1(k) = O(k^3)$, $F_2(k) = O(k)$.

Thus, for fixed x, the values of the integrals in the integral equations (33) principally determine the values of the functions $q(\xi)$ and $f(\xi)$ at the point $\xi = x$, thanks to which an approximate solution of these integral equations may be constructed relatively simply for concrete contact problems.

We note that by fixing the functions $q(x)$ and $f(x)$ and substituting them in the equations (33) one may find conditions which must be satisfied by the configurations of the compressed bodies and their elastic constants. For the case $q(x) = \text{const.}$ and $f(x) = \text{const.}$ such computations have been executed by another method by V. I. Krasnen'kov [4].

We have limited our attention here to the case of initial contact along a straight line, but our solution also applies to initial contact along a curve.

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INTERFACES BETWEEN VISCOUS FLUIDS IN NARROW PASSAGES

G. I. TAYLOR
Cambridge, England

The name of Muskhelishvili is so well known among workers on problems involving potential theory that it is with diffidence that I venture to describe in his birthday volume a very simple example of this class. The interest of the problem is that the solution is not unique, depending as it does, on a single undetermined ratio λ. Experiments show that it is only when λ has the particular value $\frac{1}{2}$ that the solution represents physical reality, but why that value is significant has not yet (Jan. 1960) been explained.

§ 1. Formulation of the problem

When viscous fluid is forced to flow through a porous medium by the pressure of a fluid of less viscosity the interface is in general unstable, though it can be stabilized by gravity, provided it is horizontal and the lower, denser, less viscous fluid is moving upwards slowly enough. In attempting to describe this instability analytically P. G. Saffman and G. I. Taylor [2] and others have assumed Darcy's law, according to which the flow velocity is proportional to the pressure gradient divided by the viscosity of the fluid. The boundary condition assumed at the interface was simply that there is a constant difference between the pressures on the two sides of it. This condition has been assumed to hold in problems concerning the water level surface in underground water flow. It leads to the conclusion that when a horizontal interface becomes unstable as the speed of upward flow increases, all wave lengths become unstable at the same flow speed and in this respect it is analogous to the instability of the surface of water when it is forced downwards by air pressure producing acceleration greater than g.

If the interface has initially the form of a simple wave of small amplitude and wave length L it might be expected to develop into a form in which "fingers" of the less viscous fluid spaced at distances L apart penetrate into the more viscous one. Experiments in which the porous medium is transparent and the lower fluid coloured show that such "fingers" are in fact formed and it is a matter of considerable interest to know the width of the invading "fingers", for if they are very narrow

they will penetrate through the more viscous fluid without forcing it upwards. In the analogous case of the accelerating fluid forced by air pressure to move, it has been shown [1] that the instability of the air-fluid interface does not prevent the main body of the fluid from being accelerated almost as though the interface had remained flat.

The great difficulty involved in the mathematical analysis of waves of finite amplitude turned my attention from small wave disturbance to explore possible final states in which "fingers" of one fluid move at uniform speed into the other. Confining attention to two-dimensional flow it may be expected that if such "fingers" originate in a small wave disturbance of wave length L they will, when fully developed, be spaced at distance L apart, if such "fingers" are all of the same length they will be separated by planes over which there is no flow. The problem is then reduced to finding the shapes of possible steadily moving interfaces when the less viscous fluid is forced into a porous medium contained between parallel impermeable sheets distance L apart. Solutions of this problem for the case when the basic flow is vertical have been given [2], using mathematically simple boundary conditions, but only the simplest case, when the viscosity of the lower fluid is negligible, is reproduced here in order to illustrate the method.

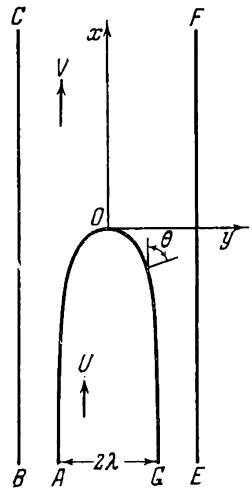

Fig. 1

Fig. 1 shows the co-ordinate system used. AOG is the interface which is assumed to move through the porous medium without change of form with velocity U. It will also be assumed for simplicity (though this is not necessary) that within the "finger" the whole of the more viscous fluid has been displaced by the less viscous. The impermeable walls are BC and EF (Fig. 1). The field equation for the flow outside "fingers" is taken as

$$u = -\frac{k}{\mu}\,\mathrm{grad}\,(p + g\rho x), \tag{1.1}$$

where μ is the viscosity and k the permeability of the porous medium. The form of (1.1) shows that the flow has a velocity potential

$$\varphi = -\frac{k}{\mu}(p + g\rho x), \tag{1.2}$$

and since the fluid is incompressible the components of velocity can be

derived from a stream function ψ such that

$$u = \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}. \tag{1.3}$$

The condition that the interface moves without change of form with velocity U in the direction x is

$$\frac{\partial \psi}{\partial s} = U \cos\theta, \tag{1.4}$$

where θ is the angle shown in Fig. 1 and s represents distance along the interface. Since $\cos\theta = dy/ds$, the condition (1.4) is

$$\psi = Uy. \tag{1.5}$$

Since the viscosity of the displacing fluid is assumed negligible the pressure condition at the interface is $p = \text{const.}$ or $\varphi = -(k/\mu)g\rho x$. Writing

$$\varphi' = \varphi + \frac{k}{\mu} g\rho x, \quad \psi' = \psi + \frac{k}{\mu} g\rho y, \tag{1.6}$$

$w' = \varphi' + i\psi'$ is a function of $z = x + iy$. The pressure condition at the interface is therefore

$$\varphi' = \text{const.} \tag{1.7}$$

§ 2. Theoretical investigation

For convenience we may take the boundaries BC, EF (Fig. 1) as $y = \pm 1$. Since these are both stream lines, ψ is constant along them, and since y is constant, ψ' is constant. Taking the velocity of the fluid far ahead of the interface as V the values of ψ on $y = \pm 1$ are $\pm V$, so that

$$\psi' = \pm(V + kg\rho/\mu) = \pm V'. \tag{2.1}$$

Since the "finger" is assumed to be of constant width, the interface must be asymptotic to the lines $y = \pm \lambda$, where 2λ is the width of the "finger". Since the total volumetric flow over sections far ahead and far behind the front of the interface must be the same

$$V = U\lambda - (1 - \lambda)kg\rho/\mu. \tag{2.2}$$

Defining

$$U' = U + kg\rho/\mu, \quad V' = \lambda U', \tag{2.3}$$

the condition (1.5) at the interface is replaced by

$$\psi' = U'y. \tag{2.4}$$

The problem is therefore to determine a function w' of z such that (1.7), (2.1) and (2.4) are satisfied on the boundaries. It may be noted that

$$\psi' \sim V'y \text{ at } x \sim \infty,$$

(2.5)

$$\psi' = V'\left(\frac{y-\lambda}{1-\lambda}\right) \text{ and } \varphi' = 0 \text{ at } x \sim -\infty.$$

Since z is a function of w', y is a harmonic function of φ' and ψ'. Consider the harmonic function

$$y = \frac{\psi'}{V'} + \sum_1^\infty A_n \sin \frac{n\pi\psi'}{V'} \exp \frac{-n\pi\varphi'}{V'}. \tag{2.6}$$

This satisfies (2.1) on $y = \pm 1$. It also satisfies (2.4) on the interface when $\varphi' = 0$, provided

$$\frac{\psi'}{U'} = \frac{\psi'}{V'} + \sum_1^\infty A_n \sin \frac{n\pi\psi'}{V'}. \tag{2.7}$$

Expanding ψ'/V' in the range $V' > \psi' > -V'$ as a Fourier series in terms of $\sin(n\pi\psi'/V')$ the value A_n is found to be

$$A_n = \frac{2}{\pi}(1-\lambda)\frac{(-1)^n}{n}. \tag{2.8}$$

Consider now the function

$$z = \frac{w'}{V'} + \frac{2}{\pi}(1-\lambda)\log \tfrac{1}{2}\left(1 + \exp \frac{-\pi w'}{V'}\right); \tag{2.9}$$

since

$$\log\left(1 + \exp \frac{-\pi w'}{V'}\right) = \sum_1^\infty \frac{(-1)^{n+1}}{n} \exp \frac{-n\pi w'}{V'}, \tag{2.10}$$

the real and imaginary parts can be separated. The imaginary part is identical with (2.6), so that (2.9) must be the required equation between z and w'. The real part of (2.9) is

$$x = \frac{\varphi'}{V'} + \frac{2}{\pi}(1-\lambda)\sum_1^\infty \frac{(-1)^{n+1}}{n} \cos \frac{n\pi\psi'}{V'} \exp \frac{n\pi\varphi'}{V'} - \frac{2}{\pi}(1-\lambda)\log 2, \tag{2.11}$$

and on the interface where $\varphi' = 0$, $y = \psi'/U'$ (2.11) becomes

$$x = \frac{2}{\pi}(1-\lambda)\sum_1^\infty \frac{(-1)^{n+1}}{n} \cos \frac{n\pi y}{\lambda} - \frac{2}{\pi}(1-\lambda)\log 2, \tag{2.12}$$

and since

$$\sum_{1}^{\infty} \frac{(-1)^{n+1}}{n} \cos \frac{n\pi y}{\lambda} = \log 2 + \log \cos \frac{\pi y}{2\lambda},$$

(2.12) can be written

$$x = \frac{2}{\pi}(1-\lambda) \log \cos\left(\frac{\pi y}{2\lambda}\right). \tag{2.13}$$

(2.13) represents the shape of the "finger". The "finger" can be brought to rest by superposing on the whole system a velocity $-U$. The stream-lines will then be identical with those in an irrotational flow past a solid body whose shape is represented by (2.13). It is of interest to compare these shapes with those used by Zhuravlev [3] in a different connection. Zhuravlev considered the set of shapes produced by a two-dimensional source of strength m/λ situated in the centre of a channel formed by walls at $y = \pm 1$ and superposed on a stream of velocity q. Zhuravlev's shapes are the stream-lines which could be replaced by a solid obstacle, that is those which separate the fluid which originates in the source m from the fluid which approaches from up-stream. To compare these shapes with those represented by (2.13) when λ is given, it is necessary to express q and m in terms of λ and V. The velocity relative to the obstacle is $q - \frac{1}{4}m$ and this is $U - V$ or $U' - V'$. In Zhuravlev's problem the velocity is the same on the two sides of the dividing stream line, so that

$$\frac{m}{2\lambda} = \frac{U-V}{1-\lambda} = \frac{q-\frac{1}{4}m}{1-\lambda}.$$

Hence

$$\frac{q}{m} = \frac{2-\lambda}{4\lambda}$$

and the equation of the dividing line referred to the vertex as origin is

$$x = \frac{1}{\pi}\left\{\log \sin \frac{\pi y}{\lambda} - \log \sin (1-\lambda)\frac{\pi y}{\lambda} + \log(1-\lambda)\right\}. \tag{2.14}$$

When $\lambda = \frac{1}{2}$, (2.14) is

$$x = \frac{1}{\pi} \log \cos \pi y \tag{2.15}$$

which is identical with (2.13) when $\lambda = \frac{1}{2}$. Thus, in the particular case $\lambda = \frac{1}{2}$, corresponding with $q/m = \frac{3}{4}$ and only in that case, the shape of

the "finger" is one of Zhuravlev's series. In trying to find out why this should be G. I. Taylor and P. G. Saffman [4] found that all the shapes represented by (2.13) could be reproduced by means of a distribution of sources on the centre line of the channel provided $\lambda < \frac{1}{2}$, but not when $\lambda > \frac{1}{2}$.

The shapes represented by (2.13) for values of $\lambda = 0.2$, 0.5 and 0.8 are shown in Fig. 2*a*. It will be noticed that they can all be derived by appropriate stretching of a single curve, say that for $\lambda = \frac{1}{2}$ which is given in (2.15); to obtain the curve for any other value of λ this must be strained

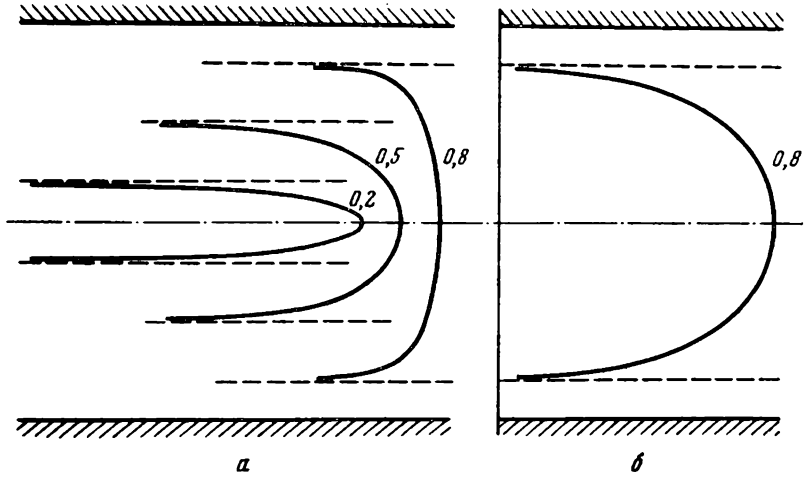

Fig. 2. Left, Interfaces according to (2.13) for $\lambda = 0.2$, 0.5, 0.8. Right, Zhuravlev's shape for $\lambda = 0.8$.

in the ratio $2(1 - \lambda)$ in the x-direction and 2λ in the y-direction, i.e., when $\lambda > \frac{1}{2}$, it is contracted in the x-direction while it is expanded in the y-direction. The difference between Zhuravlev's shapes and those represented by (2.13) is illustrated in Fig. 2 where Zhuravlev's shape for $\lambda = 0.8$ is shown for comparison with that represented by (2.13) when $\lambda = 0.8$. It appears that Zhuravlev's has a rounder end when $\lambda > \frac{1}{2}$.

It will be noticed that though the effect of gravity has been taken into account the shape of the interface does not depend on gravity. The non-dimensional parameter which might have been expected to influence the shape, namely $kg\rho/\mu U$ has no effect on the shape of the interface.

The same result is found when the viscosity of the intruding fluid is not neglected.

§ 3. Experimental investigation

Equation (2.13) contains an undetermined parameter. I have so far found no principle or hypothesis which would enable me even to guess *a priori* which value would be expected under experimental conditions. The difficulty of seeing the interface in a porous medium prevented direct experiments and recourse was had to the Hele-Shaw cell which consists of two parallel sheets of glass separated by a thin space. The flow at every point is proportional to, and in the direction of the pressure gradient. The mean velocity is $\frac{2}{3}$ of the maximum velocity at the point halfway between the plates. The velocity can therefore be related to a velocity potential. The flow, in fact, is exactly analogous to a two-dimensional flow in a porous medium so far as the field equations are concerned. The conditions at the interface between two fluids however may differ considerably both between the porous medium and the Hele-Shaw cell and between them and the simple theoretical conditions necessary to produce a mathematically soluble problem. Apparatus for injecting a fluid of small viscosity into one of larger viscosity contained in a Hele-Shaw cell was designed [2] and the shape of the "finger" so produced was photographed.

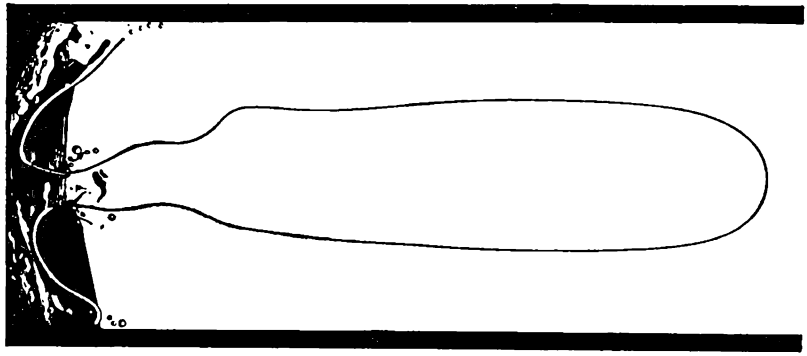

Fig. 3*a*. Air penetrating glycerine.

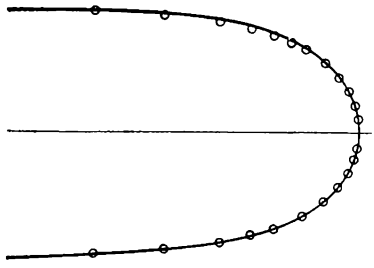

Fig. 3*b*. Enlargement of Fig. 3*a*, points calculated using (2.13).

Fig. 3*a* shows the interface when air is blown into a channel 5 cm. wide, 0.1 cm. thick which was full of glycerine. This was an early photograph and U was not measured but is reproduced here to show that the width of the interface rapidly spreads to its final form. Fig. 3*b* is an enlargement of the end of the interface of Fig. 3*a* and the calculated

points corresponding with (2.13) where $\lambda = \frac{1}{2}$ are marked on the photograph.

Fig. 4*a* shows water penetrating into oil of viscosity 0.3 poise ("Diala" oil).

Fig. 4*b* shows a high viscosity oil (4.5 poises; "Talpa" oil) penetrating into glycerine (9 poises). In both cases λ is close to $\frac{1}{2}$.

When the intruding fluid was introduced very slowly, λ was found to be greater than $\frac{1}{2}$ and several sets of corresponding measurements of λ and U, the velocity of the intruding "finger", were made. Fig. 5 shows the

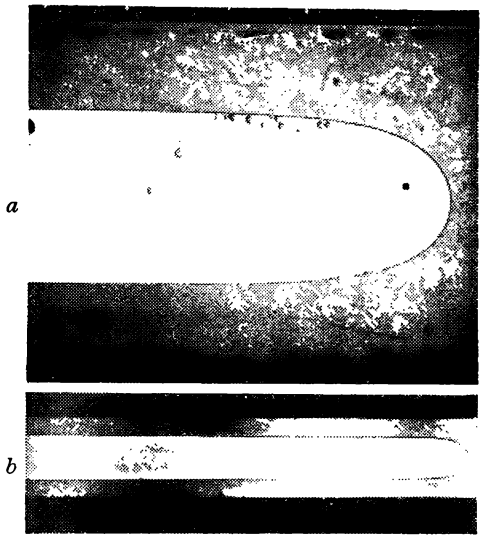

Fig. 4*a*. Finger of oil ($\mu = 4.5$) penetrating glycerine ($\mu = 9P$).
Fig. 4*b*. Water penetrating oil ($\mu = 0.3P$).

results for water penetrating into "Diala" and "Talpa" oils of viscosity 0.3 and 4.5 poises at the temperature of the experiments. These experiments were performed with the Hele-Shaw cell in a horizontal position. It will be seen that λ decreases as U increases and that the curve flattens out. The ordinates are values of $\mu U/T$, where T is the surface tension, because if the oil has "Newtonian" properties and wets the surface of the cell, surface tension is the only physical quantity besides μ and U which could be expected to affect the boundary condition at the meniscus between air and oil. $\mu U/T$ is the only non-dimensional parameter which can be formed from these three quantities when the geometry of the cell is fixed. This deduction is confirmed by the way in which the points corresponding to oils of very different viscosities (0.3 and 4.5P) lie on the same curve in Fig. 5. Fig. 3*b* shows that the observed shape of a "finger" is very close to that described by (2.13) when $\lambda = \frac{1}{2}$, and Fig. 5

shows that this value is rapidly attained as $\mu U/T$ increases. The assumptions made in the theory were

(i) that no fluid was left behind on the plate, and

(ii) that the change in pressure on passing through the meniscus is constant.

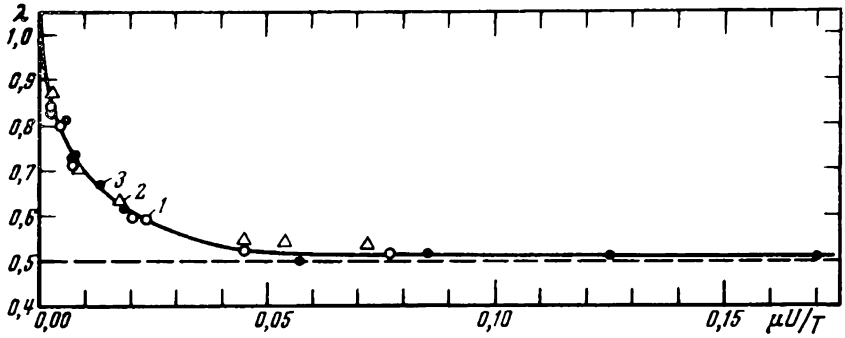

Fig. 5. Water penetrating Talpa oil ($\mu = 4.5P$) and Diala oil ($\mu = 0.3$).

(i) is nearly true when $\mu U/T$ is very small, but (ii) is not true. If the cell has thickness b and breadth a, the shape of the meniscus when the fluid is at rest is one for which the sum of the two principal curvatures is constant. When the fluid wets the glass, one of these is $2/b$ and the other $2/a$. Since b is constant, a must also be constant, so that the limiting form of the mid-plane section of the meniscus as $U \sim 0$ is a semicircle of radius $\frac{1}{2}a$ and $\lambda \sim 1$. When U is small but not zero, $1 - \lambda$ is small and the meniscus is still rounded and is unlike the curve calculated on the assumption that the fall in pressure through it is constant. Fig. 6 shows

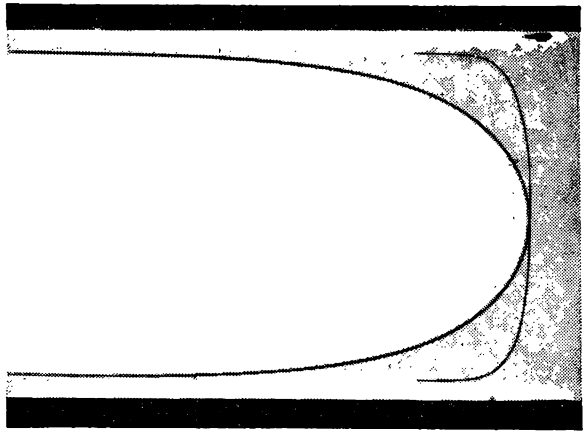

Fig. 6. Photograph of water-Talpa interface at very low speed. Calculated curve using assumptions (i) and (ii) with $\lambda = 0.87$.

a photograph in which $\lambda = 0.87$ and the theoretical curve of (2.13) is drawn on the photograph for comparison.

When $\mu U/T$ is large, assumption (i) is not true but (ii) is nearly true, since as $\mu U/T$ increases the stresses due to viscosity become large compared with the variation in pressure drop through the interface. (i) is far from being true, so that it seemed surprising that the shape calculated assuming (i) should be so very close to the observed limiting shape as $\mu U/T$ increases. It was found, however (Appendix in [2]), that the same analysis applies when any constant proportion of the fluid is left behind on the glass walls of the cell and, since experiments (not yet published) show that a constant is left behind, it seems that the agreement between the observed shape and one of the series of calculated shapes, namely that corresponding with $\lambda = \frac{1}{2}$, is explained. The reason for the occurrence of only this particular value of λ under experimental conditions is obscure; some mathematical peculiarities associated with $\lambda = \frac{1}{2}$ have been studied [4]. It may be of interest to note that, if water were used to expel oil from a Hele-Shaw cell at such a speed that $\lambda = \frac{1}{2}$, less than half the oil would have been driven out at the far end before the water first appeared there. In fact it has been found in experiments involving the upward displacement of one fluid by another in a porous bed that the invading fluid usually breaks through when less than half of the original fluid has been driven out unless the flow is very slow.

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STAGES IN THE DEVELOPMENT OF THE CONCEPT OF STRESS

C. TRUESDELL
Bloomington, U.S.A.

§ 1. Statement of the stress principle

The general theory of stress is due to Cauchy (1822). We may state his *stress principle* as follows: Imagine within a continuous body any closed diaphragm; then the action of the material without upon that within is equipollent to a field of *stress vectors* $\mathbf{t}_{(\mathbf{n})}$ defined upon the diaphragm. By combining this principle with Euler's equations expressing balance of linear momentum and moment of momentum in any kind of body, Cauchy derived a number of now familiar consequences; in particular,

$$\mathbf{t}_{(\mathbf{n})} + \mathbf{t}_{(-\mathbf{n})} = 0, \tag{1.1}$$

$$\mathbf{t}_{(\mathbf{n})} = \mathbf{t}\cdot\mathbf{n},$$

where $\mathbf{n}$ is the unit normal to the imaginary diaphragm and $\mathbf{t}$ is the *stress tensor*; moreover, in regions where the field variables are sufficiently smooth the equation

$$\rho\mathbf{a} = \operatorname{div}\mathbf{t} + \rho\mathbf{f}, \tag{1.2}$$

$\mathbf{a}$ and $\mathbf{f}$ being the fields of acceleration and body force, expresses a necessary and sufficient condition for the balance of linear momentum, while when (1.2) is satisfied, the condition

$$\mathbf{t} = \mathbf{t}', \tag{1.3}$$

where a prime denotes transposition, is equivalent to the balance of moment of momentum with respect to an arbitrary point. In deriving (1.3) it is assumed that there are no body couples; if there are couple stresses, the stress principle itself has to be modified. If body couples and/or couple stresses are present, (1.3) is modified, but (1.2) remains unchanged. Finally, at a surface of discontinuity such as a shock wave or a sound wave the application of Cauchy's stress principle and Euler's law of linear momentum yields the Rankine-Hugoniot condition

$$[\mathbf{t}_{(\mathbf{n})}] + [\rho U\mathbf{v}] = 0, \tag{1.4}$$

where $\mathbf{v}$ is the velocity vector, U is the speed of propagation, and the

bracket denotes a jump; the balance of moment of momentum is satisfied automatically if (1.4) holds.

That the stress principle is not an obvious one is shown by the well known difficulty of making it clear to students, and now that continuum mechanics is no longer included in the training of physicists in many countries, increasing numbers of younger specialists in such fields as solid state physics and the statistical mechanics of fluids find themselves at a loss in following even classical ideas in continuum mechanics, mainly from failure to master the apparently simple concept of stress. Intuition is no more than organized experience. The purpose of this note is to show how one main aspect of mechanical intuition, the subtle idea of stress, grew through the centuries.

It is unnecessary to repeat here the considerations of Cauchy, which may be found, essentially, in any good book on elasticity or the hydrodynamics of viscous fluids, or in any of the older treatises on theoretical physics. Here we shall follow in outline the dark groping from case to case in the earlier researches which gained, one by one, the individual intuitive elements which Cauchy had only to purify of superfluous special properties and unify into one conceptual framework.

Cauchy's stress principle may be separated into four independent statements:

1) The physical dimensions of stress are [force]/[area];

2) Stress is defined upon an imagined boundary dividing the material into two parts;

3) Stress is a vector or vector field equipollent to the action of one part of the material upon another;

4) The direction of the stress vector is not restricted.

These four ideas originated distinctly and underwent some measure of distinct development.

§ 2. Stevin's principle of solidification

That liquids exert forces upon walls confining them is common experience, but it was only from centuries of thought that any definite concept of *internal* pressure was distilled. The earliest reference to the force exerted by fluid on fluid occurs in Archimedes' (A. C. 250) theory of a spherical ocean covering a spherical earth. His Postulate 1 is: "Let it be supposed that a fluid is of such a character that, its parts lying evenly and being continuous, that part which is thrust the less is driven along by that which is thrust the more; and that each of its parts is thrust by the fluid which is above it in a perpendicular direction if the fluid be sunk in

anything or pressed together by anything else." What this means is not clear.

Stevin's *principle of solidification* is often stated as follows: In any fluid at rest, if any portion be replaced by a rigid solid, the force exerted by the remainder will not be altered. No statement so embracing or explicit is to be found, however, before the work of Clairaut (1743). The theory of Stevin (1586) is restricted to incompressible fluids on the surface of a flat earth. He derives what amounts to a correct formula for the pressure on a horizontal bottom; while he nowhere makes any definite assertion about lateral pressure, he succeeds in finding the total force on inclined areas by approximating them by stairshaped walls and then passing to the limit as the steps decrease to zero. Stevin's form of the principle of solidification, spread over several corollaries, refers only to the total force F on a plane bottom. In modern terms, it is the assertion that

$$F = \rho g h A, \tag{2.1}$$

where h is the depth of the plane bottom of area A below the surface of the water, independently of the shape of the vessel.

The work of Pascal (1663) seems to be entirely derivative as far as incompressible fluids are concerned; from his book it became known that air in equilibrium follows much the same laws as does water.

§ 3. Galileo's rupture stress

In the *Two New Sciences* (1638), Salviati undertakes to refute the common error that a long rope cannot hold up so great a weight as a short one. Given a particular rope, he considers attached at C a weight just sufficient to break it, and he asks Simplicio where the break will occur. Simplicio chooses a point D, "because at this point the rope is not strong enough to support, say, 100 lbs." Salviati then imagines the rope fixed at F, just above D, and at E, just below D, he proposes to attach the weight. He says the rope is still subject to the same pull at D, and thus the short segment FE will break again at D, by Simplicio's admission. This *reductio ad absurdum* is a typical example of Galileo's method of concealing his assumptions in rhetoric. In fact, he has assumed that the whole effect on the rope above D of the rope and weight DC is to exert a certain force in the direction DE. That is, *the action of the system below D on that above D is equipollent to a force at D.*

In Galileo's consideration of the rupture of cylindrical beams in tension, he asserts that the force required is as the base area, "since by so much greater are the fibres, the filaments, or the tenacious parts that hold

together the parts of the solid." That is, he assumes that rupture in tension occurs when a certain force per unit area is attained.

§ 4. Pardies' and James Bernoulli's concept of tension in a flexible line

In his study of the catenary curve and the suspension bridge, Pardies (1673) assumed that the form of a flexible line remains unchanged if we solidify any part, or, further, if we replace the parts above two points A and a by *suitable forces acting along the tangents at A and a*. This principle was used in all the later analyses of the catenary.

In a long series of researches on flexible lines of arbitrary thickness and subject to arbitrary distributed loading (1691–1704), James Bernoulli adopted Pardies' principle without any change. He introduced the *tension* (firmitas) explicitly; denoting it by T, we may write as follows one of the forms in which he gave the *general equations of equilibrium for a plane flexible line*:

$$T\frac{dx}{ds} = T_0 - \int_0^s F_x\,ds,$$

$$T\frac{dy}{ds} = -\int_0^s F_y\,ds, \tag{4.1}$$

where F_x and F_y are the components of applied force per unit length. While these results were not published until 1744, meanwhile (1716) Hermann, who had been a pupil of the great James Bernoulli, had evolved them and had published an explicit definition of the "tenacitas vel firmitas": "The tension or compression of a thread or body at any of its points or at an element of the curve is that force of the thread or body which resists that power or force growing from all the applied powers and tending by pulling the thread in opposite directions to tear it apart. This tension exactly equals or is equipollent to that tearing force resulting from all the powers applied to the body." This definition is not as inclusive as its words suggest, for Hermann considered nothing more general than a plane perfectly flexible curve.

§ 5. Variation of tension in the fibres of a beam

In an analysis of rupture by transverse force, Leibniz (1684) *allowed the tensions of the fibres to vary arbitrarily*. His ideas were expressed more

clearly and in greater detail, but without any substantial improvement, by Varignon (1702) and Bülffinger (1729).

In his first work on elastic bending, James Bernoulli (1691) replaced the action of all the fibres of a cross-section by that of a single spring on the convex side. In a series of deep and difficult later researches (1694–1705) on the neutral fibre, most of them still unpublished today, he sought to take into account the variation of tension simultaneously with bending, but he adopted an erroneous hypothesis and went astray. Here he tried in vain to achieve what he had done with entire success for perfectly flexible lines, namely, *to express the principles of mechanics in general, without use of any constitutive equation defining a particular kind of material.*

The work of James Bernoulli was taken up by Parent (1713), who was the first to consider the *equilibrium of interior forces* in addition to the equilibrium of moments in the analysis of a loaded beam. Allowing different moduli for linearly varying normal forces, and representing them on a diagram, he showed that the area of the triangle of compression must equal the area of the triangle of tension. In a single sentence, he remarked also that forces tangent to the cross-section, i.e., *shear stresses*, are necessary for equilibrium. Parent's excellent work, which really foreshadows the stress principle, remained unnoticed and had no influence.

§ 6. Early mathematical hydrostatics

Brilliant as were its successes, the early mathematical hydrostatics developed prior to the introduction of the internal pressure. It is easy to explain to a modern reader how this was possible, since he easily sees the general solution of the differential equations of hydrostatics and can imagine how one or another property of this solution was painfully sought out by the early theorists. Here, not attempting to summarize the achievements, we collect the indications that to some extent foreshadow the concept of pressure. Newton (1687) was the first to *isolate in imagination a part of the interior fluid* and to assert that it is "equally pressed on every side," and "any internal part of a fluid is in the same state as a submerged body." However, he referred only to fluid inclosed within an unmoved spherical vessel which itself is pressed equally on every side, and how near he came to grasping the essential property of *any* fluid in equilibrium subject to *any* field of force is not at all clear. No trace of the concept of pressure can be found in Newton's treatments of hydrodynamical problems, which rest on special devices.

In Taylor's theory of the equilibrium of the atmosphere (1715), the

postulate really used, though not stated, is equivalent to

$$dF = \rho g A \, dh, \tag{6.1}$$

where F is the force which would be exerted on a plane area A at height h. I.e., Taylor regarded Stevin's formula (2.1) as applicable to increments even in a fluid of varying density ρ.

Euler's treatise on hydrostatics (c. 1740) concerns only incompressible fluids near the earth's surface. Euler states explicitly that *in all cases the pressures in all directions at a given point are equal and are normal to the surfaces on which they act.* The work opens with a lemma equivalent to the equation

$$\mathbf{F} = \int_S \rho g h \, d\mathbf{S}. \tag{6.2}$$

The work of Clairaut and MacLaurin avoids use of the internal pressure.

§ 7. Hydraulics

While Daniel Bernoulli (1730) succeeded in relating the speed of a steady flow of incompressible fluid in a tube to the pressure it exerts upon the walls, in regard to the action of fluid on fluid he gave only some vague remarks.

The *internal force* acting upon a slice of fluid in a tube was introduced by John Bernoulli (1740) in his successful program of incorporating Daniel Bernoulli's results within the general framework of mechanics. This pressure, called "moving force" by John Bernoulli, represents the action of fluid upon fluid; while only incompressible fluids flowing in tubes are considered, correct results are obtained for tubes of arbitrary form and for unsteady flow. In effect, John Bernoulli applied to hydraulic *motion* the concept of tension as evolved by James Bernoulli for the equilibrium of a flexible line.

Euler enthusiastically adopted the idea of John Bernoulli and made it the basis of his clear and general presentation of hydraulics in a series of works devoted mainly to concrete problems (1749–1752). Only incompressible fluids are considered. The true *hydraulic pressure* p, having the dimensions of [*force*]/[*area*], plays in these works exactly the same role as it does today.

§ 8. Hydrodynamics

While D'Alembert's essay on hydrodynamics (1749) obtained a number of important and fairly general equations, the march of thought is obscure.

D'Alembert claims to reduce hydrodynamics to hydrostatics, which he says has been reduced in turn to "the single principle of experience, the equality of pressure in all directions," yet he nowhere introduces the pressure into the mathematics; in arriving at the equations for plane and for rotationally symmetric potential flow, he refers to "forces" in the interior but takes these as being reversed accelerations.

Euler's hydrodynamical papers (1750–1766) rest upon use of the *internal hydrodynamic pressure* in full generality, for any fluid in any circumstances. In his first paper (1750), Euler speaks of an element of fluid as being "subject to the pressures of the particles of water surrounding it;" in his second (1752), of "the pressure with which the particles of water everywhere actually act upon each other." The dynamical equation satisfied by the pressure is

$$\rho \mathbf{a} = -\operatorname{grad} p + \rho \mathbf{f}; \tag{8.1}$$

here, for the first time, the pressure is a *field* varying in space and time, and the only way in which it falls short of Cauchy's stress vector is that the force of the pressure is assumed to be normal to any surface on which it acts. Although Euler never gives any fuller or clearer explanation in words except in the case of equilibrium, the *mathematical treatment* of pressure is entirely explicit and clear. Indeed, the concept is so natural that once it is achieved, one has only to look upon the equations, which speak for themselves. Euler's own earlier writings on hydrostatics and hydraulics, where only special cases occur, include more careful physical motivation. This is but one of many examples in the history of science indicating that as soon as a definite mathematical theory has been stated, "physical intuition" begins to develop to such a point that what has been gotten in fact as end result of long struggles of thought becomes "obvious."

§ 9. Resultant shear force, and the general theory of deformable lines

While in his earliest research on elasticity (1727) Euler succeeded in taking into account a linear variation of tension over the cross-section of a bent beam, so that he derived James Bernoulli's law of bending from Hooke's law of extension, he was unsatisfied with the statical foundation of the theory of the elastica and sought for most of his lifetime, in repeated trials, to establish the equations of the elastica, like those of the general catenary, directly from the balance of forces acting on a portion of the curve. That this is impossible is one of the major pieces of evidence

that led him, finally, to perceive that the balance of moments is a mechanical law independent, in general, of the balance of forces. Before achieving this degree of clarity, however, he drew profit from his studies of systems of forces. In a passage written about 1742 in one of his unpublished notebooks, Euler calculated the force which has to be supplied at a point A on the elastica in order that the band to one side of A retain its shape when all the material to the other side of A is cut away; he found that it is not tangential; i.e., *shear force* is generally necessary in addition to the tension that suffices in the perfectly flexible case. In 1771 he finally succeeded in separating the statical equations from the constitutive equations. Introducing the *stress resultant* acting upon a plane curve as a vector having tangential and normal components T and V, Euler obtained the *fully general statical equations* when there are no distributed couples, viz,

$$\begin{aligned} \frac{dT}{ds} - V\kappa &= -F_t, \\ \frac{dV}{ds} + T\kappa &= -F_n, \\ \frac{dM}{ds} - V &= 0, \end{aligned} \tag{9.1}$$

where κ is the curvature and M is the bending moment. Later (1774) he announced the principle

$$\mathbf{S}_+ + \mathbf{S}_- = 0, \tag{9.2}$$

where $\mathbf{S} = T\mathbf{t} + V\mathbf{n}$ and where the $+$ and $-$ signs refer to the effects of the material on the opposite sides of the point in question.

§ 10. Explicit recognition of shear stress

In a celebrated paper (1773), Coulomb revived and made more explicit the considerations of Parent. He envisaged arbitrary variation of the forces, both normal and tangential, acting on the cross-section of a beam subject to transverse terminal load. Like Parent's, these forces may be regarded as *interior stresses* in the plane case. Coulomb actually wrote down, in integrated form, *all equations of equilibrium* for the cross-section; thus he concluded that *the area under the curve of pressures must equal that under the curve of tensions*, and he calculated the *resultant shear stress* in terms of the load. Going beyond Parent, Coulomb was the first to consider

the stresses acting upon *different planes* through a point; he calculated the shear stress upon a plane *arbitrarily inclined* to the direction of the applied force in a beam supposed to be in a state of pure compression, and he showed that *the shear stress is greatest on a plane inclined at* 45°.

While Coulomb's ideas were clear, his achievements should not be exaggerated. Failing by far to reach the level attained by Euler in hydrodynamics and in the theory of deformable lines, Coulomb did not obtain any differential equations, nor did he foreshadow the stress tensor.

§ 11. A remark concerning Cauchy

From the above account, it is clear that every conceptual element in Cauchy's theory of stress was to be found in one or another of the more special theories constructed in the previous century. Moreover, in researches of Fresnel done in 1821–1822, with which Cauchy must certainly have been familiar, many of Cauchy's results are more or less implied, although in Fresnel's work the concepts of stress and strain are always connected through a presumed linearly elastic response.

Thus it might seem that Cauchy's achievement in formulating and developing the general theory of stress was an easy one. It was not. Cauchy's concept has the simplicity of genius. Its profound originality is fully delineated only against the background of the century of achievement by the brilliant geometers who preceded, treating special kinds and cases of deformable bodies by complicated and sometimes incorrect ways without ever hitting upon this basic idea, which immediately became and has remained the foundation of the mechanics of distributed matter.

Nothing is harder to surmount than a corpus of true but too special knowledge; to reforge the tradition of his forebears is the greatest originality a man can have.

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ON THE BASIC EQUATIONS OF SOIL DYNAMICS

H. S. TSIEN (Quian Xue-sen)
Peking, China

The basic equations of soil dynamics were first proposed by N. M. Gersevanov [1]. He considered the soil as a two phase complex, consisting of an elastic skeleton of a solid phase filled with the water of the liquid phase. Tan Tjong-kie in his paper on consolidation of soils [2] remarked on the insufficiency of the study of a soil skeleton as an elastic body and on the necessity of introducing into the dependence between the stresses and deformations a viscous term which takes into account the property of motion of the soil. A recent study of the structure and physical properties of soils [3] seems to have confirmed Tan Tjong-kie's reasoning.

In the present paper we will construct on the basis of the equations of N. M. Gersevanov and the reasoning of Cheng Tsun-tsi new basic equations of the dynamics of soils.

Let the displacements of the soil skeleton of the solid phase in the directions of the coordinate axes x, y, z be denoted by u_x, u_y, u_z. Then the strain components of the solid phase are

$$\varepsilon_x = \frac{\partial u_x}{\partial x}, \quad \varepsilon_y = \frac{\partial u_y}{\partial y}, \quad \varepsilon_z = \frac{\partial u_z}{\partial z},$$

(1)

$$\gamma_{xy} = \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right), \quad \gamma_{yz} = \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right), \quad \gamma_{xz} = \left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right),$$

and we obtain for the change of volume

$$\bar{\varepsilon} = (\varepsilon_x + \varepsilon_y + \varepsilon_z). \tag{2}$$

The stresses in the solid phase corresponding to these strains have the components

$$\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}, \quad \bar{\sigma} = \tfrac{1}{3}(\sigma_x + \sigma_y + \sigma_z). \tag{3}$$

In accordance with Cheng Tsun-Tsi the viscous flow of the soil skeleton may be assumed to be incompressible. Denoting the shear modulus by G, Young's modulus by E, Poisson's ratio by ν and the coefficient of viscosity by η, we obtain the following relationships between the stresses and

strains:

$$G = \frac{E}{2(1+\nu)}, \tag{4}$$

$$\begin{aligned}
&\frac{\partial \varepsilon_x}{\partial t} - \tfrac{1}{3}\frac{\partial \bar{\varepsilon}}{\partial t} = \tfrac{1}{2}\left\{\frac{1}{G}\frac{\partial}{\partial t} + \frac{1}{\eta}\right\}(\sigma_x - \bar{\sigma}), \quad \frac{\partial}{\partial t}\gamma_{xy} = \left\{\frac{1}{G}\frac{\partial}{\partial t} + \frac{1}{\eta}\right\}\tau_{xy}, \\
&\frac{\partial \varepsilon_y}{\partial t} - \tfrac{1}{3}\frac{\partial \bar{\varepsilon}}{\partial t} = \tfrac{1}{2}\left\{\frac{1}{G}\frac{\partial}{\partial t} + \frac{1}{\eta}\right\}(\sigma_y - \bar{\sigma}), \quad \frac{\partial}{\partial t}\gamma_{yz} = \left\{\frac{1}{G}\frac{\partial}{\partial t} + \frac{1}{\eta}\right\}\tau_{yz}, \\
&\frac{\partial \varepsilon_z}{\partial t} - \tfrac{1}{3}\frac{\partial \bar{\varepsilon}}{\partial t} = \tfrac{1}{2}\left\{\frac{1}{G}\frac{\partial}{\partial t} + \frac{1}{\eta}\right\}(\sigma_z - \bar{\sigma}), \quad \frac{\partial}{\partial t}\gamma_{zx} = \left\{\frac{1}{G}\frac{\partial}{\partial t} + \frac{1}{\eta}\right\}\tau_{zx}.
\end{aligned} \tag{5}$$

We will subdivide the stresses into two parts: the pressure of the water p, causing filtration of the liquid phase in the skeleton of the solid phase, and the part of the deformation of the skeleton and the stress which arises due to the external forces applied to the boundary of the soil. The last is the stress expressed by the relations (3) without the water pressure p, where

$$\bar{\varepsilon} = \frac{\bar{\sigma} - p}{\theta}, \quad \theta = \frac{E}{3(1-2\nu)} = \frac{2G(1+\nu)}{3(1-2\nu)}, \tag{6}$$

and θ is the bulk modulus. Therefore we obtain the following connections between the components of the deviatorial stresses and strains:

$$\begin{aligned}
&\left(\eta\frac{\partial}{\partial t} + G\right)\sigma_x = 2G\eta\frac{\partial \varepsilon_x}{\partial t} + \left[\eta(\theta - \tfrac{2}{3}G)\frac{\partial}{\partial t} + \theta G\right]\bar{\varepsilon} + \left(\eta\frac{\partial}{\partial t} + G\right)p, \\
&\left(\eta\frac{\partial}{\partial t} + G\right)\sigma_y = 2G\eta\frac{\partial \varepsilon_y}{\partial t} + \left[\eta(\theta - \tfrac{2}{3}G)\frac{\partial}{\partial t} + \theta G\right]\bar{\varepsilon} + \left(\eta\frac{\partial}{\partial t} + G\right)p, \\
&\left(\eta\frac{\partial}{\partial t} + G\right)\sigma_z = 2G\eta\frac{\partial \varepsilon_z}{\partial t} + \left[\eta(\theta - \tfrac{2}{3}G)\frac{\partial}{\partial t} + \theta G\right]\bar{\varepsilon} + \left(\eta\frac{\partial}{\partial t} + G\right)p, \\
&\left(\eta\frac{\partial}{\partial t} + G\right)\tau_{xy} = G\eta\frac{\partial \gamma_{xy}}{\partial t}, \\
&\left(\eta\frac{\partial}{\partial t} + G\right)\tau_{yz} = G\eta\frac{\partial \gamma_{yz}}{\partial t}, \\
&\left(\eta\frac{\partial}{\partial t} + G\right)\tau_{zx} = G\eta\frac{\partial \gamma_{zx}}{\partial t}.
\end{aligned} \tag{7}$$

We will now proceed to the study of the water pressure and the

forces arising in the skeleton due to the movement of the water. If we denote the porosity of the soil by f, it may be assumed that the pressure is distributed between the skeleton and the liquid phase in the ratio $(1 - f)$ and f, i.e., a pressure $(1 - f)p$ acts on the solid phase, a pressure fp on the liquid phase. If we denote the mean velocity of the motion of the liquid phase in the pores of the skeleton by v_x, v_y, v_z, we obtain the mean velocity relative to the solid phase

$$v_x - \frac{\partial u_x}{\partial t}, \quad v_y - \frac{\partial u_y}{\partial t}, \quad v_z - \frac{\partial u_z}{\partial t}.$$

The viscous force, arising as the result of this relative velocity, is determined in the usual manner by filtration theory, except that the velocity of motion in the usual filtration theory is calculated with the help of the area of the cross-section of the continuous porous medium, and not by use of the area of the pore of the skeleton. In order to transform the velocity related to the area of clearance to the velocity of the section of the medium, one must multiply it by the porosity f. In addition, in the usual filtration calculation the so-called coefficient of permeability k_ϕ is determined by the difference of the height of the water head, and not by the difference of pressure. Therefore the volume forces acting on the skeleton have the form

$$\kappa\left(v_x - \frac{\partial u_x}{\partial t}\right), \quad \kappa\left(v_y - \frac{\partial u_y}{\partial t}\right), \quad \kappa\left(v_z - \frac{\partial u_z}{\partial t}\right),$$

where

$$\kappa = \frac{f^2 \rho_2 g}{k_\phi} \tag{8}$$

with ρ_2 the water density and g the gravitational acceleration. If we denote the density of the solid phase by ρ_1, the mass of the solid phase per unit volume of soil will be $\gamma_1 = \rho_1(1 - f)$. The equations of motion for the solid phase have now the form

$$\begin{aligned}
\gamma_1 \frac{\partial^2 u_x}{\partial t^2} &= \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} - (1 - f)\frac{\partial p}{\partial x} + \kappa\left(v_x - \frac{\partial u_x}{\partial t}\right), \\
\gamma_1 \frac{\partial^2 u_y}{\partial t^2} &= \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} - (1 - f)\frac{\partial p}{\partial y} + \kappa\left(v_y - \frac{\partial u_y}{\partial t}\right), \qquad (9) \\
\gamma_1 \frac{\partial^2 u_z}{\partial t^2} &= \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} - (1 - f)\frac{\partial p}{\partial z} + \kappa\left(v_z - \frac{\partial u_z}{\partial t}\right) - \gamma_1 g.
\end{aligned}$$

The mass of the liquid phase per unit volume of soil will be $\gamma_2 = \rho_2 f$.

Neglecting small higher order quantities, we obtain the equations of motion of the liquid phase in the form

$$
\begin{aligned}
\gamma_2 \frac{\partial v_x}{\partial t} &= -f\frac{\partial p}{\partial x} - \kappa\left(v_x - \frac{\partial u_x}{\partial t}\right), \\
\gamma_2 \frac{\partial v_y}{\partial t} &= -f\frac{\partial p}{\partial y} - \kappa\left(v_y - \frac{\partial u_y}{\partial t}\right), \\
\gamma_2 \frac{\partial v_z}{\partial t} &= -f\frac{\partial p}{\partial z} - \kappa\left(v_z - \frac{\partial u_z}{\partial t}\right) - \gamma_2 g.
\end{aligned} \tag{10}
$$

There still remains one more equation, namely the equation of continuity for the liquid

$$
\frac{\partial \gamma_2}{\partial t} + \frac{\partial \gamma_2 v_x}{\partial x} + \frac{\partial \gamma_2 v_y}{\partial y} + \frac{\partial \gamma_2 v_z}{\partial z} = 0. \tag{11}
$$

If we neglect small higher order quantities, we obtain

$$
\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = -\frac{1}{\rho_2 f}\left\{\rho_2 \frac{\partial f}{\partial t} + f\frac{\partial \rho_2}{\partial t}\right\}.
$$

In the general case, water is assumed to be incompressible; consequently ρ_2 is constant and

$$
f\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) = -\frac{\partial f}{\partial t}.
$$

If we also assume that the grains of the solid phase are incompressible, then

$$
-\frac{\partial f}{\partial t} = \frac{\partial (1-f)}{\partial t} = (1-f)\frac{\partial \bar{\varepsilon}}{\partial t}.
$$

Consequently, the equation of continuity has the form

$$
f\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) = (1-f)\frac{\partial \bar{\varepsilon}}{\partial t}. \tag{12}
$$

The equations (7), (9), (10) and (12) form the complete system which has only to be supplemented by boundary and initial conditions. The last are readily written down. It must only be noted here that, if the normal to the boundary which is impermeable to water is denoted by n, the component of the velocity of the liquid phase in the direction of the

normal by $-fv_n$, then one has on this boundary the condition

$$\frac{\partial p}{\partial n} = 0. \tag{13}$$

In addition, the water pressure p may not be a negative quantity; its minimum value is zero. If the pressure p on some surface of a soil mass is equal to zero, there may be no water at this surface, and therefore

$$fv_n = \frac{\partial n}{\partial t}. \tag{14}$$

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APPROXIMATE ANALYSIS OF SHELLS WITH HOLES BY THE METHODS OF POTENTIAL THEORY

D. V. VAINBERG and A. L. SINIAVSKII
Kiev, U.S.S.R.

In accordance with the technical bending theory of thin shells [1], the elastic equilibrium of a circular cylindrical shell of medium length is described by the system of differential equations

$$\begin{aligned}
&\frac{\partial^2 u}{\partial \alpha^2} + \frac{1-\nu}{2}\frac{\partial^2 u}{\partial \beta^2} + \frac{1+\nu}{2}\frac{\partial^2 v}{\partial \alpha \partial \beta} + \frac{\nu}{R}\frac{\partial w}{\partial \alpha} = -\frac{1-\nu^2}{Eh}X,\\
&\frac{1+\nu}{2}\frac{\partial^2 u}{\partial \alpha \partial \beta} + \frac{\partial^2 v}{\partial \beta^2} + \frac{1-\nu}{2}\frac{\partial^2 v}{\partial \alpha^2} + \frac{1}{R}\frac{\partial w}{\partial \beta} = -\frac{1-\nu^2}{Eh}Y, \qquad (1)\\
&\frac{\nu}{R}\frac{\partial u}{\partial \alpha} + \frac{1}{R}\frac{\partial v}{\partial \beta} + \frac{h^2}{12}\Delta\Delta w + \frac{1}{R^2}w = \frac{1-\nu^2}{Eh}Z.
\end{aligned}$$

Here α, β are the (dimensional) coordinates of points on the middle surface of the shell, u, v, w the displacements, X, Y, Z the components of the external loading, R the radius of curvature of the middle surface, h the thickness of the shell, E the modulus of elasticity and ν Poisson's ratio of the material.

Analytic solutions, known in the literature, of boundary problems for the system of equations (1) were obtained either in the form of trigonometric series by Fourier's method or by the variational method. However, these methods are not directly applicable to problems of the state of stress of shells with holes.

For the study of complicated problems it is convenient to apply the methods of potential theory. These methods were developed for the solution of the problems of Dirichlet and Neumann for the Poisson equation; in the theory of elasticity, they have been applied to the investigation of three dimensional problems (the method of C. Somigliana [2]). Potential theory was applied to statical problems of thin shells by N. A. Kil'chevskii [3—4].

In potential theory, one starts from the so-called Green's formula. For the system of equations (1) this formula may be obtained by application of the ordinary methods [5] and it can be represented in the form, expressing the theorem of reciprocal work,

$$(2)\quad \iint\limits_{(S)}(Xu^* + Yv^* + Zw^*)dS +$$

$$+ \oint\limits_{(L)}(T_{1p}u^* + T_{2p}v^* + Q_pw^* - M_p\varphi^*{}_p + M_{pl}\varphi^*{}_l)dL =$$

$$= \iint\limits_{(S)}(X^*u + Y^*v + Z^*w)dS +$$

$$+ \oint\limits_{(L)}(T^*_{1p}u + T^*_{2p}v + Q^*{}_pw - M^*{}_p\varphi_p + M^*_{pl}\varphi_l)dL.$$

Here S is the region on the cylindrical surface, L is the contour bounding the region S, $\mathbf{p}$, $\boldsymbol{l}$ are the normal and tangential directions with respect to the contour L, u, v, w and u^*, v^*, w^* are two arbitrary systems of displacements, φ_p, φ_l, $\varphi^*{}_p$, $\varphi^*{}_l$ are the angles of rotation corresponding to the above stated system of displacements, X, Y, Z and X^*, Y^*, Z^* are two systems of surface loads, T_{1p}, T_{2p}, T^*_{1p}, T^*_{2p} are tensile forces acting in the plane with normal $\mathbf{p}$ in the directions of the axes α and β, Q_p, M_p, M_{pl}, $Q^*{}_p$, $M^*{}_p$, M^*_{pl} are the shear forces, bending moments and torques acting in the plane with normal $\mathbf{p}$ and tangent $\boldsymbol{l}$.

For the formula (2) to be true one has to impose on the form of the contour L some limitation of the type of Liapunov conditions [6] which we will assume to be fulfilled.

In the method to be considered here the choice of the "auxiliary" system of displacements u^*, v^*, w^* is essential. In order to obtain an integral representation of the quantities u, v, w, one must apply in the capacity of auxiliary displacements those due to the effect of concentrated forces.

In [7—8] the auxiliary state was selected in such a manner that the application of the formula (2) led to a system of Fredholm equations which did not contain contour integrals.

In the problem under consideration of a shell with openings it is essential to retain the contour integrals in the formulae. Thus, one succeeds in reducing a two-dimensional problem of the theory of shells to functional equations determining the unknown functions in terms of their values on the contour.

In order to show the possibilities of the application of potential theory, we consider the example of a cylindrical shell with a hole. Let the contour of the hole be a space curve which for development of the middle surface of the shell into the plane assumes the shape of an ellipse with semi-axes

a_0 and b_0 (Fig. 1). The parametric equation of an ellipse the center of which has the coordinates $(a/2, R\pi/2)$ may be written in the form

$$\alpha = \frac{a}{2} + b_0 \sin\theta, \qquad \beta = \frac{R\pi}{2} + a_0 \cos\theta. \tag{3}$$

The quantity θ is to increase as one moves along the ellipse in a clockwise direction.

A constant tensile force T_0 is to act on the ends of the shell. At its ends we have the boundary conditions

$$T_1 = T_0, \quad w = v = M_1 = 0. \tag{4}$$

The following conditions are fulfilled on the contours of the holes:

$$T_{1p} = T_{2p} = Q_p = M_p = 0. \tag{5}$$

We set Poisson's ratio equal to zero.

The presence of holes in the shell changes in an essential manner the type of the state of deformation. In the absence of the hole and for the boundary conditions (4) there exists in the cylindrical shell under consideration a homogeneous membrane state of stress. Due to the presence of the opening, a concentration of the membrane stresses occurs. However, in contrast to the case of a plane field, transverse deflections occur which cause large bending stresses.

We turn now to the formula (2), assuming that the system of displacements u, v, w and its corresponding angles of rotation, forces, moments and surface loads refer to the above mentioned cylindrical shell with a hole. The quantities u^*, v^*, w^* and the corresponding angles of rotation, forces, moments and surface forces are to refer to the cylindrical shell without a hole, loaded by the system of normal concentrated forces, shown in Fig. 2. At the ends we will give the boundary conditions

$$T^*_1 = M^*_1 = v^* = w^* = 0. \tag{6}$$

In application to the state described above the formula (2) assumes the form

$$w = -\oint_{(L_1)} (T^*_{1p}u + T^*_{2p}v + Q^*{}_p w - M^*{}_p\varphi_p + M^*_{pl}\varphi_l)dL_1 + \oint_{(L_2)} T_0 u^* dL_2, \tag{7}$$

where L_1, L_2 are the segments of the contours, indicated in Fig. 1.

The expression (7) is an integral representation of the transverse displacements w of a shell with an opening. It is analogous to the known integral representation of harmonic functions, and likewise to C. Somigliana's formulae in the three dimensional theory of elasticity.

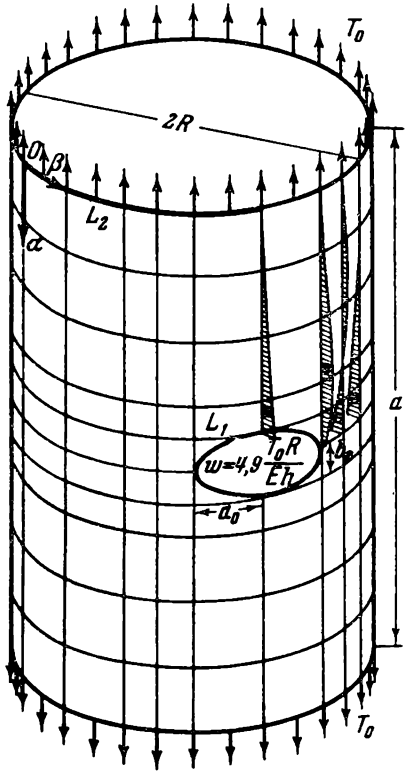

Fig. 1

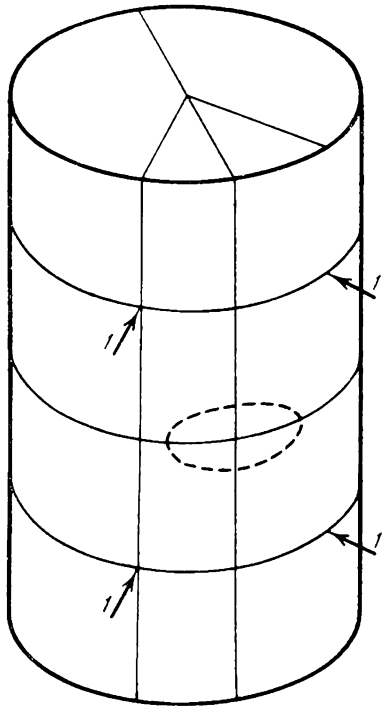

Fig. 2

The quantities T^*_{1p}, T^*_{2p}, Q^*_p, M^*_p, M^*_{pl} may be found in the form of ordinary or double trigonometric series [1]. The double series have satisfactory convergence for not too small values of h/R. [1]

A calculation shows that

$$\oint_{(L_2)} T_0 u^* dL_2 = 0.$$

The integral representation (7) contains terms involving as multipliers

[1] In the example given below we have taken $h/R = 1/30$. In the case of small values h/R one must use ordinary series.

the quantity h^2/R^2. Omitting these terms, we obtain, finally,

$$w(\alpha, \beta) = \sum_{m=1,3\cdots} \sum_{n=0,2\cdots} w_{mn} \sin \frac{m\pi\alpha}{a} \cos \frac{n\beta}{R}, \tag{8}$$

where

$$w_{mn} = \frac{32\pi}{a^3R^2} \frac{m^2}{\Delta_{mn}} \oint_{(L_1)} \left[-\frac{n^2}{R^2} \sin \frac{m\pi\alpha}{a} \cos \frac{n\beta}{R} \cos(p, \alpha) + \right.$$

$$\left. + \frac{mn\pi}{aR} \cos \frac{m\pi\alpha}{a} \sin \frac{n\beta}{R} \cos(p, \beta) \right] u dL_1 +$$

$$+ \frac{32\pi}{a^3R^2} \frac{m^2}{\Delta_{mn}} \oint_{(L_1)} \left[\frac{mn\pi}{aR} \cos \frac{m\pi\alpha}{a} \sin \frac{n\beta}{R} \cos(p, \alpha) - \right.$$

$$\left. - \frac{m^2\pi^2}{a^2} \sin \frac{m\pi\alpha}{a} \cos \frac{n\beta}{R} \cos(p, \beta) \right] v dL_1,$$

$$\Delta_{mn} = \frac{D}{Eh} \left(\frac{m^2\pi^2}{a^2} + \frac{h^2}{R^2} \right)^4 + \frac{1}{R^2} \frac{m^4\pi^4}{a^4},$$

$$\cos(p, \alpha) = -\frac{\sin\theta}{\sqrt{1 + (\varepsilon^2 - 1)\cos^2\theta}},$$

$$\cos(p, \beta) = -\frac{\varepsilon \cos\theta}{\sqrt{1 + (\varepsilon^2 - 1)\cos^2\theta}},$$

$$\varepsilon = \frac{b_0}{a_0}, \quad dL_1 = a_0 \sqrt{1 + (\varepsilon^2 - 1)\cos^2\theta}\, d\theta.$$

The relation (8) itself represents a functional equation relating the magnitudes of the displacements u and v on the contour of the ellipse to the transverse deflection w over the entire surface of the shell.

One may still combine the functional equation (8) with the first two equations of the system of differential equations (1)

$$\frac{\partial^2 u}{\partial \alpha^2} + \frac{1}{2} \frac{\partial^2 u}{\partial \beta^2} + \frac{1}{2} \frac{\partial^2 v}{\partial \alpha \partial \beta} = 0,$$

$$\frac{1}{2} \frac{\partial^2 u}{\partial \alpha \partial \beta} + \frac{\partial^2 v}{\partial \beta^2} + \frac{1}{2} \frac{\partial^2 v}{\partial \alpha^2} + \frac{1}{R} \frac{\partial w}{\partial \beta} = 0. \tag{9}$$

The boundary conditions for the equations (9) at the ends of the shell have the form

$$T_1 = T_0, \; v = 0, \tag{10}$$

while on the contour of the hole

$$T_{1p} = T_{2p} = 0. \tag{11}$$

The remaining boundary conditions of (4) and (5) were used for the construction of the functional equation (8).

Thus, the system of the differential equations of equilibrium (1) are found to have been replaced by a system consisting of two of the equilibrium equations and one integral equation, following from the theorem of reciprocal work. This replacement is found to be especially useful for the study of the transverse deflections of a shell with openings.

The advantage of this method consists of the fact that the displacements u and v enter into (8) only under the sign of the contour integral. Assuming the functions $u(\theta)$, $v(\theta)$ to have the form of trigonometric polynomials in θ

$$\begin{aligned} u(\theta) &= u_1 \sin\theta + u_3 \sin 3\theta + \cdots + u_{2k-1} \sin (2k-1)\theta, \\ v(\theta) &= v_1 \cos\theta + v_3 \cos 3\theta + \cdots + v_{2k-1} \cos (2k-1)\theta, \end{aligned} \tag{12}$$

and substituting these expressions in (8), we obtain a certain form of the representation of the function $w(\alpha, \beta)$, which contains the unknown constants $u_1, u_3, \cdots, u_{2k-1}, v_1, v_3, \cdots, v_{2k-1}$, and, in fact,

$$w = \sum_{m=1,3\cdots}^{\infty} \sum_{n=0,2\cdots}^{\infty} w_{mn} \sin \frac{m\pi\alpha}{a} \cos \frac{n\beta}{R}, \tag{13}$$

where

$$w_{mn} = \sum_{j=1,3}^{2k-1} u_j(C_1H_{1j} + C_2H_{2j}) + \sum_{j=1,3}^{2k-1} v_j(C_3H_{3j} + C_4H_{4j}),$$

$$C_1 = \frac{32\pi a_0 m^2 n^2}{a^3R^4\Delta_{mn}} (-1)^{(m+n-1)/2},$$

$$C_2 = -\frac{32\pi^2 b_0 m^3 n}{a^4R^3\Delta_{mn}} (-1)^{(m+n-1)/2},$$

$$C_3 = -\frac{32\pi^2 a_0 m^3 n}{a^4R^3\Delta_{mn}} (-1)^{(m+n-1)/2},$$

$$C_4 = \frac{32\pi^3 b_0 m^4}{a^5R^2\Delta_{mn}} (-1)^{(m+n-1)/2},$$

$$H_{1j} = \int_0^{\pi/2} \sin\theta \sin j\theta \cos(m\pi\gamma \sin\theta) \cos(\lambda n \cos\theta) d\theta,$$

$$H_{2j} = \int_0^{\pi/2} \cos\theta \sin j\theta \sin(m\pi\gamma \sin\theta) \sin(\lambda n \cos\theta) d\theta,$$

$$H_{3j} = \int_0^{\pi/2} \sin\theta \cos j\theta \sin(m\pi\gamma \sin\theta) \sin(\lambda n \cos\theta) d\theta,$$

$$H_{4j} = \int_0^{\pi/2} \cos\theta \cos j\theta \cos(m\pi\gamma \sin\theta) \cos(\lambda n \cos\theta) d\theta,$$

$$\gamma = \frac{b_0}{a}, \quad \lambda = \frac{a_0}{R}.$$

For the evaluation of the integrals H_{1j}, H_{2j}, H_{3j} and H_{4j}, one has to use Bessel's formulae

$$J_{2n}(X) = \frac{2}{\pi} \int_0^{\pi/2} \cos(x \sin\theta) \cos 2n\theta \, d\theta,$$

$$J_{2n+1}(X) = \frac{2}{\pi} \int_0^{\pi/2} \sin(x \sin\theta) \sin(2n+1)\theta \, d\theta, \tag{14}$$

and also the trigonometric expansions

$$\cos(x \sin\theta) = J_0(x) + 2\sum_{i=1}^{\infty} J_{2j}(x) \cos 2j\theta,$$

$$\sin(x \sin\theta) = 2\sum_{i=1}^{\infty} J_{2j-1}(x) \sin(2j-1)\theta. \tag{15}$$

As an example we consider the quantity H_{1j}. Taking into consideration that $j = 2p - 1$, $p = 1, 2, \cdots$, we obtain

$$H_{1,2p-1} = \tfrac{1}{2} \int_0^{\pi/2} [\cos 2(p-1)\theta - \cos 2p\theta] \cos(m\pi\gamma \sin\theta) \times \tag{16}$$

$$\times [J_0(\lambda n) + 2\sum_{q=1}^{\infty} (-1)^q J_{2q}(\lambda n) \cos 2q\theta] d\theta =$$

$$= \frac{\pi}{4} J_0(\lambda n)[J_{2(p-1)}(m\pi\gamma) - J_{2p}(m\pi\gamma)] +$$

$$+ \sum_{q=1}^{\infty} (-1)^q J_{2q}(\lambda n) \int_0^{\pi/2} [\cos 2(p-1)\theta - \cos 2p\theta] \times$$

$$\times \cos 2q\theta \cos(m\pi\gamma \sin\theta) d\theta =$$

$$= \frac{\pi}{4} J_0(\lambda n)[J_{2(p-1)}(m\pi\gamma) - J_{2p}(m\pi\gamma)] +$$

$$+ \frac{\pi}{4} \sum_{q=1}^{\infty} (-1)^q J_{2q}(\lambda n)[J_{2(q+p-1)}(m\pi\gamma) +$$

$$+ J_{2(q-p+1)}(m\pi\gamma) - J_{2(q+p)}(m\pi\gamma) - J_{2(q-p)}(m\pi\gamma)].$$

Assuming that $w(\alpha, \beta)$ has the form (13), we determine the functions $u(\alpha, \beta)$ and $v(\alpha, \beta)$ from the system of differential equations (9) for the boundary conditions (10)—(11).

The system (9) is analogous to the system of equations for the plane problems of the theory of elasticity. Therefore the determination of $u(\alpha, \beta)$ and $v(\alpha, \beta)$ may be reduced to a boundary value problem of the theory of functions of a complex variable by the method studied in §§ 29, 31 of [9].

The system (9) is conveniently written in the form

$$\begin{aligned} \frac{\partial T_1}{\partial \alpha} + \frac{\partial S}{\partial \beta} &= 0, \\ \frac{\partial T_2}{\partial \beta} + \frac{\partial S}{\partial \alpha} &= 0, \end{aligned} \tag{17}$$

where

$$T_1 = Eh \frac{\partial u}{\partial \alpha},$$

$$T_2 = Eh \left(\frac{\partial v}{\partial \beta} + \frac{1}{R} w \right),$$

$$S = \frac{Eh}{2} \left(\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha} \right).$$

We introduce the notation

$$\bar{T}_2 = T_2 - \frac{Eh}{R} w. \tag{18}$$

Then the system (17) assumes the form

$$\begin{aligned} \frac{\partial T_1}{\partial \alpha} + \frac{\partial S}{\partial \beta} &= 0, \\ \frac{\partial T_2}{\partial \beta} + \frac{\partial S}{\partial \beta} &= -\frac{Eh}{R} \frac{\partial w}{\partial \beta}, \end{aligned} \tag{19}$$

$$T_1 = Eh\frac{\partial u}{\partial \alpha},$$

$$\bar{T}_2 = Eh\frac{\partial v}{\partial \beta}, \tag{19}$$

$$S = \frac{Eh}{2}\left(\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha}\right).$$

The boundary conditions for the system (19) must be formulated in the following manner:

$$T_1 = T_0,\ v = 0 \quad \text{for} \quad \alpha = 0,\ \alpha = a, \tag{20}$$

on the contour of the developed holes

$$\bar{T}_{pp} = -\frac{\varepsilon^2 \cos^2\theta}{1 + (\varepsilon^2 - 1)\cos^2\theta}\frac{Eh}{R}w,$$

$$\bar{T}_{pl} = -\frac{\varepsilon \sin\theta\cos\theta}{1 + (\varepsilon^2 - 1)\cos^2\theta}\frac{Eh}{R}w. \tag{21}$$

Here

$$\bar{T}_{pp} = \frac{1}{1 + (\varepsilon^2 - 1)\cos^2\theta}(T_1 \sin^2\theta + \varepsilon^2\bar{T}_2\cos^2\theta + 2\varepsilon S \sin\theta\cos\theta),$$

$$\bar{T}_{pl} = \frac{1}{1 + (\varepsilon^2 - 1)\cos^2\theta}[\varepsilon(\bar{T}_2 - T_1)\sin\theta\cos\theta + S(\sin^2\theta - \varepsilon^2\cos^2\theta)]. \tag{22}$$

The system (19) with the boundary conditions (20)—(21) may be considered to be a system of differential equations which holds true for the solution of plane problems of the theory of elasticity for an infinite strip with a series of elliptic holes. On the surface of the strip one has the distributed additional loading

$$Y = \frac{Eh}{R}\frac{\partial w}{\partial \beta}. \tag{23}$$

The normal force $\bar{T}_{pp}$ and the tangential force $\bar{T}_{pl}$ act on the contour of the elliptic hole. The tensile force T_0 acts along the straight edges bounding the strip and, in addition, it is subject to the condition $v = 0$.

A particular solution of the system (19) is easily found in the form of double trigonometric series. We will effect the solution of the homogeneous problem on the basis of the known methods of §§ 32, 50, 87 of [9].

Owing to the conditions (20) on the straight edges and the assumption that Poisson's ratio is equal to zero, the width of the strip plays no role in the solution of the homogeneous problem and it may be assumed to be infinite.

If the distance between the holes is larger than three times their dimensions, one may neglect the interaction of the holes for the solution of the plane problem of the theory of elasticity. In this case, the problem reduces to the consideration of the infinite plane with one elliptic hole.

The magnitudes of the forces on the contour of the elliptic hole must be represented in the form of trigonometric series. It is convenient to use the methods of harmonic analysis for this purpose. The coefficients of these trigonometric series depend on

$$u_1, u_3, \cdots, u_{2k-1}, v_1, v_3, \cdots, v_{2k-1}.$$

As a result of the solution of the system (19) for the boundary conditions (20)—(21) we obtain the values of the quantities $u(\theta)$, $v(\theta)$ in the form of the trigonometric series

$$\begin{aligned} u(\theta) &= A_1 \sin\theta + A_3 \sin 3\theta + \cdots, \\ v(\theta) &= B_1 \cos\theta + B_3 \cos 3\theta + \cdots. \end{aligned} \tag{24}$$

The coefficients $A_1, A_3, \cdots, B_1, B_3$ are linear functions of the quantities $u_1, u_3, \cdots, u_{2k-1}, v_1, v_3, \cdots, v_{2k-1}$ and T_0. Comparing coefficients of the first $(2k-1)$ terms of the trigonometric series (24) with the corresponding coefficients of the polynomials (12), we obtain the system of linear algebraic equations

$$\begin{aligned} u_1 &= A_1(u_1, u_3, \cdots, u_{2k-1}, v_1, v_3, \cdots, v_{2k-1}, T_0), \\ u_3 &= A_3(u_1, u_3, \cdots, u_{2k-1}, v_1, v_3, \cdots, v_{2k-1}, T_0), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ u_{2k-1} &= A_{2k-1}(u_1, u_3, \cdots, u_{2k-1}, v_1, v_3, \cdots, v_{2k-1}, T_0). \end{aligned} \tag{25}$$

As an illustration, we executed the numerical work for the following values of the parameters:

$$\frac{h}{R} = \frac{1}{30}, \quad \varepsilon = \frac{b_0}{a_0} = \frac{1}{2},$$

$$\gamma = \frac{b_0}{a} = \frac{1}{16}, \quad \lambda = \frac{a_0}{R} = \frac{1}{30}.$$

As a result, the first approximation to w was obtained (retaining in the formulae only the parameters A_1 and B_1) as well as the second approximation (by retaining the parameters A_1, A_3, B_1 and B_3). The data of the first and second approximations were found to be sufficiently close. The magnitudes of the ratios wEh/T_0R are given in the following table, and the corresponding curves of the deflections are shown in Fig. 1.

TABLE 1. Relative deflections w: $\dfrac{T_0R}{Eh}$

α \ β	0	$\frac{\pi R}{8}$	$\frac{\pi R}{4}$	$\frac{5\pi R}{16}$	$\frac{3\pi R}{8}$	$\frac{\pi R}{2}$
0	0	0	0	0	0	0
$\frac{a}{8}$	— 0.9	— 0.7	— 0.08	0.4	0.7	0.9
$\frac{a}{4}$	— 1.2	— 1.5	— 0.4	0.7	1.5	1.9
$\frac{3a}{8}$	— 1.6	— 2.0	— 1.7	— 0.3	1.6	4.1
$\frac{7a}{16}$	— 1.8	— 2.1	— 2.0	— 1.1	1.2	4.9
$\frac{a}{2}$	— 2.0	— 2.4	— 2.4	— 1.7	0.8	—

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A PROJECTIVE PROPERTY OF FORCE AND DEFLECTION FIELDS

I. N. VEKUA
Novosibirsk, U.S.S.R.

This paper studies a property of the force field of membrane shells with regard to groups of projective transformations of space. An analogous property is likewise established for the field of infinitesimal deflections of a surface.

§ 1. Let S be the middle surface of a shell. The state of membrane equilibrium of the shell is determined with the help of a symmetrical contravariant second order tensor $T^{\alpha\beta}$, belonging to the surface S, when the force acting on an element with the normal $\mathbf{l}$ is given by the formula

$$\mathbf{T}_{(l)} = T^{\alpha\beta}\mathbf{r}_\alpha l_\beta,$$

where $\mathbf{r}_\alpha = \partial\mathbf{r}/\partial x_\alpha$ $(\alpha = 1, 2)$ are the base vectors of the coordinate system x^1, x^2 considered on S and l_β are the covariant components of the unit vector $\mathbf{l}$. As is known (cf., for example, [1], Chap. 6) the tensor $T^{\alpha\beta}$ satisfies the system of equations

$$\nabla_\alpha T^{\alpha\beta} + X^\beta = 0, \quad (\beta = 1, 2),$$
$$b_{\alpha\beta}T^{\alpha\beta} + Z = 0,$$

where ∇_α is the symbol for covariant differentiation, $b_{\alpha\beta}$ are the coefficients of the second fundamental form of the surface S and X^1, X^2 and Z are the components of the surface loading $\mathbf{X}$

$$\mathbf{X} = X^\alpha\mathbf{r}_\alpha + Z\mathbf{n},$$

where $\mathbf{n}$ is the unit normal to S. Consequently, X^α are the contravariant components of a vector belonging to the surface S and Z is a scalar. Obviously, $X^\alpha\mathbf{r}_\alpha$ is the tangential loading on the surface and $Z\mathbf{n}$ is the normal loading.

We consider the case of a convex shell of positive curvature. Then there exists on the surface S, which we assume to be regular, a so-called conjugate isometric net of coordinate lines $x = \text{const.}$ $y = \text{const.}$ with respect to which the second fundamental quadratic form becomes ([1], Chap. 2)

$$\text{II} = \Lambda(dx^2 + dy^2), \quad \Lambda = \pm\sqrt{aK},$$

where K is the principal curvature of the surface S and a is the discriminant of the first fundamental quadratic form. In what follows we

will always use the positive sign in front of the root. For this the normal to the surface must be directed to the convex side.

With respect to the conjugate isometric coordinate system the force field may be expressed in the form ([1], Chap. 6, § 5)

$$T^{11} - iT^{12} = \frac{w'}{a\sqrt[4]{K}} - \frac{Z}{2\sqrt{aK}}, \qquad T^{11} + T^{22} = -\frac{Z}{\sqrt{aK}}, \tag{1.1}$$

where w' is a complex function satisfying the equation

$$\partial_{\bar{z}} w' - \bar{B}\bar{w}' = F. \tag{1.2}$$

The following notation has been used here:

$$\partial_{\bar{z}} = \tfrac{1}{2}(\partial_x + i\partial_y), \quad \partial_z = \tfrac{1}{2}(\partial_x - i\partial_y),$$

$$B = \tfrac{1}{4}(\Gamma^1_{22} - \Gamma^1_{11} + 2\Gamma^2_{12}) + \frac{i}{4}(\Gamma^2_{22} - \Gamma^2_{11} - 2\Gamma^1_{12}), \tag{1.3}$$

$$F = \tfrac{1}{2}\sqrt{a\sqrt{K}}\left\{\sqrt{K}\frac{\partial}{\partial z}\left(\frac{Z}{K}\right) - \sqrt{a}(X^1 - iX^2)\right\}, \tag{1.4}$$

where $\Gamma^\lambda_{\alpha\beta}$ is the Christoffel symbol of the second kind.

Thus, by the formulae (1.1), there corresponds to every solution of the equation (1.2) a completely definite force field and, conversely, to every such field there corresponds a definite solution of the equation (1.2). We will call the function w' the complex force function, and the equation (1.2) the fundamental equation of the membrane state of equilibrium of shells. It will be shown below that this equation has the property of invariance with respect to the group of projective transformations of space.

§ 2. THEOREM 1. *The coefficient B of the equation* (1.2) *is invariant with respect to the group of projective transformations of space.*

PROOF. The name projective transformation of space is given to the transformation of the form

$$\overset{*}{x}{}^i = \frac{a^i_k x^k + a^i}{V}, \quad V = b_k x^k + b, \quad (i = 1, 2, 3),$$

where x^i and $\overset{*}{x}{}^i$ are Cartesian coordinates of space points, a^i_k, a^i, b_k and b are constants, and it is assumed that the determinant of the transformation does not vanish, i.e.,

$$D = \begin{vmatrix} a^1_1 & a^2_1 & a^3_1 & b_1 \\ a^1_2 & a^2_2 & a^3_2 & b_2 \\ a^1_3 & a^2_3 & a^3_3 & b_3 \\ a^1 & a^2 & a^3 & b \end{vmatrix} \neq 0.$$

Therefore, without harming generality, it may be assumed that $D^2 = 1$. For a projective transformation of space the surface S is transformed into a new surface S_*, and the Christoffel symbols of the second kind, the coefficients of the second fundamental quadratic form, the discriminant of the first fundamental quadratic form and the principal curvature are transformed in accordance with the formulae (cf., for example, [2], Chap. 15)

$$\overset{*}{\Gamma}{}^{\lambda}_{\alpha\beta} = \Gamma^{\lambda}_{\alpha\beta} - \delta^{\lambda}_{\beta}\frac{\partial \log V}{\partial x^{\alpha}} - \delta^{\lambda}_{\alpha}\frac{\partial \log V}{\partial x^{\beta}} + \frac{p^{\lambda}}{V}b_{\alpha\beta}, \tag{2.1}$$

$$\overset{*}{b}_{\alpha\beta} = \frac{p}{V}b_{\alpha\beta}, \tag{2.2}$$

$$a_* = \frac{1}{p^2V^6}a, \tag{2.3}$$

$$K^* = p^4V^4K. \tag{2.4}$$

Here the quantities with asterisks refer to the transformed surface S_*, and the quantities p^{λ} and p are determined uniquely from the equalities

$$p = \overset{*}{\mathbf{n}}P\mathbf{n}, \quad p^{\lambda} = -V\tilde{b}^{\lambda\alpha}\frac{\partial \log p}{\partial x^{\alpha}},$$

where P is the matrix with the components

$$p^i_j = a^i_j - \overset{*}{x}{}^i b_j$$

and $\tilde{b}^{\alpha\beta}$ is the contravariant tensor corresponding to the covariant tensor $b_{\alpha\beta}$.

It is seen from (2.2)–(2.4) that a surface of positive curvature goes over into a surface of positive curvature and that a conjugate isometric coordinate on S becomes the same net of lines on S_*. Therefore the basic equation of the membrane equilibrium state of stress of a shell, the middle surface of which coincides with S_*, will have the form

$$\partial_{\bar{z}}w_* - \bar{B}_*\bar{w}_* = F_*, \tag{2.5}$$

where

$$B^* = \tfrac{1}{4}(\overset{*}{\Gamma}{}^1_{22} - \overset{*}{\Gamma}{}^1_{11} + 2\overset{*}{\Gamma}{}^2_{12}) + \frac{i}{4}(\overset{*}{\Gamma}{}^2_{22} - \overset{*}{\Gamma}{}^2_{11} - 2\overset{*}{\Gamma}{}^1_{12}),$$

$$F_* = \tfrac{1}{2}\sqrt{a_*\sqrt{K_*}}\left\{\sqrt{K_*}\frac{\partial}{\partial z}\left(\frac{Z_*}{K_*}\right) - \sqrt{a_*}(X^1_* - iX^2_*)\right\}.$$

If we now turn to the formulae (2.1) and take into consideration that

$$b_{11} = b_{22} = \sqrt{aK},\ b_{12} = 0,$$

we will readily verify that

$$B = B_*,$$

as had to be proved.

§ 3. We will now assume that the external loading satisfies a condition of the form

$$F_* = C_0 F,\ C_0 \neq 0, \tag{3.1}$$

where C_0 is some real constant quantity. This condition may be written in a form which does not depend on the special choice of the coordinate system on S. We will introduce into the consideration the contravariant second order tensor

$$h^{\alpha\beta} = c^{\alpha\lambda} c^{\beta\gamma} b_{\lambda\gamma} = \frac{1}{K} \tilde{b}^{\alpha\beta},$$

where $c^{\alpha\beta}$ is the discriminant second order contravariant tensor

$$c^{11} = c^{22} = 0,\ c^{12} = -c^{21} = \frac{1}{\sqrt{a}}.$$

Referred to a conjugate isometric system of coordinates this tensor takes the form

$$h^{11} = h^{22} = \sqrt{\frac{K}{a}},\ h^{12} = h^{21} = 0. \tag{3.2}$$

Using also (2.3) and (2.4), the equation (3.1) may be written in the form

$$\tfrac{1}{2} h_*^{\alpha\beta} \frac{\partial}{\partial x^\alpha}\left(\frac{Z_*}{K_*}\right) - X_*^\beta = C_0 q \left\{ \tfrac{1}{2} h^{\alpha\beta} \frac{\partial}{\partial x^\alpha}\left(\frac{Z}{K}\right) - X^\beta \right\}, \tag{3.3}$$

$$(\beta = 1, 2),$$

where

$$q = pV^5. \tag{3.4}$$

Equation (3.3) is true for any coordinate system.

§ 4. If the loading on the surfaces S and S_* satisfies the condition (3.3), the solutions of the equations (1.2) and (2.5) will be connected by the equality

$$w_* = C_0 w'.$$

Consequently, the force field on S will be determined from the relations

$$(4.1)\quad \overset{*}{T}{}^{11} - i\overset{*}{T}{}^{12} = \frac{C_0 w'}{a_* \sqrt[4]{K_*}} - \frac{Z_*}{2\sqrt{a_* K_*}},\quad \overset{*}{T}{}^{11} + \overset{*}{T}{}^{22} = -\frac{Z_*}{\sqrt{a_* K_*}}.$$

Eliminating from (1.1) and (4.1) the function w', we obtain an equality which, taking into consideration (3.2), may be written in the form

$$(4.2)\quad \overset{*}{T}{}^{\alpha\beta} + \tfrac{1}{2}\overset{*}{h}{}^{\alpha\beta} Z_* = C_0 q(T^{\alpha\beta} + \tfrac{1}{2}h^{\alpha\beta} Z).$$

This equation obviously is true for any coordinate system. Thus, one has proved

THEOREM 2. *If the fields of surface loading* $\mathbf{X}$ *and* $\mathbf{X}_*$ *on* S *and* S_* *satisfy the condition* (3.3), *every field of forces of a membrane equilibrium state of stress on* S *constitutes such a field on* S_* *in agreement with the formulae* (4.2).

The condition (3.3) is obviously always fulfilled, if $\mathbf{X}$ and $\mathbf{X}_*$ are determined by loading potentials ([1], Chap. 6, § 5)

$$\mathbf{X} = \tfrac{1}{2}h^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\left(\frac{Z}{K}\right)\mathbf{r}_\beta + Z\mathbf{n},$$

$$\mathbf{X}_* = \tfrac{1}{2}\overset{*}{h}{}^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\left(\frac{Z_*}{K_*}\right)\overset{*}{\mathbf{r}}_\beta + Z_*\overset{*}{\mathbf{n}},$$

where Z and Z_* are arbitrary scalar functions on S and S_* respectively. It is sufficient if they are continuous and have piecewise continuous derivatives.

Consequently, in the presence of an external loading potential, one has always a relationship of the form (4.2) between the force fields.

§ 5. If there is no surface loading at all, i.e., if

$$\mathbf{X} \equiv 0,\ \mathbf{X}_* \equiv 0,$$

then the equality (4.2) becomes

$$(5.1)\quad \overset{*}{T}{}^{\alpha\beta} = qT^{\alpha\beta}.$$

We have written here instead of $C_0 q$ the symbol q, since in this case the force fields constitute linear manifolds. The equalities (5.1) may likewise be given a geometric interpretation.

As is known, for infinitesimal displacements of the surface, the variations of the coefficients of the second fundamental quadratic form are expressed in the form ([1], Chap. 5, § 7)

$$(5.2)\quad \delta b_{\alpha\beta} = c_{\alpha\lambda}c_{\beta\gamma}T^{\lambda\gamma},$$

where $c_{\alpha\beta}$ is the covariant discriminant second order tensor

$$c_{11} = c_{22} = 0, \; c_{12} = -c_{21} = \sqrt{a}.$$

On the basis of (2.3) it is readily verified that for projective transformations of the space the tensor $c_{\alpha\beta}$ is transformed in accordance with the formula

$$\overset{*}{c}_{\alpha\beta} = \frac{1}{pV^3} c_{\alpha\beta}. \tag{5.3}$$

By (5.1), (5.2) and (5.3) we have

$$\overset{*}{\delta b}_{\alpha\beta} = p^{-2}V^{-6}qc_{\alpha\lambda}c_{\beta\gamma}T^{\lambda\gamma} = \frac{1}{pV}\delta b_{\alpha\beta}.$$

Hence we have proved

THEOREM 3. *If the surface S_* is obtained from the surface S by help of some projective transformation of the space, then the deflection fields of these surfaces are interrelated by the formulae*

$$\overset{*}{\delta b}_{\alpha\beta} = \frac{1}{pV}\delta b_{\alpha\beta}.$$

It is likewise not difficult to obtain the transformation formulae for the displacement field in the case of infinitesimal deflections of the surface.

The displacement field on the surface may be expressed by the formula ([1], Chap. 5, § 3)

$$\mathbf{U} = \mathrm{Re}\left\{-2K^{-\frac{1}{4}}w\mathbf{n}_z + \frac{1}{K\sqrt{a}}\frac{\partial}{\partial z}(\sqrt{a}\,K^{\frac{3}{4}}w).\mathbf{n}\right\}, \tag{5.4}$$

where w is an arbitrary solution of the equation

$$\partial_{\bar{z}}w + B\bar{w} = 0.$$

Since this equation is invariant with respect to the group of projective transformations of the space, the displacement field on S_* will be expressed by the analogous formula

$$\mathbf{U}_* = \mathrm{Re}\left\{-2K_*^{-\frac{1}{4}}w\overset{*}{\mathbf{n}}_z + \frac{1}{K_*\sqrt{a_*}}\frac{\partial}{\partial z}(\sqrt{a_*}K_*^{\frac{3}{4}}w)\;\overset{*}{\mathbf{n}}\right\}. \tag{5.5}$$

Taking into consideration the formulae (cf., for example, [1], p. 404)

$$\tfrac{1}{2}\sqrt{aK}\,(\mathbf{r}^1 - i\mathbf{r}^2) = -\mathbf{n}_z, \; \tfrac{1}{2}\sqrt{a_*K_*}\,(\overset{*}{\mathbf{r}}{}^1 - i\overset{*}{\mathbf{r}}{}^2) = -\overset{*}{\mathbf{n}}_z,$$

we obtain from (5.4) and (5.5)

$$w = \frac{u_1 + iu_2}{\sqrt{a}\sqrt{K}} = \frac{\overset{*}{u}_1 + i\overset{*}{u}_2}{\sqrt{a_*}\sqrt{K_*}},$$

where

$$u_\alpha = \mathbf{U}\mathbf{r}_\alpha, \quad \overset{*}{u}_\alpha = \overset{*}{\mathbf{U}}\overset{*}{\mathbf{r}}_\alpha \quad (\alpha = 1, 2)$$

are the covariant components of the field of tangential displacements. Hence we have

$$\overset{*}{u}_\alpha = \frac{1}{V^2} u_\alpha, \quad (\alpha = 1, 2). \tag{5.6}$$

These formulae are true with respect to any coordinate system.

It is now likewise not difficult to obtain the transformation formulae of the normal component of the displacement field. In fact, the normal component of the displacement vector is expressed by the formula (cf., for example, [1], p. 398)

$$u_0 = \frac{1}{2H} \nabla^\alpha u_\alpha = \frac{1}{2H} \frac{1}{\sqrt{a}} \frac{\partial \sqrt{a}\, a^{\alpha\beta} u_\beta}{\partial x^\alpha},$$

where H is the mean curvature of the surface and $a^{\alpha\beta}$ are the contravariant components of the metric tensor of the surface. The normal displacements for the transformed surface S_* are now expressed by (5.6) by the formula

$$\overset{*}{u}_0 = \frac{1}{2H_*} \frac{1}{\sqrt{a_*}} \frac{\partial}{\partial x^\beta} \left(\frac{\sqrt{a_*}\, \overset{*}{a}^{\alpha\beta} u_\beta}{V^2} \right).$$

The above projective properties of the different fields of infinitesimal deflections have been deduced earlier by somewhat different means and in a different form (cf., for example, [3]).

We likewise note that these results hold true also for surfaces of negative and zero curvature.

§ 6. It follows from the property of projective invariance of the function B *that* $B \equiv 0$ *for all surfaces of second order positive curvature.* In fact, for a spherical surface, the Christoffel symbols of the second kind are given by

$$\Gamma^1_{11} = \Gamma^2_{12} = -\Gamma^1_{22} = \frac{\partial \log \sqrt{a}}{\partial x}, \quad \Gamma^2_{22} = \Gamma^1_{21} = -\Gamma^2_{11} = \frac{\partial \log \sqrt{a}}{\partial y}.$$

Hence it is seen directly that $B \equiv 0$ for every spherical surface. Since

the sphere is the projective image of any second order surface of positive curvature, our statement is true as a consequence of Theorem 1.

The converse statement is also true: *If* $B \equiv 0$ *and* $K > 0$, *then the corresponding surface is a second order surface of positive curvature.*

This proposition has been proved in the author's work ([1], Chap. 5, § 4). Another approach to the proof of this proposition may be given, if it is noted that one has with regard to a conjugate isometric system of coordinates the following relation between the function B and the Darboux tensor $\Theta_{\alpha\beta\lambda}$:

$$B = \frac{1}{2\sqrt{aK}}(\Theta_{111} - i\Theta_{222}). \tag{6.1}$$

In fact, the Darboux tensor is determined by the help of the following equality ([2], Chap. 15)

$$\Theta_{\alpha\beta\lambda} = b_{\alpha\beta\lambda} - \tfrac{1}{4}\left(b_{\alpha\beta}\frac{\partial \log K}{\partial x^\lambda} + b_{\lambda\alpha}\frac{\partial \log K}{\partial x^\beta} + b_{\lambda\beta}\frac{\partial \log K}{\partial x^\alpha}\right),$$

where $b_{\alpha\beta\lambda} = \nabla_\lambda b_{\alpha\beta}$ is the so-called Codazzi tensor. Taking into consideration that with regard to a conjugate isometric coordinate system $b_{11} = b_{22} = \sqrt{aK}$, $b_{12} = 0$, and likewise keeping in mind the formulae ([1], Chap. 2, § 6)

$$\frac{\partial \log \sqrt{a}}{\partial x^\alpha} = \Gamma^\beta_{\alpha\beta}, \quad (\alpha = 1, 2),$$

$$\frac{\partial \log \sqrt{K}}{\partial x} = -\Gamma^1_{11} - \Gamma^1_{22}, \qquad \frac{\partial \log \sqrt{K}}{\partial y} = -\Gamma^2_{11} - \Gamma^2_{22},$$

we obtain by means of simple manipulations

$$\begin{aligned} \Theta_{111} = \Theta_{122} &= \frac{1}{2\sqrt{aK}}(\Gamma^1_{22} - \Gamma^1_{11} + 2\Gamma^2_{12}), \\ \Theta_{222} = -\Theta_{112} &= \frac{1}{2\sqrt{aK}}(\Gamma^2_{11} - \Gamma^2_{22} + 2\Gamma^1_{21}). \end{aligned} \tag{6.2}$$

It is necessary to remember that the Darboux tensor is symmetric with respect to all three indices. We obtain (6.1) directly from (1.3) and (6.2).

We note that the invariance of the function B with regard to the group of projective transformations of space follows likewise from (6.1), if we use the formulae (2.3) and (2.4), and also known properties of the Darboux tensor [2]:

$$\Theta_{\alpha\beta\lambda}{}^* = \frac{p}{V}\Theta_{\alpha\beta\lambda}.$$

It is seen from (6.1) that B vanishes if, and only if the Darboux tensor vanishes. But it is known that the identical vanishing of the Darboux tensor is the necessary and sufficient condition for a surface to be of second order. A proof of the last statement, belonging to L. L. Verbitskii, has been presented in the book of V. F. Kagan [2], Chap. 15). This proof which is based on some theorems of Bonnet and Cesàro differs considerably in its method from the proof given in the author's book [1].

In conclusion, we will make yet the following remarks. By means of the function B one may express the fundamental quadratic and cubic forms of projective differential geometry of surfaces of positive curvature. If one excludes from the consideration surfaces of second order, these forms may be taken in the form (normalized forms of Fubini)

$$F_2 = B\bar{B}\,dz\,d\bar{z},\quad F_3 = B\bar{B}(B\,d\bar{z}^3 + \bar{B}\,dz^3).$$

Thus, the surface is determined by the function B exactly apart from an arbitrary projective transformation of space. In projective geometry the line element is determined by the formula

$$ds = \frac{F_3}{F_2} \equiv B\frac{d\bar{z}^2}{dz} + \bar{B}\frac{dz^2}{d\bar{z}}.$$

It follows directly from this formula that there pass through every point of a surface of positive curvature, where $B \neq 0$, three tangential directions to the surface along which $ds = 0$.

In addition, these directions, called the directions of Darboux, divide the tangential plane into three equal parts.

Finally, we will still note that the real and imaginary parts themselves represent the components of a mixed third order tensor, determined by

$$B^{\lambda}_{\alpha\beta} = \frac{1}{2\sqrt{K}}\,c^{\gamma\lambda}\Theta_{\gamma\alpha\beta}. \tag{6.3}$$

Referred to a conjugate isometric coordinate net we will have

$$B^2_{11} = B^1_{21} = \frac{1}{2\sqrt{Ka}}\,\Theta_{111},\quad B^1_{22} = B^2_{12} = -\frac{1}{2\sqrt{aK}}\,\Theta_{222}.$$

Consequently, the equality (6.3) is equivalent to the formula (6.1). It therefore follows that the tensor $B^{\lambda}_{\alpha\beta}$, determined by (6.3) with respect to any coordinate system on the surface, represents an invariant with respect to the group of projective transformations of space. One has to keep in mind that this property is not retained with reference to the components of other types of that tensor (e.g., for $B_{\alpha\beta\lambda}$).

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LINEAR INTEGRO-DIFFERENTIAL EQUATIONS WITH SMALL PARAMETERS FOR HIGHER DERIVATIVES

N. P. VEKUA
Tbilisi, U.S.S.R.

§ 1. Linear differential equations with small parameters for higher derivatives have been the subject of studies by many authors. A fairly complete bibliography on this problem is given at the end of the paper by M. I. Vishik and L. A. Liusternik [1].

In the present note we will consider integro-differential equations with small parameters preceding the highest derivatives which have the form

$$(1.1)\qquad L_\varepsilon \rho_\varepsilon \equiv \sum_{k=1}^{l} a_{m+k}\varepsilon^k \rho_\varepsilon^{(m+k)}(t) + a_m \rho_\varepsilon^{(m)}(t) + \sum_{j=0}^{m-1}\left[a_j(t)\rho_\varepsilon^{(j)}(t) + \int_0^t K_j(t,\tau)\rho_\varepsilon^{(j)}(\tau)\,d\tau \right] = f_\varepsilon(t),$$

$$(1.2)\qquad L_\varepsilon^* \rho_\varepsilon \equiv \sum_{k=1}^{l} a_{m+k}\varepsilon^k \rho_\varepsilon^{(m+k)}(t) + a_m \rho_\varepsilon^{(m)}(t) + \sum_{j=0}^{m-1}\left[a_j(t)\rho_\varepsilon^{(j)}(t) + \int_0^1 \frac{\Gamma_j(t,\tau)\rho_\varepsilon^{(j)}(\tau)}{\tau - t}\,d\tau \right] = f_\varepsilon(t),$$

where a_{m+k} $(k = 0, 1, \cdots, l)$ are constants, $a_{m+l} = 1$, $a_m \neq 0$, $m > 0$, ε is a small parameter $(\varepsilon > 0)$, $a_j(t)$ $(j = 0, 1, \cdots, m-1)$ are continuous functions in the interval $[0, 1]$, $f_\varepsilon(t)$ is a given continuous function admitting the decomposition

$$f_\varepsilon(t) = f(t) + O(\varepsilon),$$

$K_j(t, \tau)$ are continuous functions in the interval $[0, 1]$ and $\Gamma_j(t, \tau)$ satisfy a Hölder condition H with respect to both variables in this interval. The integrals in equation (1.2) must be understood in the sense of the Cauchy principal value.

We consider below the Cauchy problem for these equations and study the question of the limit $\varepsilon \to 0$.

One may formulate the Cauchy problem for these equations as follows: To find the solution of equation (1.1) [equation (1.2)] for the initial values

$$\rho^{(j)}(0) = D_j, \quad (j = 0, 1, \cdots, m + l - 1),$$

where D_j $(j = 0, 1, \cdots, m + l - 1)$ are arbitrarily given constants. We will call this Problem K_ε.

We consider now the degenerate equations obtained from (1.1) and (1.2) for $\varepsilon = 0$

$$(1.3) \qquad a_m\rho^{(m)}(t) + \sum_{j=0}^{m-1}\left[a_j(t)\rho^{(j)}(t) + \int_0^t K_j(t, \tau)\rho^{(j)}(\tau)\, d\tau\right] = f(t),$$

$$(1.4) \qquad a_m\rho^{(m)}(t) + \sum_{j=0}^{m-1}\left[a_j(t)\rho^{(j)}(t) + \int_0^1 \frac{\Gamma_j(t, \tau)\rho^{(j)}(\tau)\, d\tau}{\tau - t}\right] = f(t).$$

For these equations the Cauchy problem (Problem K_0) has the form

$$\rho^{(j)}(0) = D_j, \quad (j = 0, 1, \cdots, m - 1).$$

Denoting the solutions of Problems K_ε and K_0 for equations (1.1) and (1.3) by $\rho_\varepsilon(t)$ and $\rho(t)$ respectively, we will show that under definite conditions of regularity of the degeneracy, one has the equality

$$\lim_{\varepsilon \to 0} \rho_\varepsilon(t) = \rho(t).$$

We will consider the same question for the equations (1.2) and (1.4). For equations of the form

$$\sum_{k=1}^{l} a_{m+k}\varepsilon^k\rho_\varepsilon^{(m+k)}(t) + \sum_{j=0}^{m} a_j\rho_\varepsilon^{(j)}(t) = f_\varepsilon(t)$$

this problem has been studied sufficiently completely; a short survey of the results may be found in the above quoted work of M. I. Vishik and L. A. Liusternik [1].

§ 2. First we will consider Problem K_ε for the equation (1.1). Without restriction of generality one may assume that the initial values are zero, i.e., $D_j = 0$ $(j = 0, 1, \cdots, m + l - 1)$.

Introducing the notation

$$(2.1) \qquad \rho_\varepsilon^{(m-1)}(t) = \mu_\varepsilon(t),$$

we will have

$$\rho_\varepsilon^{(k)}(t) = \frac{1}{(m - k - 2)!}\int_0^t (t - \tau)^{m-k-2}\mu_\varepsilon(\tau)\, d\tau + \sum_{j=1}^{m-k-1} C_j \frac{t^{m-k-j-1}}{(m - k - j - 1)!},$$

$$(k = 0, 1, \cdots, m - 2),$$

where C_k $(k = 1, 2, \cdots, m-1)$ are arbitrary constants, $0! = 1$ and $\rho_\varepsilon^{(0)}(t) \equiv \rho_\varepsilon(t)$. Obviously we obtain from this formula

$$C_k = \rho_\varepsilon^{(m-k-1)}(0) = 0, \quad (k = 1, 2, \cdots, m-1),$$

and hence

$$\rho_\varepsilon^{(k)}(t) = \frac{1}{(m-k-2)!} \int_0^t (t-\tau)^{m-k-2} \mu_\varepsilon(\tau)\, d\tau, \tag{2.2}$$

$$(k = 0, 1, \cdots, m-2).$$

By (2.1) and (2.2) equation (1.1) may be rewritten

$$\sum_{k=0}^{l} a_{m+k}\varepsilon^k \mu_\varepsilon^{(k+1)} + a_{m-1}(t)\mu_\varepsilon(t) + \int_0^t K^0(t, \tau)\mu_\varepsilon(\tau)\, d\tau = f_\varepsilon(t), \tag{2.3}$$

where

$$K^0(t, \tau) = K_{m-1}(t, \tau) + \sum_{j=0}^{m-2} \frac{a_j(t)}{(m-j-2)!} (t-\tau)^{m-j-2} + \tag{2.4}$$

$$+ \sum_{j=0}^{m-2} \frac{1}{(m-j-2)!} \int_\tau^t K_j(t, t_1)(t_1 - \tau)^{m-j-2}\, dt_1.$$

The degenerate equation corresponding to (2.3) has obviously the form

$$a_m \frac{d\mu}{dt} + a_{m-1}(t)\mu(t) + \int_0^t K^0(t, \tau)\mu(\tau)\, d\tau = f(t). \tag{2.5}$$

If we now find the solution of the equation (2.3) for the initial Cauchy values

$$\mu_\varepsilon^{(j)}(0) = 0, \quad (j = 0, 1, \cdots, l), \tag{2.6}$$

where $\mu_\varepsilon^{(0)} \equiv \mu_\varepsilon$, we obtain by help of the formula (2.2) the solution of Problem K_ε for the equation (1.1). It is likewise obvious that, if we find the solution of the equation (2.5), satisfying the condition $\mu(0) = 0$, the solution of Problem K_0 for the equation (1.3) is also easily obtained. Further, it is obvious that if we show the truth of the equality

$$\lim \mu_\varepsilon(t) = \mu(t), \tag{2.7}$$

then we readily obtain

$$\lim_{\varepsilon \to 0} \rho_\varepsilon(t) = \rho(t),$$

where $\rho_\varepsilon(t)$ and $\rho(t)$ denote the solutions of Problems K_ε and K_0 for the equations (1.1) and (1.3) respectively.

Thus we must clarify for what conditions the equality (2.7) holds true.

The equation (2.3) will be rewritten in the form

$$\sum_{k=0}^{l} a_{m+k}\varepsilon^k \mu_\varepsilon^{(k+1)}(t) = \Phi_\varepsilon(t), \tag{2.8}$$

where

$$\Phi_\varepsilon(t) = f_\varepsilon(t) - a_{m-1}(t)\mu_\varepsilon(t) - \int_0^t K^0(t, \tau)\mu_\varepsilon(\tau)\, d\tau, \tag{2.9}$$

and for the time being we will consider the right hand side to be a known function. Then one may consider (2.8) to be a non-homogeneous linear differential equation with constant coefficients.

We will consider the homogeneous equation

$$\sum_{k=0}^{l} a_{m+k}\varepsilon^k \mu_\varepsilon^{(k+1)}(t) = 0 \tag{2.10}$$

corresponding to (2.8) and will find the fundamental system of solutions of this equation. The characteristic equation, corresponding to the equation (2.10), has the form

$$\alpha \sum_{k=0}^{l} a_{m+k}\varepsilon^k \alpha^k = 0,$$

whence $\alpha_0 = 0$ and

$$\sum_{k=0}^{l} a_{m+k}\nu^k = 0, \tag{2.11}$$

where

$$\nu = \alpha\varepsilon. \tag{2.12}$$

In agreement with the terminology of [1], the equation (2.11) will be called the supplementary characteristic equation, corresponding to the equation (1.1) [or (1.2)].

For the sake of simplicity of the exposition, we will assume that the equation (2.11) has no multiple roots. We will denote by ν_k ($k = 1, 2, \cdots, l$) the roots of this equation. We obtain by the strength of (2.12) that $\alpha_k = \nu_k/\varepsilon$.

As usual we will assume that

$$\operatorname{Re} \nu_k < 0, \tag{2.13}$$

and if these conditions are fulfilled we will call the degeneration of Problem K_ε into Problem K_0 regular (cf. [1], p. 16).

Obviously the fundamental system of solutions of the equation (2.10) has the form

$$1,\ e^{\nu_1 t/\varepsilon},\ e^{\nu_2 t/\varepsilon},\ \cdots,\ e^{\nu_l t/\varepsilon}.$$

Applying now the method of variation of parameters, we verify that the general solution of the equation (2.8) may be represented in the form

$$\mu_\varepsilon(t) = \sum_{k=0}^{l} C_k(t)\, e^{\nu_k t/\varepsilon}, \tag{2.14}$$

where $\nu_0 = 0$, $C_k(t)$ $(k = 0, 1, \cdots, l)$ is the general solution of the system

$$\frac{dC_0}{dt} + e^{\nu_1 t/\varepsilon}\frac{dC_1}{dt} + \cdots + e^{\nu_l t/\varepsilon}\frac{dC_l}{dt} = 0,$$

$$\nu_1\, e^{\nu_1 t/\varepsilon}\frac{dC_1}{dt} + \cdots + \nu_l\, e^{\nu_l t/\varepsilon}\frac{dC_l}{dt} = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\nu_1^{\,l}\, e^{\nu_1 t/\varepsilon}\frac{dC_1}{dt} + \cdots + \nu_l^{\,l}\, e^{\nu_l t/\varepsilon}\frac{dC_l}{dt} = \Phi_\varepsilon(t).$$

Solving this system, we obtain

$$C_0(t) = \frac{1}{a_m}\int_0^t \Phi_\varepsilon(\tau)\, d\tau + C_0^0,$$

$$C_k(t) = \delta_k \int_0^t e^{-(\nu_k \tau)/\varepsilon}\, \Phi_\varepsilon(\tau)\, d\tau + C_k^0 \quad (k = 1, 2, \cdots, l),$$

where δ_k are completely determined constants which do not contain ε and $C_0^0, C_1^0, \cdots, C_l^0$ are arbitrary constants. Substituting these values into (2.14), we find

$$\mu_\varepsilon(t) = \frac{1}{a_m}\int_0^t \Phi_\varepsilon(\tau)\, d\tau + \sum_{k=1}^{l} \delta_k\, e^{\nu_k t/\varepsilon}\int_0^t e^{-(\nu_k\tau)/\varepsilon}\, \Phi_\varepsilon(\tau)\, d\tau + C_0^0 + \sum_{k=1}^{l} C_k^0\, e^{\nu_k t/\varepsilon}.$$

From the initial conditions (2.6) we readily obtain that $C_k^0 = 0$ $(k = 0, 1, \cdots, l)$, and hence

$$\mu_\varepsilon(t) = \frac{1}{a_m}\int_0^t \Phi_\varepsilon(\tau)\, d\tau + \sum_{k=1}^{l} \delta_k\, e^{\nu_k t/\varepsilon}\int_0^t e^{-(\nu_k\tau)/\varepsilon}\, \Phi_\varepsilon(\tau)\, d\tau. \tag{2.15}$$

Substituting now into (2.15) the value $\Phi_\varepsilon(\tau)$, determined by (2.9), we obtain for $\mu_\varepsilon(t)$ a Volterra type equation of the form

$$\mu_\varepsilon(t) + \int_0^t K_\varepsilon(t, \tau)\mu_\varepsilon(\tau)\, d\tau = F_\varepsilon(t), \tag{2.16}$$

where

$$K_\varepsilon(t, \tau) = \frac{a_{m-1}(\tau)}{a_m} + \frac{1}{a_m}\int_\tau^t K^0(t_1, \tau)\, dt_1 + \tag{2.17}$$

$$+ \sum_{k=1}^{l} \delta_k\, e^{\nu_k t/\varepsilon} \int_\tau^t e^{-(\nu_k t_1)/\varepsilon}\, K^0(t_1, \tau)\, dt_1 + a_{m-1}(\tau) \sum_{k=1}^{l} \delta_k\, e^{\nu_k (t-\tau)/\varepsilon},$$

$$F_\varepsilon(t) = \frac{1}{a_m}\int_0^t f_\varepsilon(\tau)\, d\tau + \sum_{k=1}^{l} \delta_k\, e^{\nu_k t/\varepsilon} \int_0^t e^{-(\nu_k \tau)/\varepsilon}\, f_\varepsilon(\tau)\, d\tau. \tag{2.18}$$

We will show below that the solution of (2.16) may be presented in the form

$$\mu_\varepsilon(t) = \mu_0(t) + \varepsilon\mu_1(\varepsilon, t), \tag{2.19}$$

where $\mu_0(t)$ is a continuous function which does not depend on ε and $\mu_1(\varepsilon, t)$ is a uniformly bounded function, i.e.,

$$|\mu_1(\varepsilon, t)| \leq M,$$

where M is a constant number independent of ε.

Admitting for the present that the representation (2.19) is correct and going to the limit in (2.16) with respect to $\varepsilon \to 0$, we readily obtain

$$a_m\mu_0(t) + \int_0^t a_{m-1}(\tau)\mu_0(\tau)\, d\tau + \int_0^t \mu_0(\tau)\, d\tau \int_\tau^t K^0(t_1, \tau)\, dt_1 = \int_0^t f(\tau)\, d\tau. \tag{2.20}$$

It is obvious from this equality that $\mu_0(0) = 0$.

Further, it is easily seen that the equation (2.20) is equivalent to the equation (2.5) [for the condition $\mu(0) = 0$] and, consequently,

$$\mu_0(t) = \mu(t),$$

where $\mu(t)$ is the solution of the equation (2.5) satisfying the condition $\mu(0) = 0$.

We will show now that the representation (2.19) is actually correct.

We obtain from (2.16)

$$\mu_\varepsilon(\tau) = -\int_0^\tau K_\varepsilon(\tau, t_1)\mu_\varepsilon(t_1)\,dt_1 + F_\varepsilon(\tau).$$

Substituting this value in (2.16) under the integral sign, we find

$$\mu_\varepsilon(t) - \int_0^t K_\varepsilon^{(2)}(t, t_1)\mu_\varepsilon(t_1)\,dt_1 = F_\varepsilon^*(t), \tag{2.21}$$

where

$$K_\varepsilon^{(2)}(t, t_1) = \int_{t_1}^t K_\varepsilon(t, \tau)K_\varepsilon(\tau, t_1)\,d\tau, \tag{2.22}$$

$$F_\varepsilon^*(t) = F_\varepsilon(t) - \int_0^t K_\varepsilon(t, \tau)F_\varepsilon(\tau)\,d\tau. \tag{2.23}$$

It is easily seen that the equations (2.16) and (2.21) are equivalent.

By help of certain simple estimates one may establish that the iterated kernel $K_\varepsilon^{(2)}(t, t_1)$ may be represented in the form

$$K_\varepsilon^{(2)}(t, t_1) = K_0^{(2)}(t, t_1) + O(\varepsilon), \tag{2.24}$$

where

$$K_0^{(2)}(t, t_1) = \int_{t_1}^t K_0(t, \tau)K_0(\tau, t_1)\,d\tau \tag{2.25}$$

with

$$K_0(t, \tau) = \frac{a_{m-1}(\tau)}{a_m} + \frac{1}{a_m}\int_\tau^t K^0(t_1, \tau)\,dt_1.$$

Denoting by $R_\varepsilon(t, \tau)$ the resolvent of the equation (2.21), one may show on the basis of (2.24) that

$$R_\varepsilon(t, \tau) = R_0(t, \tau) + O(\varepsilon), \tag{2.26}$$

where $R_0(t, \tau)$ is the resolvent of the kernel $K_0^{(2)}(t, \tau)$.

The solution of the equation (2.21) may be written with the help of the resolvent in the form

$$\mu_\varepsilon(t) = F_\varepsilon^*(t) + \int_0^t R_\varepsilon(t, \tau)F_\varepsilon^*(\tau)\,d\tau. \tag{2.27}$$

Taking into consideration that $f_\varepsilon(t) = f(t) + O(\varepsilon)$, we will have from (2.18)

$$F_\varepsilon(t) = F_0(t) + O(\varepsilon),$$

where

$$F_0(t) = \frac{1}{a_m} \int_0^t f(\tau)\, d\tau.$$

After this we obtain readily from (2.23)

$$F_\varepsilon^*(t) = F_0^*(t) + O(\varepsilon), \tag{2.28}$$

where

$$F_0^*(t) = F_0(t) - \int_0^t \left[\frac{a_{m-1}(\tau)}{a_m} + \frac{1}{a_m} \int_\tau^t K^0(t_1, \tau)\, dt_1 \right] F_0(\tau)\, d\tau.$$

By the strength of (2.26) and (2.28) the formula (2.27) gives

$$\mu_\varepsilon(t) = \mu_0(t) + O(\varepsilon), \tag{2.29}$$

where

$$\mu_0(t) = F_0^*(t) + \int_0^t R_0(t, \tau) F_0^*(\tau)\, d\tau. \tag{2.30}$$

The relation (2.29) proves the truth of the representation (2.19).

The truth of the following proposition follows from the above reasoning:

THEOREM 1. *If the degeneration of Problem K_ε into Problem K_0 is regular, i.e., if the roots of the supplementary characteristic equation* (2.11) *satisfy the conditions* (2.13), *the following limiting relation holds true:*

$$\lim \rho_\varepsilon(t) = \rho(t),$$

where $\rho_\varepsilon(t)$ is the solution of Problem K_ε for the equation (1.1) *and $\rho(t)$ is the solution of Problem K_0 for the equation* (1.3).

We note that on the basis of the above results one may characterize in greater detail the behavior of the solution $\rho_\varepsilon(t)$ and its derivatives near the point $t = 0$, but we will not dwell on this aspect.

Finally, we note that if the $a_j = \text{const.}$ $(j = 0, 1, \cdots, m + l)$ and if the kernels $K_j(t, \tau)$ of the equation (1.1) depend on the difference of t and τ, i.e., $K_j(t, \tau) = K_j(t - \tau)$, then it is easily shown that the kernel of the equation (2.16) depends likewise on the difference of t and τ, i.e., $K_\varepsilon(t, \tau) = K\varepsilon(t - \tau)$, and consequently the equation (2.16) is clearly solved. Thus, in the case considered, Problem K_ε for the equation (1.1) is clearly solved.

§ 3. We will now consider Problem K_ε for the equation (1.2). Introducing the notation (2.1) and proceeding as in the case of the equation (1.1) we obtain for $\mu_\varepsilon(t)$ the equation

$$\mu_\varepsilon(t) + \int_0^1 \Gamma_\varepsilon(t, \tau)\mu_\varepsilon(\tau)\, d\tau = F_\varepsilon(t), \tag{3.1}$$

where

$$F_\varepsilon(t) = \frac{1}{a_m}\int_0^t f_\varepsilon(\tau)a\tau + \sum_{k=1}^{l} \delta_k\, e^{\nu_k t/\varepsilon}\int_0^t e^{-(\nu_k\tau)/\varepsilon}\, f_\varepsilon(\tau)\, d\tau, \tag{3.2}$$

$$\begin{aligned}\Gamma_\varepsilon(t, \tau) = \frac{a_{m-1}(\tau)}{a_m}\,\omega(\tau, t) + \frac{1}{a_m}\int_0^1 \frac{\omega(t_1, t)\Gamma^0(t_1, \tau)\, dt_1}{\tau - t_1} + \\ + \omega(\tau, t)a_{m-1}(\tau)\sum_{k=1}^{l}\delta_k\, e^{\nu_k(t-\tau)/\varepsilon} + \\ + \sum_{k=1}^{l}\delta_k\, e^{\nu_k t/\varepsilon}\int_0^1 \frac{\omega(t_1, t)\Gamma^0(t_1, \tau)}{\tau - t_1}\, e^{-(\nu_k t_1)/\varepsilon}\, dt_1,\end{aligned} \tag{3.3}$$

with $\omega(\tau, t) = 1$ for $0 \le \tau \le t$ and $\omega(\tau, t) = 0$ for $\tau > t$ and

$$\begin{aligned}\Gamma^0(t, \tau) = \Gamma_{m-1}(t, \tau) + (\tau - t)\sum_{j=0}^{m-2}\frac{a_j(t)\omega(\tau, t)}{(m-j-2)!}(t-\tau)^{m-j-2} + \\ + (\tau - t)\sum_{j=0}^{m-2}\frac{1}{(m-j-2)!}\int_0^1 \frac{\Gamma_j(t, t_1)}{t_1 - t}\,\omega(\tau, t_1)(t_1 - \tau)^{m-j-2}\, dt_1.\end{aligned}$$

On the basis of a study of the equation (3.1) one may prove the following theorem:

THEOREM 2. *If Problem K_0 for the equation* (1.4) *has a unique solution* (cf. [2], [3]) *and the degeneration of Problem K_ε into K_0 is regular, then for sufficiently small ε Problem K_ε for the equation* (1.2) *likewise has a unique solution, and the limit of its solution as $\varepsilon \to 0$ gives the solution of Problem K_0 for the degenerated equation* (1.4).

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